Műszaki Tudományos Közlemények vol. 18. (2023) 58–62.
DOI: Hungarian: https://doi.org/10.33895/mtk-2023.18.11
English: https://doi.org/10.33894/mtk-2023.18.11

KINEMATICS ANALYSIS OF CRANK MECHANISM WITH CONSTRAINT EQUATIONS

Izolda POPA-MÜLLER,[1] István PAPP[2]

[1] *Sapientia Hungarian University of Transylvania, Faculty of Technical and Human Sciences,, Department of Mechanical Engineering. Târgu Mureș, Romania, ipmuller@ms.sapientia.ro*

[2] *Transylvanian Museum-Society, Department of Technical Sciences, Cluj-Napoca, Romania, pappistvan1944@yahoo.com*

Abstract

The equation system, described for kinematic pairs of Crank mechanism, consist of general constraint equations. For Crank mechanism, the position of the center of gravity for the components, and the directions of the main axis of inertia can be determined for the entire kinematic cycle. To determine the positions, we use a coordinate system fixed to the center of gravity of the mechanism. The coordinates of the center of gravity of the members of the mechanism are always constant in relation to the geometric system attached to the component. The vector contour created in this way, defined by position vectors, must always be closed.

Keywords: *constraints, kinematic pairs, constraint equations.*

1. Crank Mechanism

The crank mechanism consists of (**Figure 1**) :

– engine fuselage (1);
– crankshaft (2);
– crank arm (3);
– piston (4).

OXYZ is the fixed system, $G_iX_iY_i,Z_i, i \in \overline{1,4}$ are the crankshaft, crank arm, piston central axis of inertia system, $O_iX_i^*Y_i^*Z_i^*, i \in \overline{1,4}$, is the auxiliary systems of the crankshaft components.

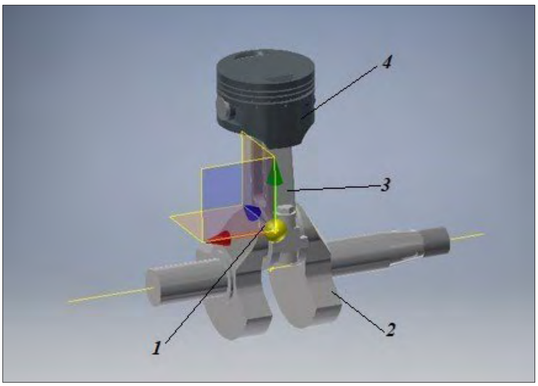

Fig. 1. *3D model of the crank mechanism and visibility of its center of gravity*

The vector contour defined by position vectors, must always be closed (**Figure 2**).

$$\vec{r}_{12} - \vec{r}_{21} + \vec{r}_{23} - \vec{r}_{32} + \vec{r}_{34} - \vec{r}_{43} + \vec{r}_{41} - \vec{r}_{14} = 0 \quad (1)$$

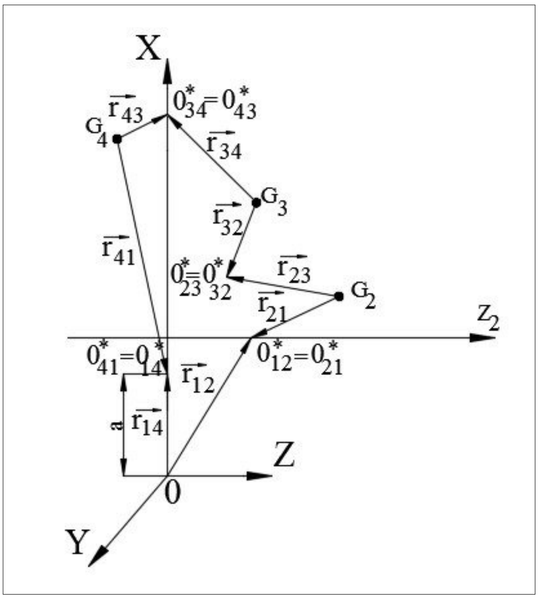

Fig. 2. *Vector contour of crank mechanism.*

1.1. Determination of mechanism constants

The center of gravity of the mechanism, in the auxiliary system:

