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Christoffel functions, L^p Markov inequalities and Marcinkiewicz-Zygmund type discretization for homogeneous polynomials on convex bodies

András Kroó[1]

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Abstract

In this paper we will study various extremal problems related to multivariate homogeneous polynomials on convex bodies. We will construct certain homogeneous needle polynomials on convex bodies which will play a central role in the study of the asymptotics of Christoffel functions and L^p Markov type estimates for homogeneous polynomials. These new results will be applied in order to derive asymptotically optimal Marcinkiewicz-Zygmund type meshes and cubature formulas for homogeneous polynomials on convex bodies

Keywords Multivariate homogeneous polynomials · Bernstein-Markov type inequalities · Convex bodies · Christoffel functions · Marcinkiewicz-Zygmund type discretization

Mathmatics subject classification 41A17

Introduction

In this paper we shall solve several extremal problems related to multivariate homogeneous polynomials. The space of real homogeneous polynomials of $d \geq 2$ variables and degree n denoted by H_n^d is spanned by the monomials $x_1^{k_1} \cdot ... \cdot x_d^{k_d}, \ k_1 + ... + k_d = n, \ x = (x_1, ..., x_d) \in \mathbb{R}^d$. Homogeneous polynomials arise naturally as a fundamental tool in problems related to neural networks and approximation by ridge functions. The question of density of homogeneous polynomials in the space of continuous functions received a considerable attention in the past two decades, see the survey [11]. Our main goal in the present paper is to study the asymptotic behaviour of Christoffel

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András Kroó
kroo@renyi.hu

1 Alfréd Rényi Institute of Mathematics, Budapest, Hungary

functions and L^p Markov type estimates for homogeneous polynomials. In order to tackle these questions we will introduce below homogeneous "needle" polynomials which attain value 1 at a given point and rapidly decrease as we move away from this point (Theorem 1). These homogeneous needle polynomials will play a crucial role in the study of the asymptotics of the Christoffel functions and L^p Markov type estimates for homogeneous polynomials.

Denote by $P_n^d := \sum_{j=0}^n H_j^d$ the set of real algebraic polynomials of d variables and total degree $\leq n$. Let μ be a positive measure with compact support $K \subset \mathbb{R}^d$, $\text{Int}K \neq \emptyset$ and density $\mu' := w \geq 0$. Then the reproducing kernel (Christoffel–Darboux kernel) $K_n(x, y)$ of measure μ for P_n^d satisfies the property

$$q(x) = \int_K K_n(x, y)q(y)w(y)dy, \quad \forall q \in P_n^d.$$

The n-th Christoffel function of measure $\mu = wdy$ on domain K is defined as

$$\lambda_n(x, K) := \frac{1}{K_n(x, x)}, \quad x \in \mathbb{R}^d.$$

It is well known that Christoffel functions admit the following extremal property

$$\lambda_n(x, K) = \inf \left\{ \int_K q^2(y)w(y)dy : \; q \in P_n^d, \; q(x) = 1 \right\}, \quad x \in \mathbb{R}^d.$$

The Christoffel functions play a fundamental role in the study of orthogonal polynomials, see for instance Stahl-Totik [16]. In addition, as exhibited in the recent monograph by Lasserre, Pauwels and Putinar [15] they turned out to also provide an efficient and easy-to-use tool to help solve some problems in data analysis, e.g. outlier detection, support inference, and density estimation.

It is known that on some model domains like simplices, cubes for certain standard (Jacobi type) weights we have $K_n(x, x) = \frac{1}{\lambda_n(x,K)} = O(n^{2d}), \quad x \in K$. On smooth domains (for instance balls), a smaller order of magnitude $K_n(x, x) = O(n^{d+1})$ is known to hold. In general, the polynomial behaviour may differ throughout the domain with the fastest growth typically at the most singular points, see for instance [1, 2, 6, 14, 19, 20]. However outside the domain the same quantity in general increases exponentially (see [15]. p. 50).

Let us introduce now the generalized L^p, $p > 0$ Christoffel function for the space of *homogeneous polynomials* H_n^d on the set $K \subset \mathbb{R}^d$ with respect to d-dimensional Lebesgue measure

$$\lambda_n^*(x, K) = \inf \left\{ \int_K |h(y)|^p dy : \; h \in H_n^d, \; h(x) = 1 \right\}, \quad x \in \mathbb{R}^d.$$

Note that in general the quantity $\lambda_n^*(x, K)$ depends on L^p, $p > 0$ as well, but we prefer not to reflect this in order to keep the notation simpler.

