

Multivariate Bernstein-Markov Type Inequalities in L^p Norm

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This survey paper is dedicated to summarizing the progress made in the last 25–30 years in the study of L^p Bernstein-Markov type inequalities for univariate and multivariate polynomials. It turned out that compared to the uniform norm this problem is considerably harder in the L^p norm. In addition, extending the study of Bernstein-Markov type inequalities to multivariate case invariably leads to more complicated deeper results which involve the geometry of the underlying domains. The significance of Bernstein-Markov type inequalities in the L^p norm is emphasized by the fact that they provide an effective tool in establishing Marcinkiewicz-Zygmund type discretization results in the integral norms.

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1. Introduction

This survey is intended to be a contribution to the fascinating story of **Bernstein-Markov type polynomial inequalities**. This story started about 170 years ago initiated by a question of Mendelev and eventually resulted in hundreds of papers with numerous important applications in Approximation Theory and Analysis in general.

The classical Bernstein and Markov inequalities for univariate algebraic polynomials p_n of degree $\leq n$ give the following sharp upper bounds for their derivatives in the uniform norm:

$$\|\sqrt{1-x^2}p'_n(x)\|_{C[-1,1]} \leq n \|p_n\|_{C[-1,1]} \quad (1)$$

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and

$$\|p'_n\|_{C[-1,1]} \leq n^2 \|p_n\|_{C[-1,1]}. \quad (2)$$

A remarkable feature of the Bernstein inequality (1) consists in the fact that it gives a much smaller order n estimate for the derivatives compared to the order n^2 bound (2). Of course one has to pay a price for this improvement by inserting the weight $\sqrt{1-x^2}$ into the derivative estimate which reduces the weighted measure of the derivatives at the endpoints of the interval. This effect of the boundary of the domain will systematically reappear in the discussion below.

The Bernstein inequality (1) is clearly equivalent after a standard substitution to its trigonometric version

$$\|t'_n\|_{C[-\pi,\pi]} \leq n \|t_n\|_{C[-\pi,\pi]} \quad (3)$$

with t_n being any trigonometric polynomial of degree n .

Arestov [1] gave an elegant extension of the Bernstein inequality (3) for L^p , $0 < p < \infty$,

$$\|t'_n\|_{L^p[-\pi,\pi]} \leq n \|t_n\|_{L^p[-\pi,\pi]}.$$

It is also well known (see for instance [27]) that inequalities (2) and (1) extend to the L^p norm on $[-1, 1]$ with some constants depending on p

$$\begin{aligned} \|\sqrt{1-x^2} p'_n(x)\|_{L^p[-1,1]} &\leq c_p n \|p_n\|_{L^p[-1,1]}, \\ \|p'_n\|_{L^p[-1,1]} &\leq c_p n^2 \|p_n\|_{L^p[-1,1]}. \end{aligned}$$

Above inequalities and their numerous generalizations play crucial role in various branches of analysis and approximation theory, it would be difficult to overstate their importance to the field. With respect to their widespread application we would like to point out just three major areas where they are widely used:

- A. Inverse Bernstein-Jackson type theorems on the rate of best approximation;
- B. Marcinkiewicz-Zygmund type results on discretization of norms;
- C. Study of the rate of Christoffel functions in the theory of orthogonal polynomials.

This survey paper is dedicated to summarizing the progress made in the last 25–30 years in the study of L^p Bernstein-Markov type inequalities for univariate and multivariate polynomials. It turned out that compared to the uniform norm this problem is considerably harder in the L^p norm. In addition, extending the study of Bernstein-Markov type inequalities to multivariate case invariably leads to more complicated deeper results which involve the geometry of the underlying domains. The significance of Bernstein-Markov type inequalities in the L^p norm is emphasized by the fact that they provide an effective

tool in establishing Marcinkiewicz-Zygmund type discretization results in the integral norms, see for instance [17].

This paper is organized as follows. Section 2 is devoted to a brief survey of the history of Bernstein-Markov type inequalities and most relevant univariate L^p Bernstein-Markov type results. Section 3 deals with the intriguing L^2 case in multivariate setting. As it turned out for the L^2 norm with certain weights exact solutions of the Bernstein-Markov type problems can be established. Finally, in Section 4 we discuss some recent multivariate L^p results which provide asymptotically exact solutions of the Bernstein-Markov problem.