$$G_1 = \begin{pmatrix} 50 \\ 0 \\ 0 \end{pmatrix} \cdot 10^{-3} \tag{2}$$

Directional factors of the main axis of inertia of the mechanism, in relation to the auxiliary system:

$$A_1 = \begin{pmatrix} 0.999277 & 0 & 0 \\ 0 & 0.999097 & 0 \\ 0 & 0 & 0.99978 \end{pmatrix} \tag{3}$$

The direction factors of the auxiliary system, in relation to the coordinate system of the main axis of inertia of the mechanism **[1]**:

$$\begin{pmatrix} \cos\alpha_{1i}^0 & \cos\beta_{1i}^0 & \cos\gamma_{1i}^0 \\ \cos\alpha_{2i}^0 & \cos\beta_{2i}^0 & \cos\gamma_{2i}^0 \\ \cos\alpha_{3i}^0 & \cos\beta_{3i}^0 & \cos\gamma_{3i}^0 \end{pmatrix} = A_1^T \tag{4}$$

The $\vec{r}_{12}$ position vector defines the coordinates of the point O_{12}^* in relation to the auxiliary system placed in the center of gravity of the mechanism:

$$\vec{r}_{12} = O_{12}^* - G_1 \tag{5}$$

The $\vec{r}_{12I}$ position vector expresses the coordinates of the point O_{12}^* in relation to the coordinate system of the central axes of inertia placed in the center of gravity of the mechanism:

$$\vec{r}_{12I} = A_1^T \cdot \vec{r}_{12} \tag{6}$$

The $\vec{r}_{14}$ position vector expresses the coordinates of the point O_{14}^* in the auxiliary system placed in the center of gravity of the mechanism:

$$\vec{r}_{14} = O_{14}^* - G_1 \tag{7}$$

The fixed point O_{14}^* on the mechanism that coincides with the origin of the auxiliary system fixed on the mechanism and is located on the axis of the 4th piston:

$$O_{14}^* = \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

1.2. Determination of the crankshaft constants

From the Inventor window (**Figure 4**) you can read the moments of inertia of the crankshaft and the coordinates of its centre of gravity.

The O_{21}^* point on the main axis of the mechanism (**Figure 3**) the position vector relative to the auxiliary system placed at the center of gravity G_2 can be written:

$$\vec{r}_{21} = O_{21}^* - G_2, \; O_{21}^* = \begin{pmatrix} 0.03 \\ 0 \\ -0.0125 \end{pmatrix} \tag{9}$$

The center of gravity of the axle, in relation to the auxiliary system on the main axle (**Figure 4**):

$$G_2 = \begin{pmatrix} 1.677 \\ -0.004 \\ -5.574 \end{pmatrix} \cdot 10^{-3} \tag{10}$$

Directional factors of the main axes of inertia in relation to the auxiliary system on main axis 2:

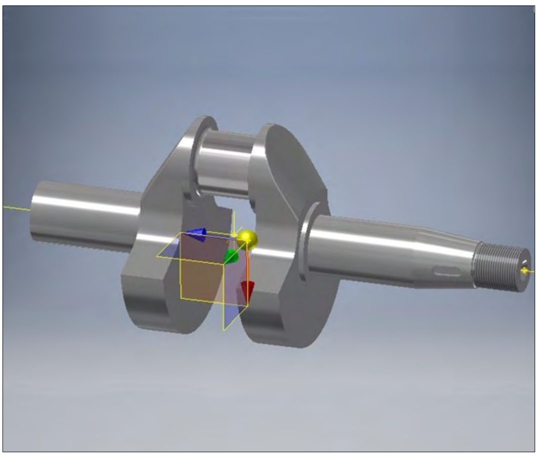

Fig. 3. *3D model of the crankshaft and visibility of its center of gravity.*

Fig. 4. *Data of inertia and center of gravity of crankshaft in Inventor.*

$$A_2 = \begin{pmatrix} 0.99997563 & 0 & 0 \\ 0 & -1.744444 \cdot 10^{-4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