It will be shown below that for any convex body $K \subset \mathbb{R}^d$, $d \geq 2$ central symmetric about the origin and every $x \in K$

$$\frac{c_2}{n^d \log^d n} \leq \lambda_n^*(x, K) \leq \frac{c_1}{n^d}.$$

Moreover, under the C^{1+} regularity condition (which in particular, holds for any C^α, $1 < \alpha \leq 2$ convex domain) we have that $\lambda_n^*(x, K) \sim n^{-d}$ for any $x \in K$, see Theorem 2.

The classical Markov problem consists in estimating the size of the derivatives of polynomials. The univariate L^p, $1 \leq p \leq \infty$ Markov type inequality for algebraic polynomials q of degree at most n states that

$$\|q'\|_{L^p[a,b]} \leq \frac{cn^2}{b-a}\|q\|_{L^p[a,b]}.$$

This inequality and its extensions to the multivariate case play a central role in various approximation problems. It is well known that for Lip1 and thus, in particular convex domains the upper bound of order n^2 for L^p, $1 \leq p \leq \infty$ Markov inequality extends for multivariate polynomials, as well (see e.g. [10]). Typically multivariate Markov-type inequalities in L^∞ norm are proved by inscribing suitable polynomial curves into the domain and reducing the problem to the univariate setting on these curves. In case of L^p, $1 \leq p < \infty$ norm this reduction of dimension technique does not work and more delicate considerations are needed.

The study of Markov type problem for multivariate homogeneous polynomials in case of uniform norm was initiated by Harris [8] who verified that for any 0-symmetric convex body in $\mathbb{R}^d$

$$\|Dh\|_{L^\infty(K)} \leq cn \log n \|h\|_{L^\infty(K)},$$

where $Dh := |\nabla h|$ stands for the Euclidian norm of the gradient of h. Subsequently, it was shown in [9] that in general the $\log n$ term in the above estimate can not be omitted in case of *non regular* convex bodies, but under an additional mild C^{1+} smoothness condition on the convex domain $n \log n$ can be replaced by $O(n)$.

In a recent paper [13] extension of these results to the L^p case was studied on *star like domains*. Let $K \subset \mathbb{R}^d$ be a star like domain given by $K := \{x = tur(u), u \in S^{d-1}, 0 \leq t \leq 1\}$ where $r \in C(S^{d-1})$ is a positive continuous function on the unit sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ with $|x|$ being the Euclidian norm. K is said to be C^α, $1 < \alpha \leq 2$ if r is continuously differentiable and its gradient ∇r is Lip$(\alpha - 1)$ on S^{d-1}. Recall that function g is Lipβ on K with some $0 < \beta \leq 1$ if its modulus of continuity satisfies the property

$$w(g, t) := \sup\{|g(x) - g(y)| : x, y \in K, |x - y| \leq t\} \leq Mt^\beta$$

with some $M > 0$.

As shown in [13] whenever K is a C^α, $1 \le \alpha \le 2$ star like domain in $\mathbb{R}^d$ with non degenerate outer normal then for any $h \in H_n^d$ and every $0 < p < \infty$ we have

$$\|Dh\|_{L^p(K)} \le cn^{\frac{1}{\alpha}+\frac{1}{2}}\|h\|_{L^p(K)}. \tag{1}$$

In case when $\alpha = 2$ upper bound (1) leads to an asymptotically optimal order n L^p Markov type estimate for homogeneous polynomials on C^2 star like domains. However, when K is C^α with $1 \le \alpha < 2$ estimate (1) did not appear to be sharp. One of the main new results of this paper (see Theorem 3 below) will establish an order n L^p Markov type estimate for homogeneous polynomials on any C^α, $1 < \alpha < 2$ *convex* domain.

Given a subspace $U \subset L^p(K)$ the Marcinkiewicz-Zygmund type problem for $1 \le p < \infty$ consists in finding discrete point sets $Y_N = \{x_1, ..., x_N\} \subset K$ and corresponding positive weights $w_j > 0$, $1 \le j \le N$ such that for any $g \in U$ we have

$$c_1 \sum_{1\le j\le N} w_j|g(x_j)|^p \le \|g\|^p_{L^p(K)} \le c_2 \sum_{1\le j\le N} w_j|g(x_j)|^p \tag{2}$$

with some constants $c_1, c_2 > 0$ depending only on p, d and K. In case when $p = \infty$ the above relation is replaced by

$$\|g\|_{C(K)} \le c \max_{1\le j\le N} |g(x_j)|, \quad g \in U.$$

The discrete nodes satisfying above property are usually called norming sets.

