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NON APPROXIMATIVE MODELLING OF INVOLUTE GEAR PAIR SLIDING

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Abstract

One of the main problems with the operation of involute tooth gears is that during engagement, the tooth profiles slide. The point of contact is the only point where pure rolling occurs among the engaging tooth profiles. This sliding, coupled with abrupt changes in load, collectively contribute to the failure and various malfunctions of the gears. In this research, the theoretical behavior of gears during sliding is studied. Existing mathematical models are examined for describing sliding and the results analysed. Then, a precise mathematical model developed for this research is presented. In creating the model, the aim was to avoid any geometric approximations and only consider the profile points where actual engagement between the gears could occur. Therefore, the contact points are interpreted within the real engagement zone. The overall goal is to create a mathematical solution that is equally applicable to simple and general involute tooth gear drives and which provides an accurate, realistic description of the relative sliding of gears.

Keywords*: gear, sliding, exact method, gear engagement.*

1. Introduction

In the case of gear drives, except for the cycloidal drive, the teeth do not roll cleanly over each other, slippage occurs and this can lead to rapid tooth wear. Common tooth failures include tooth surface damage such as progressive wear, pitting and chipping. Slippage between tooth surfaces causes wear, especially during the "break-in" phase after assembly, when prominent microgeometric irregularities in tooth surfaces are worn away by sliding with the counter gear. This type of wear diminishes and then disappears over time. It is not considered a failure because it leads to an increase in the contact area of the gears, which in turn increases the life expectancy of the drive. The problem is progressive wear, which occurs when wear increases after the running-in phase. Causes include: inadequate or insufficient lubrication, inadequate lubricant or dirt in the lubricant, or insufficient hardness of the tooth surfaces. Pitting occurs around the intersection of the tooth surface and the rolling surface. It occurs when the shear stress induced by the surface compressive stress and the sliding of the tooth surfaces against each other exceeds a limit value related to the number of cycles. First, small hairline cracks appear in the vicinity of the rolling surface tooth alignment, below the surfaces where the composite stress is greatest. As these hairline cracks reach the surface parts of the material fragment, forming dimples with rough edges. Pitting is mainly caused by surface fatigue of the material due to inadequate sizing or uneven surface load distribution, and inadequate hardness of tooth surfaces. Scratches on the tooth surface are short, straight, very shallow grooves in the direction of tooth profile sliding, caused by dust particles or other small impurities in the lubricating oil. The grooves are deeper than the scratches and occur in groups. Inadequate lubrication causes metallic contact between the tooth surfaces, causing them to momentarily weld together. However, due to the relative sliding of the tooth surfaces, the welded particles are torn out of one of the tooth surfaces. These particles can subsequently contaminate the lubricating oil and cause further damage to the tooth surface. This phenomenon is called chewing. In the 20th century, many studies were carried out to increase the load capacity and service life of gears. The sliding between tooth surfaces and the sliding speed have been studied extensively. Vidéki [1] found a close relationship between service life and sliding speed. He found that the sliding velocity decreases when the spindle distance is increased as much as the gearing allows. Diker [2], Szeniczei [3] and Bolotovszkij [4] based their design of gear drives on relative sliding. Szeniczei developed a model describing the relative slip for gear drives with an involute profile. This model is based on an approximation, since instead of the involute arc length, he used the arc length of the simulation circle, where the radius of the simulation circle is equal to the radius of curvature. According to his calculations, the length of the circular arcs can be obtained as the product of the radius of curvature, denoted ρ and the angular rotation of the gear, denoted γ, which, as above, is $\rho_1 \cdot \gamma_1$, for one gear and $\rho_2 \cdot \gamma_2$ for the other gear. Thus, the value of the relative slip from Figure 1 is:

$$\chi = \frac{\rho_1 \cdot \gamma_1 - \rho_2 \cdot \gamma_2}{\rho_1 \cdot \gamma_1} \tag{1}$$

The relative slip is described by a hyperbolic function, which shows that the relative slip is zero at the principal point and increases steadily away from the principal point (Figure 2).