1.3. Determination of the crank arm constants

The coordinates of the center of gravity relative to the auxiliary system taken on the axis line of the upper hole of the crank arm connected to the crankshaft (**Figure 5**):

$$G_3 = \begin{pmatrix} -61.158 \\ -0.004 \\ -2.586 \end{pmatrix} \cdot 10^{-3} \tag{12}$$

The coordinates of the point O_{34}^* taken on the axis line of the lower hole of the crank arm relative to the auxiliary system of the crank arm. The point coincides with the point O_{43}^* on the piston pin.

$$O_{34}^* = \begin{pmatrix} 89.3 \\ 0 \\ 0 \end{pmatrix} \cdot 10^{-3} \tag{13}$$

1.4. Determination of the piston constants

Direction factors of the main inertial axes relative to the auxiliary system attached to the piston (**Figure 6**):

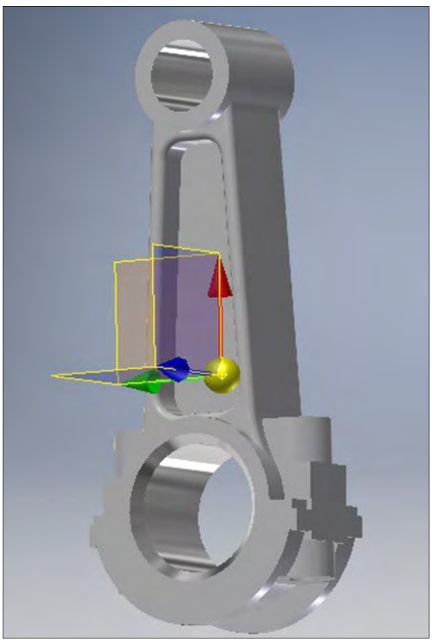

Fig. 5. *3D model of the crank arm and visibility of its center of gravity.*

$$A_4 = \begin{pmatrix} 0.99979172 & 0 & 0 \\ 0 & 0.99999998 & 0 \\ 0 & 0 & 0.99999945 \end{pmatrix} \tag{14}$$

The coordinates of the point O_{43}^* taken on the axis line of the piston pin coincide with the point taken on the crank arm during operation.

$$O_{43}^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot 10^{-3} \tag{15}$$

2. Incorporating the constraint equations into a system

We define the components in 37 positions. The driving parameter of the mechanism ψ_2^*, knowing its direction factors formed with the inertial crankshafts for the components' auxiliary systems and its rotation angles. [1, 2].

2.1. Constraint equations of the crank mechanism discussed as a space mechanism

The number of constraint equations depends on the type of joint, in the case of a rotational joint we have 5 constraint equations.

The constraint equation written on the rotating joint of the mechanism [3–5]:

$$\begin{aligned} & x_{G_1} + x_{12} \cdot \alpha_{110} + y_{12} \cdot \alpha_{210} + z_{12} \cdot \alpha_{310} \\ & -x_{G_2} - x_{21} \cdot \alpha_{12}(\psi_2, \theta_2, \phi_2) + \\ & -y_{21} \cdot \beta_{12}(\psi_2, \theta_2, \phi_2) - z_{21} \cdot \gamma_{12}(\psi_2, \theta_2)) = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} & y_{G_1} + x_{12} \cdot \beta_{110} + y_{12} \cdot \beta_{210} + z_{12} \cdot \beta_{310} \\ & -y_{G_2} - x_{21} \cdot \alpha_{22}(\psi_2, \theta_2, \phi_2) - y_{21} \cdot \beta_{22}(\psi_2, \theta_2, \phi_2) + \\ & -z_{21} \cdot \gamma_{22}(\psi_2, \theta_2) = 0 \end{aligned} \tag{17}$$