Evidently we must have $N \ge \dim U$ in order for these estimates to be possible for each $g \in U$. These equivalence relations turned out to be an effective tool used in numerous applications. Naturally, the main goal here is to find discrete point sets $Y_N = \{x_1, ..., x_N\} \subset K$ of possibly *minimal cardinality* N. When $U = \Pi_n^d$, and the domain $K \subset \mathbb{R}^d$ has nonempty interior, we have $\dim \Pi_n^d \sim n^d$ and therefore asymptotically optimal discrete meshes for Π_n^d must be of order n^d. For $p = \infty$ such discrete points sets are called optimal meshes. In [11] existence of L^∞ optimal meshes was verified for any C^α, $2-\frac{2}{d} < \alpha \le 2$ star like domain. Subsequently new tangential Bernstein type inequalities were applied in [12] and [3] in order to verify the *existence of optimal meshes in any convex body in* $\mathbb{R}^d$ in case when $d = 2$ and any $d > 2$, respectively. Recently, in [5] the existence of optimal meshes on C^α, $2-\frac{2}{d} < \alpha \le 2$ star like domains was extended for L^p, $d-1 < p < \infty$. Applying the L^p Markov type estimates of optimal order n verified in this paper for homogeneous polynomials we will construct explicit discrete point sets of optimal cardinality $cn^{d-1} \sim \dim H_n^d$ on the boundary of any C^α, $1 < \alpha < 2$ *convex* domain which provide Marcinkiewicz-Zygmund type meshes for homogeneous polynomials for every $p > 0$. (Theorem 4). In addition, cubature formulas of asymptotically optimal cardinality will be also given in Theorem 5.

New results

Homogeneous polynomials $h \in H_n^d$ have the symmetry property of being even or odd depending on n. Therefore throughout this paper it is natural to consider convex bodies $K \subset \mathbb{R}^d$ which are **central symmetric about the origin**, i.e., $-x \in K, \forall x \in K$. As usual the convex body is called **regular** if it has a unique supporting plane at every point of its boundary ∂K. In case when K is regular its outer normal $\eta(z), z \in \partial K$ is continuously differentiable on ∂K and thus the modulus of continuity of the gradient of $\eta(z)$ denoted by

$$w_K(t) := \sup\{|\nabla\eta(z_1) - \nabla\eta(z_2)| : z_1, z_2 \in \partial K, |z_1 - z_2| \leq t\}, \quad t > 0$$

satisfies $w_K(t) \to 0, t \to +0$. We will say that K is a C^{1+} convex body if it is regular and w_K satisfies a Dini-Lipschitz type condition

$$\int_0^1 \frac{w_K(t)}{t} dt < \infty.$$

Evidently, this condition holds in particular when $\nabla\eta(z)$ is Lipα, $0 < \alpha \leq 1$, i.e., K is a $C^{1+\alpha}$. A model case of a $C^{1+\alpha}$, $0 < \alpha \leq 1$ convex body is the usual $l^{1+\alpha}$ ball in $\mathbb{R}^d$ given by

$$\{x = (x_1, ..., x_d) \in \mathbb{R}^d : |x_1|^{1+\alpha} + ... + |x_d|^{1+\alpha} \leq 1\}.$$

Let us denote by $B^d(0, a) \subset \mathbb{R}^d$ the closed ball of radius $a > 0$ centered at the origin. In what follows we will always assume that convex body K is properly dilated so that

$$B^d(0, r) \subset K \subset B^d(0, 1),$$

where $r > 0$ is the radius of the largest ball inscribed into K. Note that John's theorem [7] on characterization of largest inscribed ellipsoids for convex bodies allows to assume $r \geq d$ for an appropriate linear transform of K.

Some of the results and estimates given in this paper are domain independent with various corresponding constants depending only on d, p and r. In order to simplify the presentation these possibly distinct positive constants will be denoted by c. In the statements involving C^{1+} property of the convex body K the possibly distinct positive constants depending only on the domain K and d, p, r will be denoted by c_K. As usual, we will write $A \sim B$ if $c_1 A \leq B \leq c_2 A$ with c_1, c_2 depending only on K, d, p and r.

