



MAGIC SQUARE

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Abstract

As a state exam thesis, we are implementing an interactive, electronic, but non-computer-based fourth-order magic square as a logic game. To solve the magic square, we use an improved version of a method originally published in an old book [1], which we present here. The game is produced using 3D printing. Its interactive component consists of tiles displaying the numbers to be placed, which contain transmitters, a microcontroller that receives their signals, and a connected touchscreen display.

Keywords: *magic square, interactive logic game.*

1. Introduction

What is a Magic Square?

A magic square is a tabular arrangement of natural numbers in which the sum of the numbers in each row, each column, and both main diagonals is always the same. This sum is called the magic constant, or alternatively, the magic sum. Magic squares are thus square matrices that possess these properties, and their study falls within the field of recreational mathematics.

The order of a magic square is the number of rows and columns, denoted by n . The construction of the square involves n^2 numbers. Depending on the arrangement of these numbers, different types of magic squares can be distinguished. Strictly structured magic squares consist of consecutive numbers without gaps or repetitions. If there are missing numbers, the square is gapped, and if it contains repeating numbers, it is repetitive.

The history of magic squares has faded into the mists of the past. The first confirmed mention of a third-order magic square dates back to the first century in China, although some theories suggest that it may have been known as early as two millennia before that. Whether due to their seemingly mystical properties or as a form of entertaining mental exercise, magic squares have been studied in various cultures throughout history. They

eventually reached Europe in the 11th century through Arabic transmission. In the early 1300's, the Byzantine scholar Manuel Moschopoulos wrote a mathematical treatise on magic squares, omitting the Middle Eastern mysticism associated with them and providing guidance on their construction. However, in the Western world, magic squares remained shrouded in occultism for a long time: they were known and described, but no records were made of their construction methods. Their perceived mystical nature is reflected in the way strictly structured magic squares were associated with celestial bodies: the third-order square was linked to Saturn, the fourth-order to Jupiter, and so on, with the fifth through ninth orders corresponding to Mars, the Sun, Venus, Mercury, and the Moon, respectively. This was not unusual, as metals, gemstones, and even the days of the week were similarly linked to celestial bodies according to the traditions of astrology and alchemy. Nevertheless, this association contributed to the growing European interest in magic squares, and from the 16th century onward, various construction techniques began to emerge.

Writings on the history of magic squares often mention Albrecht Dürer's engraving, Melencolia I from 1514 (Fig. 1 and 2.), which features a fourth-order magic square among many other symbolic elements. A modern example can be found at the Sagrada Família in Barcelona, next

to the southwest entrance. This repetitive magic square (Fig. 3) has a magic sum of 33, representing the number of years spent by Jesus Christ on the Earth. Interestingly, this square is a slightly modified version of the one in Dürer's engraving.

2. Creating a Magic Square

The construction of a magic square is based on the concept of the magic sum. If the first term of the sequence is 1, then the numbers involved in the construction of the n -th order square are the natural numbers from 1 to n^2 . For simplicity, let us denote the largest number n^2 by m . The sum of the numbers involved in the square can be determined using the formula for the sum of an arithmetic series, which gives us $M = m \cdot (m + 1) / 2$. Since the square consists of n rows, and the magic sum must be the same in each row, it follows that the magic sum for each row, N , is simply $N = M / n$. Since we are adding natural numbers, both N and M must also be natural numbers.

If the number sequence does not start from 1, but from a larger number k (or, if we accept that zero or negative numbers can also appear, starting from a smaller number), then each number in the sequence will be shifted by $k - 1$. Consequently, the magic sum will increase (or decrease) by $n \cdot (k - 1)$. This does not cause any significant change in the construction of the square, so for the sake of simplicity, we will focus on squares that start from 1.

Based on the mentioned properties of magic squares, we can observe that a first-order magic square would simply be a table containing only the number 1. We can also immediately notice that a strictly structured second-order square matrix does not exist. For example, if we place the number 1 in the top-left corner, we would need to place the same number next to it, below it, and diagonally as well in order to ensure that the magic sum is consistent in all directions. However, this second number must be the same as the one in the top-left corner, because otherwise, the magic sum would not work in the second row, second column, or along the other diagonal. Therefore, a second-order magic square can only be a trivial repetitive square, where the same number appears four times.