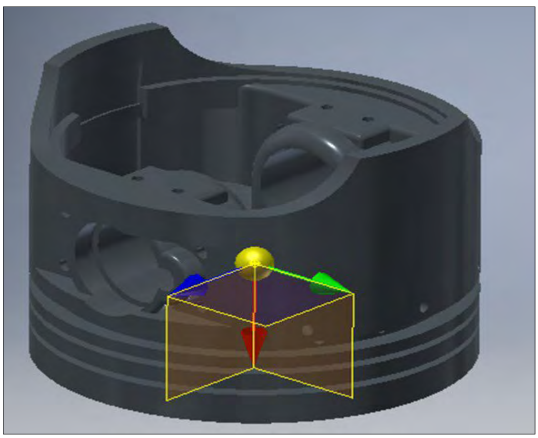

Fig. 6. *3D model of the piston and visibility of its center of gravity.*

$$\begin{aligned}&z_{G_1}+x_{12}\cdot\gamma_{110}+y_{12}\cdot\gamma_{210}+z_{12}\cdot\gamma_{310}\\&-z_{G_2}-x_{21}\cdot\alpha_{32}(\psi_2,\phi_2)-y_{21}\cdot\beta_{32}(\psi_2,\phi_2)\\&-z_{21}\cdot\gamma_{32}(\psi_2,\theta_2)=0\end{aligned}\tag{18}$$

$$b_{131}-\left(\begin{aligned}&\alpha_{12}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{120}+\beta_{12}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{220}+\\&\gamma_{12}(\psi_2,\theta_2)\cdot\gamma_{320}\end{aligned}\right)=0\tag{19}$$

$$b_{231}-\left(\begin{aligned}&\alpha_{22}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{120}+\beta_{22}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{220}+\\&\gamma_{22}(\psi_2,\theta_2)\cdot\gamma_{320}\end{aligned}\right)=0\tag{20}$$

Constraint equation written on the rotating joint of the crankshaft:

$$\begin{aligned}&x_{G_2}+x_{23}\cdot\alpha_{12}(\psi_2,\theta_2,\phi_2)+y_{23}\cdot\beta_{12}(\psi_2,\theta_2,\phi_2)+\\&+z_{23}\cdot\gamma_{12}(\psi_2,\theta_2)-x_{G_3}-(x_{32}\cdot\alpha_{13}(\psi_3,\theta_3,\phi_3)+\\&+y_{32}\cdot\beta_{13}(\psi_3,\theta_3,\phi_3)+z_{32}\cdot\gamma_{13}(\psi_3,\theta_3))=0\end{aligned}\tag{21}$$

$$\begin{aligned}&y_{G_2}+x_{23}\cdot\alpha_{22}(\psi_2,\theta_2,\phi_2)+y_{23}\cdot\beta_{22}(\psi_2,\theta_2,\phi_2)+\\&+z_{23}\cdot\gamma_{22}(\psi_2,\theta_2)-y_{G_3}-(x_{32}\cdot\alpha_{23}(\psi_3,\theta_3,\phi_3)+\\&+y_{32}\cdot\beta_{32}(\psi_3,\theta_3,\phi_3)+z_{32}\cdot\gamma_{32}(\psi_3,\theta_3))=0\end{aligned}\tag{22}$$

$$\begin{aligned}&z_{G_2}+x_{23}\cdot\alpha_{32}(\theta_2,\phi_2)+y_{23}\cdot\beta_{22}(\theta_2,\phi_2)+\\&+z_{23}\cdot\gamma_{32}(\theta_2)-z_{G_3}-(x_{32}\cdot\alpha_{33}(\theta_3,\phi_3)+\\&+y_{32}\cdot\beta_{33}(\theta_3,\phi_3)+z_{32}\cdot\gamma_{33}(\theta_3))=0\end{aligned}\tag{23}$$