1. Homogeneous needle polynomials on convex bodies.

We start by constructing certain homogeneous needle polynomials on convex bodies. These polynomials which are of independent interest will be needed in the study of the asymptotics of Christoffel functions and for verifying L^p Markov type estimates for homogeneous polynomials. Needle polynomials (sometimes also called peaking polynomials) attain value 1 at a given point $z \in \partial K$ of the boundary while their integrals over underlying domain is small. The size of the integrals of these needle polynomials can be considered on all of K or on its boundary ∂K endowed with the usual Lebesgue surface measure in $\mathbb{R}^d$, $d \geq 2$ denoted by $d\nu(z)$.

Theorem 1 *Let $K \subset \mathbb{R}^d$, $B^d(0,r) \subset K \subset B^d(0,1)$, $d \geq 2$ be a convex body centrally symmetric about the origin. Then for any $p > 0$ and any $z \in \partial K$ there exist polynomials $Q_z(x) \in H_n^d$, $n \in \mathbb{N}$ such that $Q_z(z) = 1$ and*

$$\int_K |Q_z(x)|^p dx \leq \frac{c}{n^d}, \quad n \in \mathbb{N}. \tag{3}$$

In addition, for every $x \in \partial K$

$$\int_{\partial K} |Q_z(x)|^p d\nu(z) \leq \frac{c}{n^{d-1}}, \quad n \in \mathbb{N}. \tag{4}$$

It should be noted that for all homogeneous polynomials $h \in H_n^d$ of degree $n \in \mathbb{N}$ we have

$$\int_K |h_n(x)|^p dx \sim \frac{1}{n} \int_{\partial K} |h_n(x)|^p d\nu(x), \tag{5}$$

see Lemma 8 below. Therefore for each $z \in \partial K$ relation (3) can be equivalently written as

$$\int_{\partial K} |Q_z(x)|^p d\nu(x) \leq \frac{c}{n^{d-1}}, \quad n \in \mathbb{N}.$$

This last upper bound provides an estimate similar to (4). However, it should be noted the proof of (4) involving integration over variable z instead of x is much more delicate.

2. Christoffel functions for homogeneous polynomials on convex bodies.

Next we give upper and lower bounds for L^p Christoffel functions for homogeneous polynomials on convex bodies. The distinctive feature of these bounds is the fact that up to a log factor we get the same asymptotics of Christoffel functions for every point of the domain. This is quite different from the case of polynomials of total degree for which the size of Christoffel functions differs throughout the domain with the fastest growth at the most singular points.

Theorem 2 *Let $K \subset \mathbb{R}^d$, $d \geq 2$ be a central symmetric convex body, $B^d(0,r) \subset K \subset B^d(0,1)$. Then for any $x \in K$ and every $p > 0$*

$$\frac{c_2}{n^d \log^d n} \leq \lambda_n^*(x, K) \leq \frac{c_1}{n^d}, \quad n \in \mathbb{N}. \tag{6}$$

Moreover, if in addition K is a C^{1+} then

$$\frac{c_K}{n^d} \leq \lambda_n^*(x, K) \leq \frac{c_1}{n^d}, \quad \forall x \in K, \quad n \in \mathbb{N}. \tag{7}$$

Taking into account relations (5) Theorem 2 can be reformulated for L^p Christoffel functions on ∂K endowed with the Lebesgue surface measure.

Corollary 1 *Under conditions of Theorem 2 for every $x \in \partial K$*

$$\frac{c_2}{n^{d-1}\log^d n} \le \lambda_n^*(x, \partial K) \le \frac{c_1}{n^{d-1}}, \quad n \in \mathbb{N}.$$

In addition, the log term in the above lower bound can be omitted if K is C^{1+}.

Thus for C^{1+} domains we have that

$$n\lambda_n^*(x, K) \sim \lambda_n^*(x, \partial K) \sim \frac{1}{n^{d-1}}, \quad \forall x \in \partial K.$$

It would be interesting to determine if the log term in the lower bound (6) is in general necessary in case of *non regular* convex bodies. It seems to be plausible that $\lambda_n^*(x, K) \le \frac{c}{n^d \log^d n}$ for any non regular point $x \in \partial K$. This remains an open problem.