2. Univariate Bernstein-Markov Type Inequalities in L^p Norm

It is considerably harder to establish exact explicit Bernstein-Markov type inequalities similar to (1) and (2) in case of the L^2 norm. In fact even in the univariate case this was accomplished only for certain weights w . Denote by P_n^d the set of real algebraic polynomials of degree at most n in d variables and let $Dp := |\partial p| = \left(\sum_{1 \leq j \leq d} \left(\frac{\partial p}{\partial x_j}\right)^2\right)^{1/2}$ be the l_2 norm of the gradient ∂p of $p \in P_n^d$. Then the solution to the L^p Markov problem with respect to the weight w is given by the quantity

$$M_n^d(w, p) := \sup_{p_n \in P_n^d} \frac{\|w Dp\|_{L^p(\mathbb{R}^d)}}{\|wp\|_{L^p(\mathbb{R}^d)}},$$

where either the weight w has compact support or it properly vanishes at ∞ so that norms $\|wp\|_{L^p(\mathbb{R}^d)}$ are finite for $p_n \in P_n^d$.

Even in the basic univariate case when $p = 2$ and $w = \chi_{[-1,1]}$ is the weight 1 supported on $[-1, 1]$ the quantity $M_n^1(\chi_{[-1,1]}, 2)$ is given only implicitly as the largest positive root of a certain (quite complicated) algebraic equation of order n , see [10, 13]. Schmidt [29] showed that $M_n^1(e^{-x^2}, 2) = \sqrt{2n}$ with Hermite polynomials being the corresponding extremal functions. For the Laguerre weight e^{-x} on $\mathbb{R}_+$ by an elegant result of Turan [31] we have $M_n^1(e^{-x}\chi_{[0,\infty)}, 2) = \frac{1}{2 \sin(\pi/(4n+2))}$. However for general Laguerre and Gegenbauer weights the exact L^2 constants in the Markov inequality are still undetermined explicitly, even though they can be given as solutions of certain polynomial equations or matrix eigenvalue problems, see e.g. [26].

On the other hand, it should be also mentioned that sharp asymptotic values of $M_n^1(e^{-|x|^\alpha}, p)$ are known to be of order $\sim n^{1-\frac{1}{\alpha}}$ when $1 < \alpha < \infty$ (Freud [8], Levin and Lubinsky [23]) and of order $\log n$ if $\alpha = 1$ (Nevai and Totik [24]).

3. Sharp L^2 Bernstein-Markov Type Inequalities for Multivariate Polynomials

This section is devoted solely to those multivariate L^2 Bernstein-Markov problems whose exact solution can be given explicitly.

First consider the case when w_G is the Gaussian weight

$$w_G(x) = e^{-|x|^2}, \quad |x|^2 = \sum_{1 \leq j \leq d} x_j^2, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

For this weight the weighted polynomials $e^{-|x|^2} p_n$, $p \in P_n^d$ vanish at ∞ , i.e., at the boundary of the domain which makes the question considerably easier to handle. It was shown in [22] that $M_n^d(w_G, 2) = \sqrt{2n}$ and the supremum is attained if and only if

$$p(x) = \sum_{k_1 + \dots + k_d = n} c_k \prod_{j=1}^d H_{k_j}(x_j), \quad c_k \in \mathbb{R}, \quad x = (x_1, \dots, x_d),$$

with H_{k_j} being the Hermite polynomial of degree k_j .

Turan's case of the Laguerre weight $w_T(x) = e^{-x_1 - \dots - x_d}$, $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$ turned out to be essentially more difficult. As it was proved in [22] $M_n^d(w_T, 2) = \frac{1}{2 \sin(\pi/(4n+2))}$. Moreover, the explicit form of extremal polynomials was also given therein.