$$\begin{aligned}&(\alpha_{12}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{120}+\beta_{12}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{220}+\\&\gamma_{12}(\psi_2,\theta_2)\cdot\gamma_{320})-(\alpha_{13}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{130}+\\&+\beta_{13}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{230}+\gamma_{13}(\psi_3,\theta_3)\cdot\gamma_{330})=0\end{aligned}\tag{24}$$

$$\begin{aligned}&(\alpha_{22}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{120}+\beta_{22}(\psi_2,\theta_2,\phi_2)\cdot\gamma_{220}+\\&\gamma_{22}(\psi_2,\theta_2)\cdot\gamma_{320})-(\alpha_{23}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{130}+\\&+\beta_{23}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{230}+\gamma_{23}(\psi_3,\theta_3)\cdot\gamma_{330})=0\end{aligned}\tag{25}$$

Constraint equation written on the rotating joint of the crank arm:

$$\begin{aligned}&x_{G_3}+x_{34}\cdot\alpha_{13}(\psi_3,\theta_3,\phi_3)+y_{34}\cdot\beta_{13}(\psi_3,\theta_3,\phi_3)+\\&+z_{34}\cdot\gamma_{13}(\psi_3,\theta_3)-(x_{G_4}+x_{43}\cdot\alpha_{14}(\psi_4,\theta_4,\phi_4)+\\&+y_{43}\cdot\beta_{14}(\psi_4,\theta_4,\phi_4)+z_{43}\cdot\gamma_{14}(\psi_4,\theta_4))=0\end{aligned}\tag{26}$$

$$\begin{aligned}&y_{G_3}+x_{34}\cdot\alpha_{23}(\psi_3,\theta_3,\phi_3)+y_{34}\cdot\beta_{23}(\psi_3,\theta_3,\phi_3)+\\&+z_{34}\cdot\gamma_{23}(\psi_3,\theta_3)-(y_{G_4}+x_{43}\cdot\alpha_{24}(\psi_4,\theta_4,\phi_4)+\\&(y_{G_4}+x_{43}\cdot\alpha_{24}(\psi_4,\theta_4,\phi_4)+\\&+y_{43}\cdot\beta_{24}(\psi_4,\theta_4,\phi_4)+z_{43}\cdot\gamma_{24}(\psi_4,\theta_4))=0\end{aligned}\tag{27}$$

$$\begin{aligned}&z_{G_3}+x_{34}\cdot\alpha_{33}(\theta_3,\phi_3)+y_{34}\cdot\beta_{33}(\theta_3,\phi_3)+\\&+z_{34}\cdot\gamma_{33}(\theta_3)-(z_{G_4}+x_{43}\cdot\alpha_{34}(\theta_4,\phi_4)+\\&+y_{43}\cdot\beta_{34}(\theta_4,\phi_4)+z_{43}\cdot\gamma_{34}(\theta_4))=0\end{aligned}\tag{28}$$

$$\begin{aligned}&(\alpha_{13}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{130}+\beta_{13}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{230}+\\&\gamma_{13}(\psi_3,\theta_3)\cdot\gamma_{330})-(\alpha_{14}(\psi_4,\theta_4,\phi_4)\cdot\gamma_{140}+\\&+\beta_{14}(\psi_4,\theta_4,\phi_4)\cdot\gamma_{240}+\gamma_{14}(\psi_4,\theta_4)\cdot\gamma_{340})=0\end{aligned}\tag{29}$$

$$\begin{aligned}&(\alpha_{23}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{130}+\beta_{23}(\psi_3,\theta_3,\phi_3)\cdot\gamma_{230}+\\&\gamma_{24}(\psi_3,\theta_3)\cdot\gamma_{330})-(\alpha_{24}(\psi_4,\theta_4,\phi_4)\cdot\gamma_{140}+\\&+\beta_{24}(\psi_4,\theta_4,\phi_4)\cdot\gamma_{240}+\gamma_{24}(\psi_4,\theta_4)\cdot\gamma_{340})=0\end{aligned}\tag{30}$$