Creating a third-order magic square is already a more complicated process. If we attempt to fill the square using the first nine natural numbers, the large number of possibilities will present a challenge. This is because there are $9! = 362,880$



Fig. 1. Albrecht Dürer: *Melancolia I* (engraving from 1514)

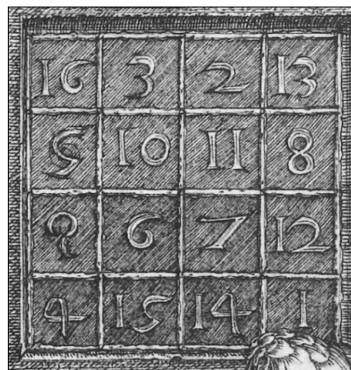


Fig. 2. Enlarged detail with the magic square.

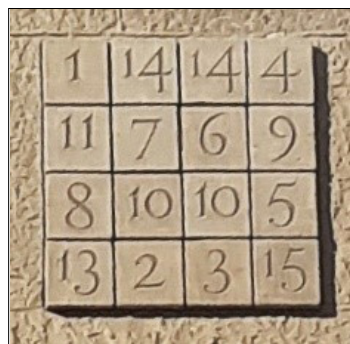


Fig. 3. The magic square at the entrance of the Sagrada Família in Barcelona.

different ways to arrange 9 elements. This is the number of permutations of nine elements, which, in human terms, is large enough that relying on trial and error to find a solution becomes unreasonable.

The difficulty of solving through mere trial and error can be illustrated with the magic hexagon. (Fig. 4). This is also a type of table of consecutive numbers, but in this case, the numbers are arranged in three directions. The numbers must be placed in such a way that the sum is the same in each direction. The sum of the 19 numbers, when divided between the five "rows" in each direction, results in the magic sum of 38.

According to records, the puzzle was published in 1910 in a weekly local newspaper titled *The Pathfinder* in Washington D.C. At the time, 19-year-old Clifford W. Adams took an interest in the problem and decided to solve it. To speed up the process, he used hexagonal ceramic tiles on which he wrote the numbers. He worked on the puzzle during his free time, and eventually, in 1957, while recovering in hospital, he found the solution. He wrote it down on a piece of paper, but it was lost, so it took another five years – more than half a century in total – before the solution was found again. Adams then sent his solution to a mathematician, who conducted thorough investigations of the puzzle, and that is how it came to the attention of the mathematical community. Over time, it was discovered that Adams' solution, excluding mirrored and rotated versions, was the only possible one – this was proven through computer-aided trials. It was also determined that no

smaller or larger magic hexagon exists. A method was developed that significantly reduced the number of configurations to be examined, making it relatively easy to reach the solution even without a computer. Interestingly, this puzzle had been previously published by Ernst von Haselberg in 1888 in a mathematical journal (*Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*), with the solution being published a year later, in the same journal.

A possible targeted solution for creating a third-order magic square is as follows: the magic sum of a strict, first-order third-order square is 15. We write down the possible triplets whose sum equals this value. For easier reference, we list them in such a way that for each number from 1 to 9, we include the other two possible numbers (which causes the list to contain repetitions, Fig. 5).

Well, from this list, it is clear that only the number 5 appears in four different arrangements, so this number must be placed in the center of the square (since it must appear in a row, a column, and both diagonals), Fig. 6.

Similarly, it can be noticed that only 2, 4, 6, and 8 can be placed in the corners, as these numbers appear in three different arrangements. The corners are at the intersection of a row, a column, and a diagonal. If we place the number 2 in the top left corner, then the number 8 must go in the bottom right corner, as the magic sum can only be achieved this way. If we place the number 4 in the top right corner, then the number 6 must go in the bottom left corner (Fig. 7).