Now let us turn our attention to the fascinating case of the unit ball $B^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$. In case of the uniform norm Baran [3] and Sarantopoulos [28] (Bernstein type estimate) and Kellogg [11] (Markov type estimate) generalized (1)–(2) for $p \in P_n^d$ as follows

$$\begin{aligned} \|\sqrt{1-|x|^2} Dp(x)\|_{C(B^d)} &\leq n \|p\|_{C(B^d)}, \\ \|Dp\|_{C(B^d)} &\leq n^2 \|p\|_{C(B^d)}. \end{aligned} \tag{4}$$

As it was already mentioned above it is considerably harder to verify exact Bernstein-Markov type inequalities in case of the L^2 norm. One remarkable exception is the sharp Bernstein type inequality for the weight $w^* := (1-x^2)^{-1/4}$

$$\|\sqrt{1-x^2} w^* p'\|_{L^2[-1,1]} \leq n \|w^* p\|_{L^2[-1,1]}, \quad p \in P_n^1 \tag{5}$$

which follows by substitution $x = \cos t$ from the Arestov inequality (3).

Recently above inequality was extended for multivariate polynomials on the d dimensional ball B^d . It turned out that for $L^2(B^d)$ the sharp analogue of (5) holds with constants $\sqrt{n(n+d-1)}$ and $\sqrt{n(n+d-1)} - d + 1$ when n is even or odd, respectively. It comes as a surprise that the exact Bernstein-Markov factor for $L^2(B^d)$ depends on the dimension d of the space! Note that as seen from (4) in $C(B^d)$ the upper bound is dimension independent.

Given any $\mu > -1$ let us consider the weight $w_\mu(x) := (1 - |x|^2)^{\frac{\mu}{2}}$. Then the solution of the L^2 Bernstein problem on B^d is given by

$$B_n^d(w_\mu, 2) := \sup_{p \in P_n^d \setminus \{0\}} \frac{\|(1 - |x|^2)^{\frac{\mu+1}{2}} Dp\|_{L^2(B^d)}}{\|(1 - |x|^2)^{\frac{\mu}{2}} p\|_{L^2(B^d)}}. \quad (6)$$

It turns out that the exact values of $B_n^d(w_\mu, 2)$ can be determined using the classical Jacobi polynomials $J_m^{\alpha, \beta}(t)$, see [30].

Theorem 1 ([18, 20]). *Let $\mu > -1, d \geq 1, n \in \mathbb{N}$. Then*

$$B_n^d(w_\mu, 2) = \begin{cases} \sqrt{n(n+d+2\mu)}, & \text{if } n \text{ is even,} \\ \sqrt{n(n+d+2\mu)-d+1}, & \text{if } n \text{ is odd.} \end{cases} \quad (7)$$

Moreover, the upper bound (7) is attained if and only if

$$p(x) = \begin{cases} c J_{\frac{n}{2}}^{(\mu, \frac{d}{2}-1)}(2|x|^2 - 1), & \text{even } n, \\ J_{\frac{n-1}{2}}^{(\mu, \frac{d}{2})}(2|x|^2 - 1)(a_1 x_1 + \dots + a_d x_d), & \forall a_j \in \mathbb{R}, \text{ odd } n. \end{cases}$$

Theorem 1 presents a rare occasion when the exact solution of the multivariate L^2 Bernstein-Markov type problem can be found explicitly. Note that contrary to the uniform norm whenever $d > 1$ the values of $B_n^d(w_\mu, 2)$ are distinct for even and odd n and in addition they depend on d , too. The univariate case (5) corresponds to (7) with $d = 1$ and $\mu = -\frac{1}{2}$. For $\mu = 0$ Theorem 1 provides the L^2 analogue of the Baran-Sarantopoulos result (first inequality in (4)).

A key ingredient of the proof is the duality which connects the solution of the extremal problem (6) to the partial differential equation

$$(1 - |x|^2)\Delta p - 2(\mu + 1)\langle x, \partial p \rangle + M^2 p = 0, \quad M > 0, \quad (8)$$

where Δ stands for the Laplace operator.

Let $M_n^*(d, \mu)$ be the sup of all those $M > 0$ for which (8) has a nontrivial solution $p \in P_n^d$. It turns out that

$$B_n^d(w_\mu, 2) = M_n^*(d, \mu).$$

Moreover, it can be shown that for $M := M_n^*(d, \mu)$ equation (8) has the **unique solution** specified in the above theorem.

As mentioned before the exact explicit solution to the L^2 Markov problem on the ball for polynomials of total degree $\leq n$ has not been found so far even for $d = 1$. When $d > 1$ we will present now the solution of the L^2 Markov problem for the **homogeneous polynomials** H_n^d on the unit sphere S^{d-1} . As usual, H_n^d stands for the space of *homogeneous polynomials* of degree n in d variables.