Constraint equation written on the rotating joint of the piston:

$$\begin{aligned}&x_{G_4}+x_{41}\cdot\alpha_{14}(\psi_4,\theta_4,\phi_4)+y_{41}\cdot\beta_{14}(\psi_4,\theta_4,\phi_4)+\\&+z_{41}\cdot\gamma_{14}(\psi_4,\theta_4)+a\cdot b_{114}(\psi_4,\theta_4,\phi_4)-\\&-(x_{G_1}+x_{14}\cdot\alpha_{110}+y_{14}\cdot\alpha_{210}+z_{14}\cdot\alpha_{310}=0\end{aligned}\tag{31}$$

$$\begin{aligned}&y_{G_4}+x_{41}\cdot\alpha_{24}(\psi_4,\theta_4,\phi_4)+y_{41}\cdot\beta_{24}(\psi_4,\theta_4,\phi_4)+\\&+z_{41}\cdot\gamma_{24}(\psi_4,\theta_4)-\\&-(y_{G_1}+x_{14}\cdot\beta_{110}+y_{14}\cdot\beta_{210}+z_{14}\cdot\beta_{310})=0\end{aligned}\tag{32}$$

$$\begin{aligned}&z_{G_4}+x_{41}\cdot\alpha_{34}(\theta_4,\phi_4)+y_{41}\cdot\beta_{34}(\theta_4,\phi_4)+\\&+z_{41}\cdot\gamma_{34}(\theta_4)-\\&-(z_{G_1}+x_{14}\cdot\gamma_{110}+y_{14}\cdot\gamma_{210}+z_{14}\cdot\gamma_{310})=0\end{aligned}\tag{33}$$

$$\begin{aligned}&\alpha_{14}(\psi_4,\theta_4,\phi_4)\cdot\alpha_{140}+\beta_{14}(\psi_4,\theta_4,\phi_4)\cdot\alpha_{240}+\\&+\gamma_{14}(\psi_4,\theta_4)\cdot\alpha_{340}-b_{111}=0\end{aligned}\tag{34}$$

$$\begin{aligned}&\alpha_{24}(\psi_4,\theta_4,\phi_4)\cdot\alpha_{140}+\beta_{24}(\psi_4,\theta_4,\phi_4)\cdot\alpha_{240}+\\&+\gamma_{24}(\psi_4,\theta_4)\cdot\alpha_{340}-b_{211}=0\end{aligned}\tag{35}$$

3. Conclusions

The following can be determined most effectively with the method of constraint equations:

-the position of the centre of gravity for each component of the crank mechanism;

-the positions of inertial crankshafts.

The values were determined relative to a fixed system.

This advantageous method can be used to determine the position of the main axis of inertia of the components.

References

[1] Papp I.: *Mechanizmusok elmélete*. Scientia Kiadó, Kolozsvár, 2010. 226 oldal.

[2] Popa-Müller I., Papp I.: *Négyoldalú térmechanizmus kinematikai vizsgálata kényszeregyenletek segítségével*. XXII OGÉT Nemzetközi Gépész Találkozó, Nagyszeben, Románia, 2014, ISSN 2068-1267, 301–305.

[3] Popa-Müller I., Papp I.: *Ötoldalú térmechanizmus kinematikai vizsgálata*. Műszaki Tudományos Közlemények 6. (2017) 147–150. https://doi.org/10.33895/mtk-2017.06.16

[4] Popa-Müller I., Papp I.: *Ötoldalú térmechanizmus, sebesség meghatározás*. XXV. OGÉT Nemzetközi Gépész Találkozó, Kolozsvár, Románia, 2017, ISSN 2068-1267, 340–344.

[5] Popa-Müller I., Papp I.: *Ötoldalú térmechanizmus, gyorsulás meghatározás*. Műszaki Tudományos Közlemények, 9. (2018) 199–202. https://doi.org/10.33895/mtk-2018.09.45