3. Markov type estimates for homogeneous polynomials on convex bodies.

Now we will consider the L^p, $p > 0$ Markov type problem of estimating norms of derivatives of homogeneous polynomials on the boundaries of convex bodies. The appropriate L^p "norm" in this case is

$$\|h\|_{L^p(\partial K)} := \left(\int_{\partial K} |h|^p d\nu\right)^{\frac{1}{p}}, 0 < p < \infty$$

with ν being the Lebesgue surface measure on ∂K. The goal is to give L^p upper bounds for the gradients

$$Dh := \max_{u \in S^{d-1}} |D_u h| = |\nabla h|, \quad h \in H_n^d,$$

where as usual D_u stands for the derivative in direction u. Since $D_x h(x) = nh(x), \forall x \in \mathbb{R}^d$ for every $h \in H_n^d$ it immediately follows by $K \subset B^d(0, 1)$ that

$$Dh(x) \ge \frac{1}{|x|}|D_x h(x)| \ge |D_x h(x)| = n|h(x)|.$$

In particular, this easily yields the lower bound for the norms of derivatives

$$\|Dh\|_{L^p(\partial K)} \ge n\|h\|_{L^p(\partial K)}, \quad \forall h \in H_n^d.$$

Our next theorem provides L^p, $p > 0$ Markov type upper bounds for the derivatives of homogeneous polynomials on the boundaries of convex bodies.

Theorem 3 *Let $K \subset \mathbb{R}^d, d \ge 2$ be a central symmetric convex body satisfying $B^d(0, r) \subset K \subset B^d(0, 1)$. Then for any $h \in H_n^d, n \in \mathbb{N}$ and any $0 < p < \infty$ we have with $\gamma := 1 + \frac{d-1}{p}$*

$$\|Dh\|_{L^p(\partial K)} \le cn \ln^\gamma(n + 1)\|h\|_{L^p(\partial K)}.$$

Moreover, if in addition K satisfies the C^{1+} property then the optimal upper bound

$$\|Dh\|_{L^p(\partial K)} \leq c_K n \|h\|_{L^p(\partial K)}$$

holds for every $h \in H_n^d$, $n \in \mathbb{N}$.

Let us note that a standard iteration yields that Theorem 3 admits an appropriate extension for higher order gradients

$$D^k f(x) := \max_{u_j \in S^{d-1}, 1 \leq j \leq k} \left| D_{u_1} ... D_{u_k} f(x) \right|, \quad k \in \mathbb{N}.$$

In particular, for C^{1+} convex bodies we have for each $h \in H_n^d$ and any $n, k \in \mathbb{N}$

$$\|D^k h\|_{L^p(\partial K)} \leq (c_K n)^k \|h\|_{L^p(\partial K)}.$$

Evidently, above upper bound of order n given for C^{1+} convex bodies is the best possible. An interesting open problem related to Theorem 3 consists in verifying an L^p lower bound of order $n \ln^\gamma(n+1)$ in case of a non regular convex body.

***4. Optimal Marcinkiewicz-Zygmund type meshes and cubature formulas for homogeneous polynomials on convex bodies*.**

It turns out that the optimal order Markov type estimate provided by Theorem 3 for homogeneous polynomials on C^{1+} convex domains can be applied in order to construct explicit Marcinkiewicz-Zygmund type meshes of asymptotically optimal cardinality $\sim n^{d-1} \sim \dim H_n^d$ for every $p \geq 1$. As usual, we will say that pairwise disjoint sets G_j, $1 \leq j \leq N$, form a partition of ∂K if $\partial K = \cup_{1 \leq j \leq N} G_j$. Let us also denote by $\text{diam}(G) := \sup_{x,y \in K} |x - y|$ the diameter of the set G.