Theorem 2 ([20]). *Let $d \geq 2$, $n \in \mathbb{N}$. Then for any $h_n \in H_n^d$ we have the sharp estimates*

$$\xi_n \|h_n\|_{L^2(S^{d-1})} \leq \|Dh_n\|_{L^2(S^{d-1})} \leq \sqrt{n(2n+d-2)} \|h_n\|_{L^2(S^{d-1})}$$

with $\xi_n = n$ if n is even and $\xi_n = \sqrt{n^2 + d - 1}$ if n is odd.

Moreover, the upper bound is attained if and only if h_n is a spherical harmonic polynomial of order n , while the lower bound is attained if and only if $h_n(x) = c|x|^n$, $\forall c \in \mathbb{R}$ or $h_n(x) = |x|^{n-1}q(x)$, $\forall q \in H_1^d$ when n is even or odd, respectively.

4. L^p Bernstein-Markov Type Inequalities on Graph Domains

In this section we will review some recent results on L^p Bernstein-Markov type inequalities on convex and star like domains. Of course, in this generality there is little hope for establishing exact up to the constant estimates. So we aim, instead for providing asymptotically sharp Bernstein-Markov type upper bounds.

It is well known that for Lip 1 and thus, in particular convex domains the upper bound of order n^2 for L^p , $1 \leq p \leq \infty$ Markov inequality extends for multivariate polynomials, as well, see e.g., [25, 14]. However, when the underlying domain is cuspidal the order of magnitude can change drastically. In particular, for Lip γ , $0 < \gamma < 1$ cuspidal domains in $\mathbb{R}^d$ the sharp order in L^∞ Markov inequality is known to be $n^{\frac{2}{\gamma}}$ (see for instance, [7, 2, 21]). Typically multivariate Markov-type inequalities in L^∞ norm are proved by inscribing suitable polynomial curves into the domain and reducing the problem to the univariate setting on these curves. In case of L^p , $1 \leq p < \infty$ norm this reduction of dimension technique does not work and more delicate considerations are needed.

In a recent paper [4] an example of 2-dimensional Lip $\frac{1}{2}$ domain $\{(x, y) \in \mathbb{R}^2 : |x| < y + 1, x^2 > 4y\}$ was considered and the asymptotically sharp order n^4 of L^p Markov inequality was verified.

Now we will present a complete solution to the above problem by providing the asymptotically sharp Markov type inequality in L^p , $1 \leq p < \infty$ norm for general Lip γ , $0 < \gamma < 1$ cuspidal domains in $\mathbb{R}^d$. It is remarkable that this order turned out to be of the same magnitude $\sim n^{\frac{2}{\gamma}}$ as in L^∞ norm.

Let $K \subset \mathbb{R}^d$ be a compact set with nonempty interior. We denote by

$$B(a, r) := \{x \in \mathbb{R}^d : |x - a| \leq r\}$$

the closed ball of radius r and center a , and $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere. For any $r > 0$, $a \in \mathbb{R}^d$ and $u \in S^{d-1}$ the cylinder $L_a(r, u)$ of radius

$r > 0$, center a and axis u is given by

$$L_a(r, u) := \{x \in \mathbb{R}^d : |x - a|^2 \leq r^2 + \langle x - a, u \rangle^2\}.$$

Furthermore, $\ell_x(u)$ will denote the line in $\mathbb{R}^d$ in direction $u \in S^{d-1}$ through point $x \in \mathbb{R}^d$.

Definition 1. K is called a **graph domain** with respect to the cylinder $L_a(r, u)$ if for every $x \in B(a, r)$ we have that $\ell_x(u) \cap K = [A_1(x), A_2(x)]$ with $A_i(x)$, $i = 1, 2$ being continuous for $x \in B(a, r)$ and $|A_1(x) - A_2(x)| > 0$, $x \in B(a, r)$.

Moreover, $K \subset \mathbb{R}^d$ is a **piecewise graph domain** if it can be covered by finite number of cylinders so that K is a graph domain with respect to each of them.