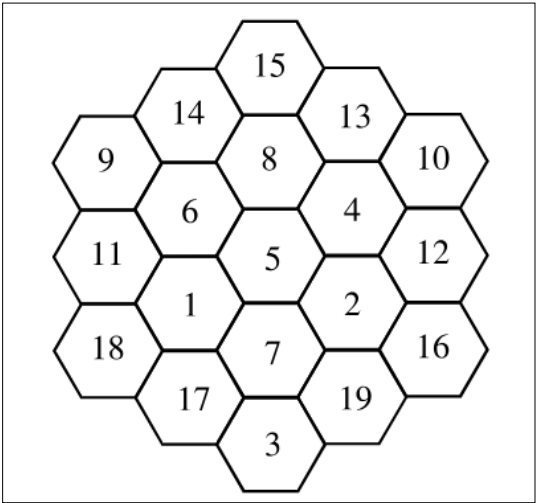


Fig. 4. Magic hexagon.

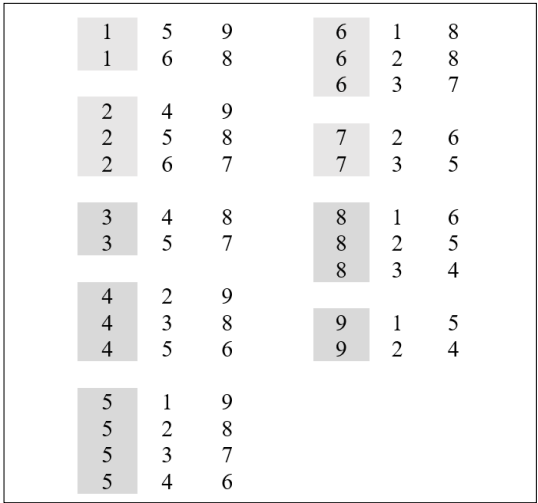


Fig. 5. Triplets that sum to 15.

	5	

Fig. 6. The center of the third-order magic square.

2		4
	5	
6		8

Fig. 7. The corners of the third-order magic square.

2	9	4
7	5	3
6	1	8

Fig. 8. The third-order magic square.

The remaining numbers appear in only two arrangements, and they are placed at the intersection of a row and a column (excluding the diagonals). Keeping the magic sum in mind, we can simply fill in the remaining four spots (Fig. 8).

If we write different numbers in the two top corners, the resulting arrangement will be the rotated and/or reflected versions of it (a total of eight different versions exist). However, if we disregard these, we can state that there is only one type of tightly fitting magic square (the reflection in both directions simultaneously corresponds to a 180-degree rotation, and the second direction's reflection also results in the rotated version of the square reflected in the first direction).

3. The Fourth-order Magic Square

Creating larger magic squares is no longer such a simple task. Our work was based on the observation published by György Berger in his 1986 book "Magic Squares". [1] According to this, in the case of a fourth-order magic square, if we start the number sequence with 0, the 16 elements of the sequence, when written in base 4, form numbers like 00₄, 01₄, ... 33₄. These numbers consist of two digits: the first (right-hand) digit is between 0 and 3, and the second (left-hand) digit is also between 0 and 3. In the complete sequence, there are four numbers with the first digit being 0, four with the first digit being 1, and so on, and the same applies to the second digit (therefore, there are four numbers with the second digit being 0, and so on). To calculate the magic sum, we need to sum four numbers, meaning the sum will be the sum

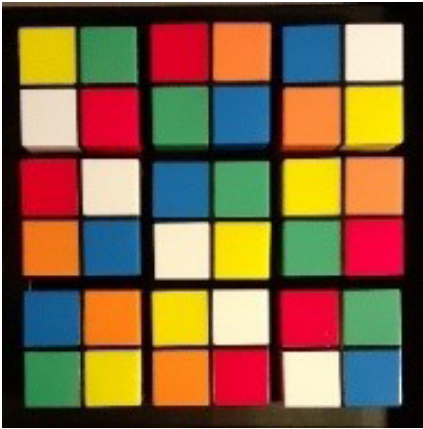


Fig. 9. Color magic square (ThinkFun: Color Cube Sudoku).

of four first and four second place values. If the sum of the place values in each row, column, and along the two diagonals is the same number, then the sum of the numbers will also be the same: the magic sum. Therefore, considering that each digit occurs four times in its place value, if we fill the table in such a way that each digit only appears once per place value in any row, column, or diagonal, the resulting square will be a magic square. However, this is not as simple as it first appears because we must exclude repetitions.