However, the model also shows that the relative slip is zero at the principal point of coupling. This would imply that there is no wear at that point. This statement was contradicted by the experiments of Gavrilenko [5] Through his experiments he established a new method for sizing gears. He used slip accelerations to determine the location and magnitude of tooth surface wear. He described the tooth surface failure by taking the positional derivative of the relative slip along the contact line. The Ganz-Botka evolutionary tooth system is also based on relative slip. Botka [6] sought to compensate for relative slip when designing gears. He proved that if the relative slip values at the interface points are equalized, this implies the equalization and minimization of the instantaneous temperature rise and Hertzian stress and slip velocity multiplications of the tooth surfaces.

Research on the sizing of gear drives using relative slip and slip accelerations is still ongoing [7].

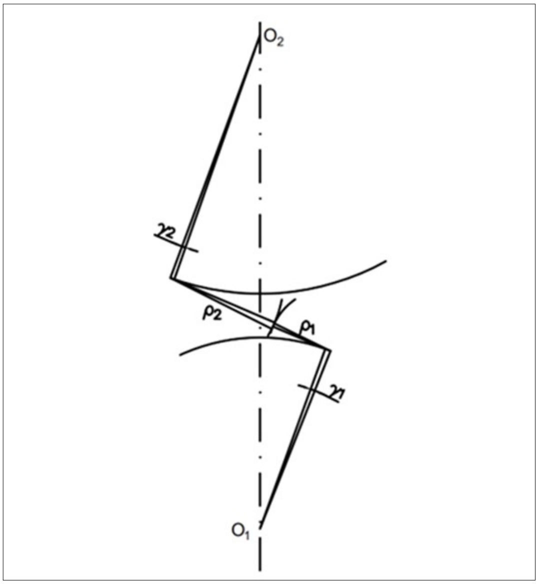

Fig. 1. *Gears in the moment of engagement.*

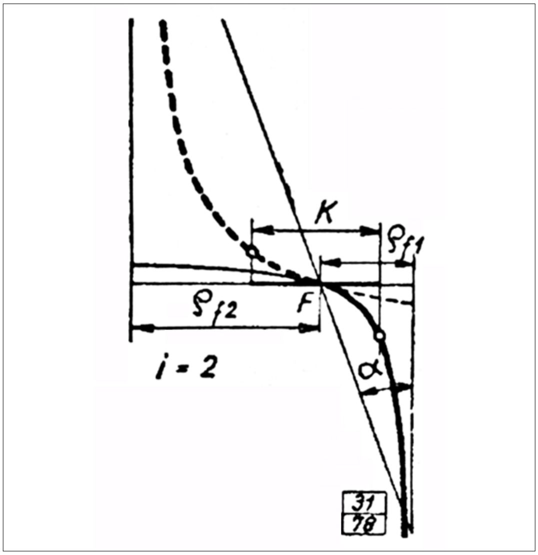

Fig. 2. *The relative sliding curve.* [3]

Many studies based on relative slip are matched with the model proposed by Szeniczei, which only approximates the real geometric ratios. In this thesis, our aim was to rethink this model in a way that avoids geometric approximations. Specifically, we wished to obtain a more accurate and realistic value for the relative slip. The hypothesis is that this may help in the future to design gear drives with increased gear life.

2. The proposed mathematical model

The basis of this research was to avoid any kind of geometric approximation when writing up relative slip. It was intended to investigate the points where the coupling between tooth surfaces could actually occur, so the coupling points within the real coupling section were interpreted. To achieve this the Erney diagram was used, as it contains the essential geometric elements of the gear drive without the need to draw circles [8]. The figure shows that the connecting line is the common tangent of the base circles. The actual coupling is established at A_1A_2 which is defined by the intersection of the coupling line and the head circles A_1 and A_2 (Figure 3). The theoretical coupling section is defined by the points T_1 and T_2 of the base circles tangent to the coupling line.

Figure 4.a. shows the relationship between the switching line and the position of the teeth. It can be seen that in all cases the points of contact are located in the plane of the coupling. The message in Figure 4. c. . is equivalent to that in Figure 2: the relative slip is zero at the principal point and increases steadily away from the principal point.