If all functions $A_i(x)$ specified above are in $\text{Lip } \gamma$, $0 < \gamma \leq 1$ then the piecewise graph domain K is called $\text{Lip } \gamma$. A model example of $\text{Lip } \gamma$ piecewise graph domain is provided by the ℓ_γ ball in $\mathbb{R}^d$ defined by $B_\gamma^d := \{x \in \mathbb{R}^d : |x_1|^\gamma + \dots + |x_d|^\gamma \leq 1\}$. Our next results gives the asymptotically sharp Markov type inequality in L^p , $1 \leq p < \infty$ for graph domains.

Theorem 3 ([16]). *Let $K \subset \mathbb{R}^d$ be a $\text{Lip } \gamma$, $0 < \gamma \leq 1$ piecewise graph domain. Then for any $q_n \in P_n^d$, $1 \leq p < \infty$ and $n \in \mathbb{N}$*

$$\|Dq_n\|_{L^p(K)} \leq c(K, p) n^{\frac{2}{\gamma}} \|q_n\|_{L^p(K)}.$$

Moreover, if K is imbedded in an affine image of the ℓ_γ ball having one of its vertices on ∂K then there exist $g_n \in P_n^d$, $n \in \mathbb{N}$ such that

$$\|Dg_n\|_{L^p(K)} > c_1(K, p) n^{\frac{2}{\gamma}} \|g_n\|_{L^p(K)}, \quad n \in \mathbb{N}.$$

By inserting a proper weight function vanishing at the boundary of the domain one can ensure a smaller Bernstein type upper bound of order n for derivatives of polynomials of degree n . Given any $x \in K$ we denote by

$$\tau_K(x) := \inf_{y \in \partial K} |x - y|$$

the Euclidean distance from x to the boundary ∂K of the domain. As shown by the next theorem the quantity $\tau_K(x)^{\frac{1}{\gamma} - \frac{1}{2}}$ provides the proper weight function needed in order to drive a Bernstein type upper bound for $\text{Lip } \gamma$, $0 < \gamma \leq 1$ piecewise graph domains.

Theorem 4 ([19]). *Let $1 \leq p < \infty$. Then given any $\text{Lip } \gamma$, $0 < \gamma \leq 1$ piecewise graph domain $K \subset \mathbb{R}^d$ we have*

$$\|\tau_K(x)^{\frac{1}{\gamma} - \frac{1}{2}} Dq_n\|_{L^p(K)} \leq c n \|q_n\|_{L^p(K)}, \quad q_n \in P_n^d.$$

Note that in the recent papers [32] and [33] the problem of finding the proper measure of the distance to the boundary of the domain was considered in case of the **simplex** in $\mathbb{R}^d$. As shown in these papers in case of estimating directional derivatives on the simplex a more delicate measure of the distance to the boundary can be used which in certain cases even yields sharp up to the constants L^2 Bernstein type upper bounds. Let us present now a typical result of this nature. Let

$$\Delta^d := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_j \geq 0, |x|_1 = x_1 + \dots + x_d \leq 1\}$$

be the d dimensional simplex and set

$$\phi_j(x) := \frac{x_j(1 - |x|_1)}{1 + x_j - |x|_1}.$$

Theorem 5 ([33]). *For any $q_n \in P_n^d$ and $1 \leq j \leq d$ we have for the simplex Δ^d*

$$\left\| \sqrt{\phi_j} \frac{\partial q_n}{\partial x_j} \right\|_{L^p(\Delta^d)} \leq c n \|q_n\|_{L^p(\Delta^d)}.$$

Note that for Δ^d Theorem 4 used with $\gamma = 1$ gives a Bernstein type estimate for the gradients of polynomials with the distance function $\sqrt{\tau_{\Delta^d}(x)}$ where

$$\tau_{\Delta^d}(x) = \min\{x_j, 1 \leq j \leq d, 1 - |x|_1\}, \quad x \in \Delta^d.$$

The distance function $\sqrt{\phi_j(x)}$ of Theorem 5 which takes into account the specific nature of the boundary of the simplex leads to sharper estimates for individual *directional derivatives* on the simplex. However, if one aims to estimate the *gradients* of polynomials Theorem 5 essentially gives an upper bound similar to Theorem 4.