The process is easier to follow if we substitute the numbers with colors and work with separate tables for each place value. In other words, we create special Latin squares using colors: ones where a color appears only once not just in the rows and columns, but also along the diagonals. In the mentioned book [1] the author calls these "color magic squares." Creating such squares is also a logic puzzle, and one version available commercially is the "Color Cube Sudoku" by the brand ThinkFun, one possible solved state of which is shown in Fig. 9.

We need to create two squares such that when placed on top of each other, every possible color combination appears only once.

The procedure developed in this way includes the following steps:

- 1. Place the four colors along the main diagonal (from the top-left corner to the bottom-right corner):

0			
	1		
		2	
			3

2. Place the same four colors along the secondary diagonal (from the bottom-left corner to the top-right corner), but in a different order, ensuring that no color is repeated in the same row or column. There are only two possible arrangements (in our example, the second arrangement is blue-red-yellow-green):

0			2
	1		
	0		
1			3

3. The remaining empty spaces can be easily filled while ensuring that no colors are repeated. In the provided example, the red can only appear in the third row of the first column, after which the red in the second column must be placed in the first row. The blue must go into the remaining space in the first column, and similarly, the blue will complete the second column. In the third column, the yellow can only be placed at the bottom, and then the green goes above it. By now, the remaining spaces for the yellow and green in the fourth column will already be determined:

0	3	1	2
2	1	3	0
3	0	2	1
1	2	0	3

4. The main diagonal of the second square can be the same as the first one, but the colors on the secondary diagonal must be arranged in a different order (as mentioned in the second possibility):

0			1
	1		
	3		
2			3

5. In the second square, we fill in the remaining empty spaces. If we keep the main diagonal, then the second square will be the transpose of the first one.

0	2	3	1
3	1	0	2
1	3	2	0
2	0	1	3

6. To create the magic square from the two color-magic squares, we proceed as follows: if in the intersection of the i -th row and the j -th column, the first square contains the number A_{ij} , in base-4, and the second contains B_{ij} we write the number $A_{ij} + B_{ij} + 1$ in base-10 in the

corresponding place of the magic square. The unit is added to ensure that the sequence starts with 1, rather than 0. The result is showed in Fig. 10.

If we swap the first two (0 and 1) or the last two (2 and 3) elements on the main diagonal, the square will become *pandiagonal*, meaning that the "broken diagonals" parallel to the main diagonal will also sum to the magic constant.(Fig. 11)

It should be noted that this method of creating color-magic squares is just one possibility among many, and only a portion of the possible magic squares can be created in this way. However, it is a relatively simple, visual solution.

Not all magic squares correspond to magic color squares (i.e., they do not have the properties of color-magic squares). Here is the case of the Dürer's square, Fig. 12.

However, we can notice that the sum of the numbers (place values) vertically, horizontally, and along both diagonals is always 6 (more precisely: 12_4), with each of the four numbers appearing four times. Additionally, there are no repeating pairs at the overlap of the two squares. For example, in the first row of the first square, we have $3_4 + 0_4 + 0_4 + 3_4 = 12_4$, while at the same time also we have $0_4 + 1_4 + 2_4 + 3_4 = 12_4$.

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

Fig. 10. Fourth-order magic square.

0	3	2	1
2	1	0	3
1	2	3	0
3	0	1	2

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

0	2	1	3
3	1	2	0
2	0	3	1
1	3	0	2

Fig. 11. Pandiagonal magic square.

3	0	0	3
1	2	2	1
2	1	1	2
0	3	3	0

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

3	2	1	0
0	1	2	3
0	1	2	3
3	2	1	0

Fig. 12. Dürer's magic square.