Subsequently, the optimal sizing of the gear drive was checked. We investigated where the point cut out by the base circles lies on the coupling line. This is significant because if its location is not correct, the task would require re-sizing the drive pair since, within these circles, instead of the involute profile ensuring correct coupling, is the derived looped involute or epicycloid of the root curve, and the involute profile should not be coupled with the root curve. It can therefore be stated that the coupling is correct if the head circle of the steering wheel intersects the coupling section outside the section defined by the base and base circle points of the wheel.

The first step is to add the points T_1 and T_2 which are at a tangent to the base circle. Next, we record the intersection points A1 and A2 of the head circles and the connecting line. From Figure 3 the following relationships can be written:

$$T_1A_1 = \sqrt{r_{a1}^2 - r_{b1}^2} \tag{2}$$

$$T_2A_2 = \sqrt{r_{a2}^2 - r_{b2}^2} \tag{3}$$

$$T_1\,T_2 = a_w \cdot \sin\alpha \tag{4}$$

$$T_1\,A_2 = T_1\,T_2 - T_2\,A_2 \tag{5}$$

$$T_2\,A_1 = T_1\,T_2 - T_1\,A_1 \tag{6}$$

$$A_1\,A_2 = T_1\,T_2 - T_1\,A_2 - T_2\,A_1 \tag{7}$$

Consider the point P, which is the intersection of the line of contact and the rolling circle, observing Figure 5, we can write down the following equations:

$$OT = \frac{m \cdot z}{2} \cdot \cos\alpha \tag{8}$$

$$LP = (h_0 - \xi + c_0 - \delta) \cdot m \tag{9}$$

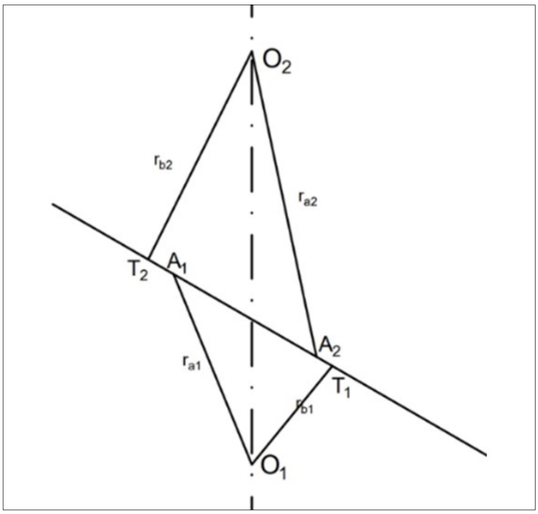

Fig. 3. *The simplified model of the Erney diagram.*

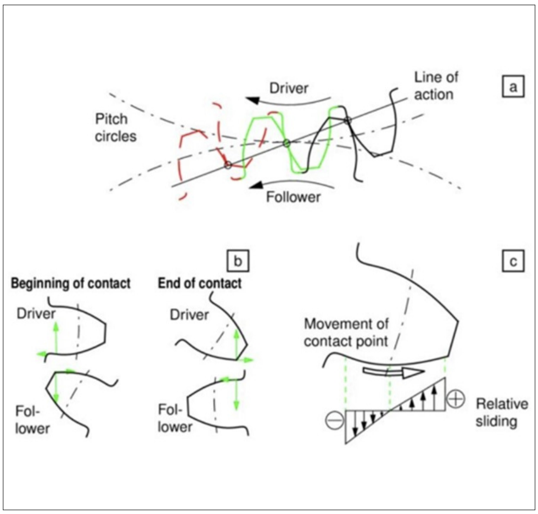

Fig. 4. *The sliding of gears,* [9] *a) The meshing of the teeth during transmission; b) The position of the teeth at the beginning and end of engagement; c) The relative sliding occurring on the teeth.*