Now we turn our attention to some recent results on L^p Markov type inequalities for homogeneous polynomials on graph domains. Theorem 2 gives a sharp upper bound in the Markov inequality for homogeneous polynomials H_n^d on $L^2(S^{d-1})$. It is remarkable that this bound is of order $\sim n$ even though for polynomials of total degree n the corresponding order is known to be of magnitude $\sim n^2$. This raises the natural question if this improvement in the rate of Markov factor for homogeneous polynomials holds on other domains, as well? The study of Markov type problem for homogeneous polynomials was initiated by Harris [9] who verified that Markov factor for homogeneous polynomials of degree n for uniform norm on convex bodies is of order $n \log n$. Subsequently, it was shown in [12] that under an additional mild smoothness condition on the convex domain this order $n \log n$ can be replaced by $O(n)$. Our goal now is to present proper extension of these results for L^p norm. In particular it will be shown below that the upper bound in the L^p Markov inequality for homogeneous polynomials on general C^2 star like domains is indeed of order n .

Let $K \in \mathbb{R}^d$ be a star like domain given by $K := \{x = \text{tur}(u) : u \in S^{d-1}, 0 \leq t \leq 1\}$ where $r \in C(S^{d-1})$ is a positive continuous function on

the unit sphere. We shall say that K is C^α , $1 < \alpha \leq 2$ if r is continuously differentiable in an open neighborhood of K and $\partial r \in \text{Lip}(\alpha - 1)$. Then K possesses a unique outer normal $\eta_K(x) \in S^{d-1}$ at every point $x \in \partial K$ on its boundary. We will assume below that the outer normal of the C^α , $1 < \alpha \leq 2$ set K is *non degenerate* in the sense that

$$\langle x, \eta_K(x) \rangle > 0, \quad \forall x \in \partial K.$$

Moreover $K \in \mathbb{R}^d$ is called a Lip 1 star like domain if $r \in \text{Lip} 1$.

Theorem 6 ([20]). *Assume that K is C^α , $1 < \alpha \leq 2$ star like domain in $\mathbb{R}^d$ with non degenerate outer normal. Then for any $h \in H_n^d$ and every $1 \leq p < \infty$ we have*

$$\|Dh\|_{L^p(K)} \leq c n^{\frac{1}{\alpha} + \frac{1}{2}} \|h\|_{L^p(K)}. \quad (9)$$

Moreover, if K is Lip 1 then for every $h \in H_n^d$ and any $k \in \mathbb{N}$

$$\|Dh\|_{L^p(K)} \leq c n^{\frac{3}{2}} \|h\|_{L^p(K)}.$$

In case when $\alpha = 2$ that is K is a C^2 star like domain with non degenerate outer normal Theorem 6 evidently yields an optimal order n Markov type estimate.

Corollary 1. *For any C^2 star like domain $K \in \mathbb{R}^d$ with non degenerate outer normal and every $h \in H_n^d$,*

$$\|Dh\|_{L^p(K)} \leq c n \|h\|_{L^p(K)}, \quad 1 \leq p < \infty.$$

The above corollary establishes an optimal order n L^p Markov type estimate for homogeneous polynomials on C^2 star like domains. Similarly, in case when K is Lip α with $1 \leq \alpha < 2$ it is plausible that the upper bound $n^{\frac{1}{\alpha} + \frac{1}{2}}$ in (9) can be improved in order to obtain estimate analogous to the L^∞ bound of Harris mentioned above. Thus we would like to propose the next

Conjecture 1. *For any Lip 1 star like domain $K \subset \mathbb{R}^d$ and every $h \in H_n^d$,*

$$\|Dh\|_{L^p(K)} \leq c n \log n \|h\|_{L^p(K)}, \quad 1 \leq p < \infty.$$

The main tool to verify Theorem 6 is the following recent result of Feng Dai, Kroó and Prymak [6] establishing a *tangential* L^p Bernstein type inequality for C^α , $1 < \alpha \leq 2$ star like domains, (see also [5] for the case $\alpha = 2$). Given a C^α , $1 < \alpha \leq 2$ star like domain $K \subset \mathbb{R}^d$ let D_T be the maximal derivative in tangential directions. Then $\forall g \in P_n^d$

$$\|\text{dist}(x, \partial K)^{\frac{1}{\alpha} - \frac{1}{2}} D_T g(x)\|_{L^p(K)} \leq c n \|g\|_{L^p(K)}.$$

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