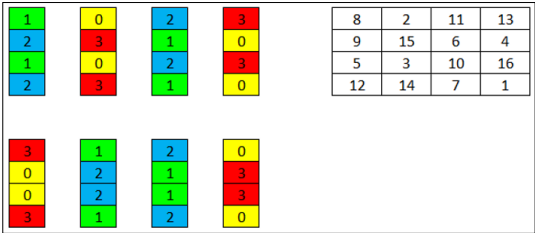


Fig. 13. A more general form of a magic square.

Based on these observations, we can create more general forms of magic squares, such as the one in Fig. 13.

4. The Magical Properties of a Fourth-Order Square

In relation to fourth-order magic squares, it is worth mentioning the fact that the magic sum is not only found along rows, columns, diagonals, and possibly broken diagonals. A provable general property of these squares is that the magic sum is also obtained:

- as the sum of four symmetrically placed numbers (referring to the four corners and the four numbers in the center of the square);
- in the case of 4×4 partitions aligned with the corners;
- if the square is pandiagonal, then for any 4×4 partition;
- as the sum of the two middle elements of the first row and the two middle elements of the bottom row;
- also as the sum of the two middle elements of the left column and the two middle elements of the right column

(so it's no surprise that these squares were thought to be magical.)

If we want to create a fourth-order magic square through trial and error, we can also use these properties.

5. The Extension of the Method to Magic Squares of Other Orders

The method, in its initial version based on the first, easily algorithmizable color-magic square, cannot be applied to the third-order magic square. However, by allowing repetitions, we can solve this problem as well (the sum of the place values must be the same along each row, column, and the two diagonals, Fig. 14).

The creation of a fifth-order magic square with this technique is somewhat more complicated but achievable. We need to work in the base-5 num-

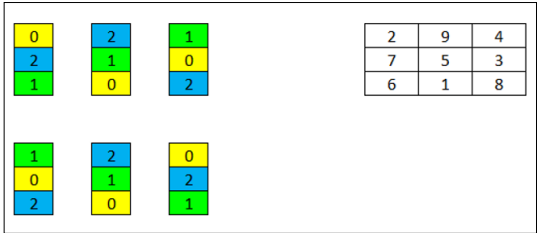
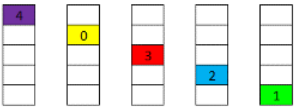


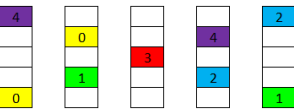
Fig. 14. Application to the third-order magic square.

ber system, meaning with five colors. This can happen through the following steps:

1. In the first square, we place the numbers between 0 and 4 along the main diagonal:

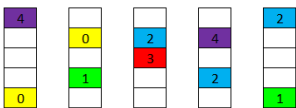


2. On the secondary diagonal, we place the numbers in such a way that the same pair of the colors does not appear in the same row or column. This constraint means that the elements can only be placed in one specific way on the secondary diagonal. The color at the intersection of the diagonals is not counted twice; it will only occupy the middle row and the middle column. By "pair of colors," we refer to the two colors that are symmetrically positioned in a column – at the top and bottom, as well as the second and fourth positions:



3. We fill the triangles above and below the main diagonal in such a way that each color appears twice in each triangle. Additionally, no color should appear more than once in the same row or column. We must proceed step by step, following the order determined by the colors already placed, as they will dictate where the next colors go.