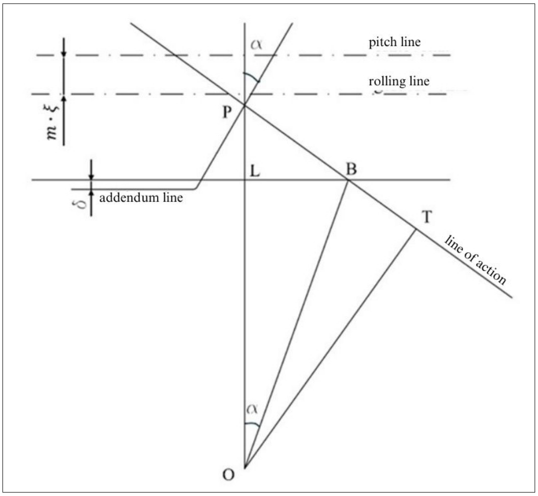

Fig. 5. *The calculation of the root circle radius.*

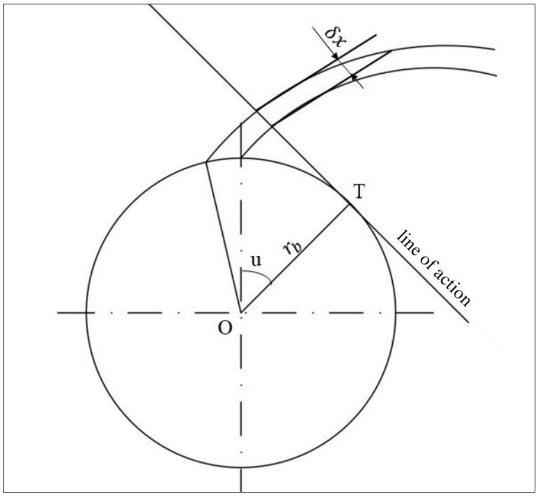

Fig. 6. *The calculation of the involute arc length.*

$$PB = \frac{LP}{\sin\alpha} \tag{10}$$

$$BT = PT - PB = OT \cdot tg\,\alpha - PB \tag{11}$$

$$r_t = \sqrt{BT^2 + OT^2} \tag{12}$$

It follows that the distance between the points cut by the base circles and the head circles on the contact line and the points of contact of the base circles is:

$$B_1T_1 = \sqrt{r_{t1}^2 - r_{b1}^2} \tag{13}$$

$$B_2T_2 = \sqrt{r_{t2}^2 - r_{b2}^2} \tag{14}$$

The points of contact of the base circles and the intersection of the root and head circles define the real working contact. The connection is good if the head circle of the counterpart wheel intersects the coupling section outside the section defined by the root and base circle points of the respective wheel, i.e

$$T_1A_2 \geq T_1B_1 \text{ and } T_2A_1 \geq T_2B_2 \tag{15}$$

The slip is defined here as the difference between the evolutionary arc lengths $\Delta\ell_1$ and $\Delta\ell_2$ corresponding to the distance between two finitely spaced attachment points. Formulas (16) and (17) are used to calculate the evolution arc lengths. To write the evolute arc length, we need to use the central angle corresponding to the rolling of the line generating the evolute on the base circle, i.e. a parameter u. Under the angular rotation u, the arc length of the circle affected by the generating line is equal to the product of the base circle of radius r_b and the angular rotation u The evolute generating line makes a planar motion with respect to the base circle, the instantaneous pole of which is the point of contact, i.e., for a given decoiling position, the radius of the evolute curve is given by exactly the segment of length $r_b u$, *so that the elementary arc length of the evolute curve is* $d\ell = r_b u\, du$. This expresses two decoiled evolute arc lengths corresponding to finite values of angular rotation u (**Figure 6**):

$$\Delta\ell_1 = \int_{u_1}^{u_2} r_{b1} \cdot u \cdot du = \frac{r_{b1} \cdot (u_2^2 - u_1^2)}{2} \tag{16}$$

$$\Delta\ell_2 = -\int_{u_1'}^{u_2'} r_{b2} \cdot u \cdot du = \frac{r_{b2} \cdot ({u_1'}^2 - {u_2'}^2)}{2} \tag{17}$$

The value of the relative slip can be obtained by relating the difference in the evolutionary arc length to the smaller evolutionary arc length, which in our case is the one associated with the driving wheel.

$$\chi = \frac{\Delta\ell_1 - \Delta\ell_2}{\Delta\ell_1} \tag{18}$$

The calculation function of the evolutionary arc length is written according to the following pseudocode:

<N> = 100	the number of points along the contact line
$< \delta x > = \dfrac{A_1A_2}{N-1}$	the length of the section travelled by the interface on the contact line

cicle from <i> = <0> to <N – 2>	definition of the loop variable
$< u_1 > = \dfrac{T_1A_2 + változó1 \cdot \delta x}{r_{b1}}$	lower parameter of the driving wheel
$< u_2 > = \dfrac{T_1A_2 + változó1 \cdot \delta x + \delta x}{r_{b1}}$	upper parameter of the driving wheel
$< u_1' > = \dfrac{T_1T_1 - (T_1A_2 + változó1 \cdot \delta x)}{r_{b2}}$	lower parameter of the driven wheel
$< u_2' > = \dfrac{T_1T_1 - (T_1A_2 + változó1 \cdot \delta x + \delta x)}{r_{b2}}$	upper parameter of the drivien wheel
$< dl_1 > = \dfrac{r_{b1} \cdot (u_2^2 - u_1^2)}{2}$	length of the arc in contact with the driving wheel
$< dl_2 > = \dfrac{r_{b2} \cdot (u_1'^2 - u_2'^2)}{2}$	length of arc in contact with driven wheel
$< \chi > = \dfrac{dl_1 - dl_2}{dl_1}$	relative slip value
$a_{i,0} = i \cdot \delta x$	first column of the output, length of the distance travelled on the contact line
$a_{i,1} = \chi$	second column of output, value of relative slip
cycle end	
return a	

According to the code, we obtained the relative slip model for elementary gears. Subsequently, we wished to investigate the use of the model for general-purpose involute gear drives. To achieve this, the maximum and minimum possible profile offsets were calculated, according to the literature, and then applied to the number of teeth of our choice. The maximum profile shift from the second law of evolutionary trigonometry and the minimum profile shift from the known geometric model of spline gearing were calculated, [3].

Table 1. *The parameters of the gears*

Title	Indication	Value
Module [-]	m	4
Pressure angle [rad]	a	π/9
Drive gear tooth number [-]	z_1	17
Driven gear tooth number [-]	z_2	34
Drive gear pitch radius [mm]	r_{w1}	34
Driven gear pitch radius [mm]	r_{w2}	68
Drive gear root radius [mm]	r_{f1}	29
Driven gear root radius [mm]	r_{f2}	163
Drive gear tip radius [mm]	r_{a1}	38
Driven gear tip radius [mm]	r_{a2}	72
Drive gear base radius [mm]	r_{a1}	31.94
Driven gear base radius [mm]	r_{a2}	63.89
Tooth height [mm]	h	9
Tooth thickness [mm]	c	1
Center distance [mm]	a_w	102

2.1. Numerical evaluations

The next part of the study is the numerical evaluation of the models. The mathematical models were applied to a specific gear drive in the MathCad environment, the parameters of which are shown in Table 1. The MathCad environment was chosen because it allowed easy verification of the calculations and the creation of graphs.

Using the relations (2) - (14), the length of the radius of the root circles was calculated taking into account the values defined in Table 1 and the position of points B_1, B_2. Since we aim for simplification, we do not consider profile shift, so tip clearance $h_0 = 1$ mm, root clearance $c_0 = 0.25$ mm, profile shift $\xi = 0$ mm and tooth root height $\delta = 0.15$ mm. Applying the equations to both the driving and driven wheels, we see that the radius of the root circle of the driving wheel is $r_{t1} = 31.97$ mm and the radius of the base circle of the driven wheel is $r_{t2} = 4.73$ mm.

Therefore, since $A_1T_2 = 14.31$ mm, $A_1T_1 = .23$ mm and $A_3T_1 = 1.70$ mm, $B_2T_2 = 10.39$ mm, it follows that the points intersected by the root circle are between the points intersected by the tip circle and the points intersected by the base circle, i.e. point B_1 is between T_1 and A_2, and point B_2 is between T_2 and A_1. In conclusion, we can say that the fold under consideration is correctly connected, i.e. it does not require re-measurement.