– In the example, the first color that can be placed in the upper triangle is the second blue, which can only go in the second position of the third column:



– In the upper triangle, green can only be placed in the third and fourth columns. It must be placed at the top of the third column, as there are no other empty spots left there:

4		1		2
	0	2	4	
		3		
	1		2	
0				1

– Thus, in the upper triangle, the second green must be placed in the fourth column, in the third position:

4		1		2
	0	2	4	
		3	1	
	1	2		
0				1

– In the upper triangle, there must be two yellows. These can appear in the fourth and fifth columns. The first yellow must go to the top of the fourth column, but we cannot yet determine the exact location of the second yellow:

4		1	0	2
	0	2	4	
		3	1	
	1	2		
0				1

– In the upper triangle, the positions of the other elements are not determined, so we move down to the lower triangle. We notice that the bottom of the fourth column is missing only one red:

4		1	0	2
	0	2	4	
		3	1	
	1	2		
0			3	1

– There must also be two reds in the upper triangle. The first red can only be placed at the top of the second column:

4	3	1	0	2
	0	2	4	
		3	1	
	1	2		
0			3	1

– We can't place any more elements in the upper triangle at this point, but we notice that in the lower triangle, one of the yellows must go in the fourth position of the third column:

4	3	1	0	2
	0	2	4	
		3	1	
	1	2		
0			3	1

– The only color missing from the bottom of the third column is purple, so why not continue with this?

4	3	1	0	2
	0	2	4	
		3	1	
	1	0	2	
0			3	1

– In the bottom row, only one space remains empty, the second one. Blue should go here:

4	3	1	0	2
	0	2	4	
		3	1	
	1	0	2	
0	2	4	3	1

– Only one element is missing from the second column: its missing middle element can only be purple:

4	3	1	0	2
	0	2	4	
	4	3	1	
	1	0	2	
0	2	4	3	1

– We check if there is a color that appears four times already. One of these is yellow. It must be placed in the center of the last column:

4	3	1	0	2
	0	2	4	
	4	3	1	0
	1	0	2	
0	2	4	3	1

– The other such color is blue. Similarly, in the first column, blue can only be placed in the center:

4	3	1	0	2
	0	2	4	
2	4	3	1	0
	1	0	2	
0	2	4	3	1

– The fifth green fits in the second position of the first column:

4	3	1	0	2
1	0	2	4	
2	4	3	1	0
	1	0	2	
0	2	4	3	1

– The first column is completed by the red placed in the fourth position:

4	3	1	0	2
1	0	2	4	
2	4	3	1	0
3	1	0	2	
0	2	4	3	1

– The last element of the second row must be red; this would be the fifth red element:



–The last column, and thus the square, is completed by the purple color placed in the fourth position:



We can observe that the order of the numbers on the main diagonal quite clearly determines the structure of the square. For the given main diagonal, we can create the corresponding secondary diagonal in two ways, but once that is done, the square can only be filled in one way. By changing the order of the elements on the diagonal, we obtain a new square: the five elements on the main diagonal can be arranged in 120 different ways, and with two possible secondary diagonals, we get a total of 240 variations (including rotated and mirrored versions), but the structure of the color-magic square remains the same for all of them.

4. . The second square can be obtained from the first by transposing it, or by reflecting its rows horizontally or its columns vertically. Moreover, by performing full pivoting, we can create another color-magic square: this involves swapping the i -th row with the j -th row, and the i -th column with the j -th column (simply swapping two rows or two columns does not result in the magic sum along the diagonals). These methods can also be combined. In the diagram, the second square is the vertical mirror image of the first.
5. The magic square is created from the two color-magic squares in the already known way (See Fig. 15)

6. Interactive Magic Square

With the resurgence of the Rubik's Cube cult, interest in various puzzles and other logic games has increased. An entire industry has developed around this, with its focal point in China. The magic square, magic hexagon, and a close relative, the magic star, have also appeared on the market as wooden puzzles (the first as the „Wooden Number 1-16 Puzzle” (Fig. 16), the second as the „Aristotle's Number Puzzle” (Fig. 17), and the third as the „Roman Numeral Puzzle” (Fig. 18).

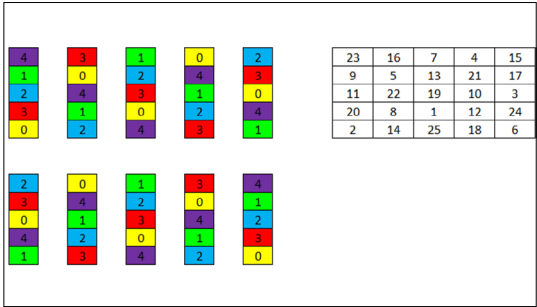


Fig. 15. A fifth-order magic square created with color-magic squares.