In the following, the mathematical solution of Szeniczei has been transferred to the MathCad environment, in order to make it easier to compare the results with our proposed model. We calculated the relative slip between the teeth using our mathematical model. The minimum and maximum profile shifts of the chosen gear drive were examined (Table 2) in order to investigate how the models behave for general gearing..

Table 2. *The limits of profile shifts for the selected number of teeth*

Tooth number	Minimum profile shift	Maximum profile shift
17	0	0.83
34	-0.98	1.71

Table 3. *The examined profile shifts*

ξ_1	ξ_2
0	−0.3
0	0.3
0.3	0
0.5	0

It is shown that both negative and positive profile shifts are allowed on the driven wheel, while only positive shifts are allowed on the drive wheel. Values from the allowed ranges were chosen as profile offsets therefore we set up four cases for general gearing (Table 3), for which the model behaviour is investigated. According to the profile shift, the gear drive parameters and the relative slip both from the Szeniczei model and from the model were re-calculated.

3. Discussion

Figure 7 illustrates the results obtained for elementary gears. The functions obtained are hyperbolic, as expected. However, there are clearly visible differences between the hyperbolas plotted by the two relative slip functions. Although they both show that as we move forward along the contact line, the relative slip between the gears decreases rapidly, their starting point and the moment when they reach zero differ significantly. In the Szeniczei solution, the initial value of the relative slip is considerably larger, almost double that in our model, and the zero point is reached at a point further along the line of contact.

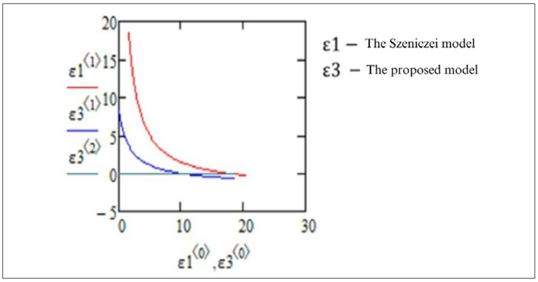

Fig. 7. *Models describing relative sliding for elementary gearing.*

Applying a profile shift to the drive wheel, the curves are visibly flattened, especially in the case of a positive profile shift (Figure 8). The curves representing the two models behave similarly when compared to the results obtained for the elementary gear. In the case of negative profile shifting, both curves show a reduction in the initial relative slip, reaching zero at almost the same point. For the positive profile shift, both curves start from a higher relative slip value, although the hyperbola illustrating the solution of Szeniczei has a much higher relative slip value compared to the elementary value and reaches the zero point much earlier than in the previous curves. The spatial difference between the two curves still remains: in our model, the value of relative slip, especially at the initial moment, is significantly smaller and reaches the moment of zero slip sooner at the contact line.

In the case of profile shifting on the driving gear, in both cases, the relative slip is reduced compared to the elementary gear, although it is reduced almost double in the case of the Szeniczei model compared to our model (Figure 9). Furthermore, observing the Szeniczei model, there does not seem to be any change relative to the elementary gear when the zero moment is reached, while in our solution the zero slip moment is more easily reached by positively shifting the profile of the driving wheel.

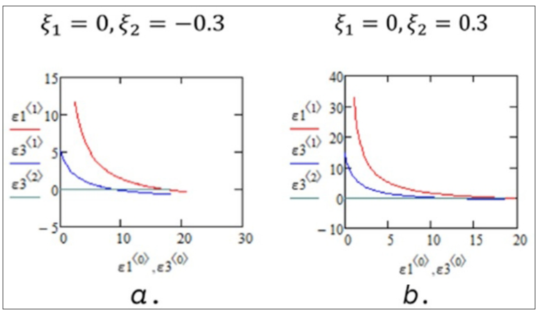

Fig. 8. *Models describing relative sliding for elementary gearing, profile shifting performed on the driven gear a) negative profile shifting b) positive profile shifting.*