Fig. 16. Fourth-order magic square as "Wooden Number 1-16 Puzzle" (Temu, unbranded product).



Fig. 17. Magic hexagon as "Aristotle's Number Puzzle" (Professor Puzzle, "Great Minds" series).

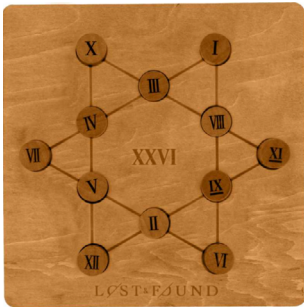


Fig. 18. Magic star as "Roman Numeral Puzzle" (Professor Puzzle, "Lost & Found" series).

However, many other logic games have also been released in electronic versions: this means that the game itself is tangible, so it's not a computer game, but a built-in circuit connects the game and the player. This usually means that the puzzle to be solved appears on a small screen, and the game has a transmitter that detects the successful solution. This may be associated with timing, sound and light effects, or the display of helpful tips. As an example, we show the GiiKER "Super Slide" (known in the Western world as klotski, with the Chinese name huarong dao) (Fig. 19), where the small screen in the top left corner shows the current starting position to be solved, and the device is controlled with push buttons. The red block's reaching the goal is detected by a sensor, and when this happens, the next level automatically appears on the screen.

Based on this idea, it occurred to us to create the fourth-order magic square as an interactive game, where the built-in electronics automatically calculate the sums of rows, columns, and diagonals. For this, each of the 16 numbered tiles must have a unique transmitter, and the tray used to place the tiles must have an equal number of sensors. A microcontroller will process the data, and a suitable display will need to be connected to show the results.

The main challenge lies in solving the sensing issue. We analyzed several solutions: the most elegant would be to identify the placed tiles using an RFID system, but placing and operating 16 RFID readers is quite a difficult task. Another approach would be to use a capacitive or inductive sensor in a contactless manner, but these typically only detect the presence of an object and cannot distinguish between them. The same applies to Hall sensors. As a more affordable, safer, and easier-to-implement solution, we use voltage dividers (Fig. 20): one resistor (R_1) is placed in the tray, and its resistance is the same in all positions. The other, unique resistance (R_2), which is different for each tile, is embedded in the tile, and the connection to the tray is made with pins hidden from the user. The two resistance connection points are connected to an analog sampling input through a current-limiting resistor. To prevent this input from floating when no tile is inserted, the other end of the R_1 resistor is connected to the power supply, and the voltage will appear on the analog input. The other end of the R_2 resistor is connected to the ground through the connector. The 16 resistors required for the voltage dividers must be chosen so that there are 17 distinguishable val-



Fig. 19. The GiiKER „Super Slide” interactive logic game.

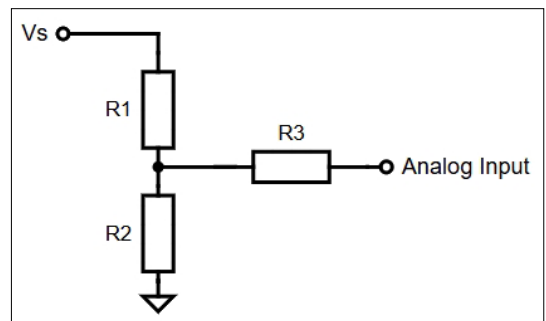


Fig. 20. The voltage divider used for detecting the inserted tile.

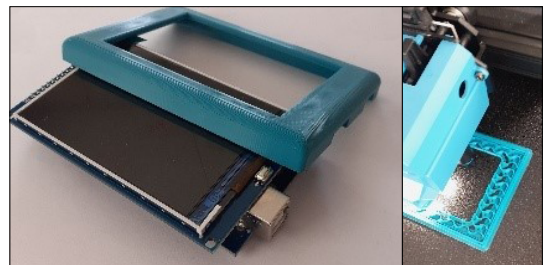


Fig. 21. Arduino Mega, LCD display, and its frame made using 3D printing.

ues across the entire voltage range, with roughly equal spacing (the seventeenth value is necessary for the empty positions).