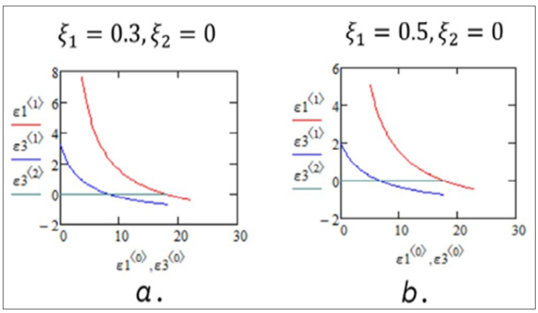

Fig. 9. *Models describing relative sliding for elementary gearing, profile shifting performed on the driving gear a) 0.3 mm profile shifting b) 0.5 mm profile shifting.*

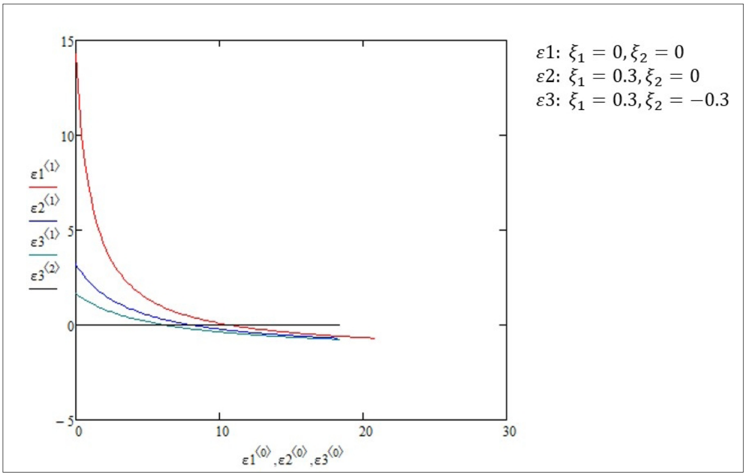

Fig. 10. *The proposed model describing relative sliding for various profile shifts.*

Figure 10 illustrates a proposed model for different profile shifts, which highlights how relative slip varies as a function of profile shift. It can be seen that the hyperbola flattens more and more as the profile shift increases. The model shows that the relative slip value can be significantly reduced by profile shifts on both gears. As shown in the figure, the relative slip value is then less than one third of the relative slip present in the case of elementary gears. The zero point is also reached sooner with a larger profile shift. At the same time, it is observed that the curves do not follow a new hyperbola after reaching the zero point, but almost converge. This allows us to conclude that the proposed model requires further, more detailed mathematical analysis, including the examination of specific cases.

4. Conclusions

Based on the evaluation of the results, it is observed that the relative slip is significantly reduced in the exact model compared to the previously established model. Moreover, the zero-slip condition is achieved earlier. Additionally, both elementary and general gears exhibit similar behavior across various profile shifts, indicating that the exact model of slip for involute gears applies universally.

The key mathematical difference between the Seniczei model and my proposed model is that in the classical (Seniczei) mathematical model, the angular rotations γ1 and γ2 satisfy the transmission ratio, i.e. $i_{12} = \dfrac{\omega_1}{\omega_2} = \dfrac{\gamma_2}{\gamma_1}$, so that the slip can be interpreted as an instantaneous limit, i.e $\chi = \dfrac{\rho_1 - i_{12}\,\rho_2}{\rho_1} = 1 - i_{12}\dfrac{\rho_2}{\rho_1}$, where the slip is only affected by the ratio of the transmission ratio and the profile curvature radii corresponding to the instantaneous contact, my proposed model follows the path of the contact along the real contact section and gives the ratio of the differences between the relative contact sections corresponding to discrete time intervals. Considering that the micro-geometric, i.e. real contact of the surfaces can be interpreted in terms of finite rather than elementary displacements, I believe that the formulation of the model I have derived is appropriate.

Further investigation into how these new findings impact gear drive sizing, operation, and longevity is deemed crucial. Experimental studies on machine elements are necessary to validate these results and explore their practical implications. There is also potential to extend this methodology to accurately predict relative slip not only for involute profiles but also for other tooth profiles such as cosine or sinusoidal profiles.

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