An Arduino Mega was deemed suitable as the microcontroller, as it has 16 analog inputs. A 3.5-inch, 480×320-pixel LCD screen can be directly connected to it, on which the 16 inserted tiles and the calculated sums can be displayed. (Fig. 21).

The tiles are made using 3D printing and have dimensions of 30×30 mm. They need to be pinned onto rectangular truncated pyramid-shaped supports on a tray, also made with 3D printing. The support houses the connector nut, and the tile's truncated pyramid-shaped indentation hides the two pins. The surface of the tile features the number, along with color-coded marks representing the first and second place digits in base-4 numeral system. (Fig. 22)

The tray, together with the frame of the screen, forms a single body, which serves as the top of the box containing the entire electronics.

With this logic game, our intention is for the player to first try solving the puzzle using the method of color-magic squares (for which the color codes on the tiles are necessary), as a fun brain exercise. However, other magic square solving strategies can also be tested with it. The automatic calculation and display of the sums accelerates the process, eliminating the need for paper and pencil. It also enables the introduction of easier game modes, where the microcontroller predefines the positions of a few tiles, and the player only needs to place the remaining ones.

7. Conclusions

With the expiration of the Rubik's Cube patent, a new golden age for twisty puzzles began: not only are traditional Rubik's Cube versions manufactured and sold, but a large number of puzzles with different patterns, shapes, rotation methods, and mechanisms have also been introduced. This resurgence has reignited the Rubik's Cube cult, attracting a large number of collectors, amateur developers, and competitors to this type of logic game. An entire industry has been built around it, and the puzzles are widely available for purchase. The most popular products are found in online stores with a diverse and abundant selection. Alongside this expansion of the product range, other logic games have also been released to the market (sold under the non-twisty category by Rubik's Cube distributors), including the fourth-order magic square.

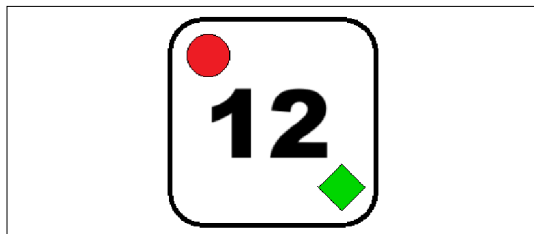


Fig. 22. One of the tiles.

Since solving the magic square is a relatively difficult task, we envisioned creating it as an interactive game. An electronic circuit automatically calculates the sum of the inserted numbers, which speeds up the solution process. Since the electronics are built around a microcontroller, it may even become possible to work out the solution on the screen, with the tiles only being placed after the solution is found.

The game logic is based on color magic squares. These are special Latin squares where repetition cannot occur even along the diagonals. This is how the tiles are labeled and displayed on the screen when placed. However, this does not exclude trying other solution methods that are not based on color magic squares.

In this work, the method based on color magic squares was extended to a higher-order square, but its implementation as an interactive game was not addressed.

Consequently, we can say that we have developed a new, easily understandable algorithm for constructing magic squares (which can only create certain types of squares), and by further developing the idea, we have created a new interactive game, bringing joy to those who enjoy logical puzzles.

References

Magic squares are discussed in many articles that are available on the internet. The most comprehensive summary of the knowledge related to them can be found on Wikipedia. If you are looking for a book on this topic, the most recent one, besides the one referenced below, is "Magic Squares and Cubes" by W. S. Andrews (1917). Andrews' very detailed book is available online:

https://djm.cc/library/Magic_Squares_Cubes_Andrews_edited.pdf.

However, we refer to only one work specifically dedicated to magic squares, as it contains all the essential information:

[1] Berger György: *Bűvös négyzetek*. Dacia Könyvkiadó, Cluj-Napoca, 1986.