Theorem 4 *Let $n \in \mathbb{N}$, $d \geq 2$, $p \geq 1$ and consider any C^{1+} central symmetric convex body $K \subset \mathbb{R}^d$ satisfying $B^d(0,r) \subset K \subset B^d(0,1)$, $r > 0$. Then for every $0 < \epsilon < 1$ there exists a partition $\partial K = \cup_{1 \leq j \leq N} G_j$ with $N \leq c_K \left(\frac{n}{\epsilon}\right)^{d-1}$, $\text{diam}(G_j) \leq \frac{c_K \epsilon}{n}$ such that for every $h_n \in H_n^d$ and any $x_j \in G_j$, $1 \leq j \leq N$ we have*

$$(1-\epsilon) \sum_{1 \leq j \leq N} \nu(G_j)|h_n(x_j)|^p \leq \int_{\partial K} |h_n(z)|^p d\nu(z) \leq (1+\epsilon) \sum_{1 \leq j \leq N} \nu(G_j)|h_n(x_j)|^p.$$

As mentioned in the introduction for polynomials in P_n^d of total degree $\leq n$ a similar statement was verified in [5] for any C^α, $2 - \frac{2}{d} < \alpha \leq 2$ convex domain and $d - 1 < p < \infty$. Thus assumptions of Theorem 4 are less restrictive both in terms of required C^{1+} smoothness and condition $p \geq 1$.

It should be also noted that since the partition in Theorem 4 satisfies $\text{diam}(G_j) \leq \frac{c_K \epsilon}{n}$, $1 \leq j \leq N$ it follows that

$$\nu(G_j) \leq \left(\frac{c_K \epsilon}{n}\right)^{d-1} \leq \frac{c_K}{N}, \quad 1 \leq j \leq N.$$

Thus the upper bound in Theorem 4 admits the following equivalent form: for every $h_n \in H_n^d$

$$\int_{\partial K} |h_n(z)|^p d\nu(z) \leq \frac{c_K}{N} \sum_{1\leq j\leq N} |h_n(x_j)|^p, \;\; N \sim n^{d-1}.$$

The last estimate corresponds to the **equal weight** version of Marcinkiewicz discretization, see [18] for details.

In addition, the Markov type estimate of Theorem 3 can be also used to verify existence of positive cubature rules of optimal cardinality for homogeneous polynomials on C^{1+} central symmetric convex bodies. Since homogeneous polynomials of odd degrees are odd, and thus their integrals on central symmetric domains equal zero we consider cubature formulas for even homogeneous polynomials.

Theorem 5 *Let $K \subset \mathbb{R}^d, d \geq 2$ be a C^{1+} central symmetric convex body satisfying $B^d(0,r) \subset K \subset B^d(0,1), r > 0$. Then given any $n \in \mathbb{N}$ there exists a partition $\partial K = \cup_{1\leq j\leq N} G_j$ with $N \sim n^{d-1}$, $\operatorname{diam}(G_j) \leq \frac{c_K}{n}, 1 \leq j \leq N$ such that for arbitrary $x_j \in G_j, 1 \leq j \leq N$ there exist positive λ_j satisfying $\lambda_j \leq \frac{c}{N}, 1 \leq j \leq N$ so that*

$$\int_{\partial K} h_n(z) d\nu(z) = \sum_{1\leq j\leq N} \lambda_j h_n(x_j), \quad \forall h_n \in H_{2n}^d.$$

Proofs

Our starting point will be the construction of proper needle polynomials. These polynomials will be based on certain properties of Jacobi polynomials $J_n^{(\alpha,\beta)}(x), \alpha, \beta > -1$ verified in [17], (7.34.1), p. 173. It is essentially shown therein that for any $\mu > -1, p > 0$

$$\int_0^1 (1-x)^\mu \left| P_n^{(\alpha,\beta)}(x) \right|^p dx \sim n^{-2\mu-2+\alpha p}, \;\; 2\mu < \alpha p - 2 + \frac{p}{2}. \tag{8}$$

Moreover, the lower bound in the above asymptotic relation holds for integration over any interval $[a, 1], 0 < a < 1$, see [17], (7.34.4). (In fact, in [17] these asymptotic relations are verified for $p = 1$ but they follow analogously for any $p > 0$.)

Lemma 1 *For any $\gamma > -1$ and arbitrary $a, p > 0$ there exist even polynomials $G_{2n} \in P_n^1, n \in \mathbb{N}$ such that with some $c > 0$ depending only on p and γ we have*

$$1 = G_{2n}(0) \leq \|G_{2n}\|_{L^\infty[-a,a]} \leq c, n \in \mathbb{N}$$

and

$$\int_{-a}^{a} |x|^\gamma |G_{2n}(x)|^p dx \leq c \left(\frac{a}{n}\right)^{\gamma+1}, \;\; n \in \mathbb{N}. \tag{9}$$

Proof Set

$$\alpha=\beta=\frac{\gamma+1}{p}>0,\quad \mu:=\frac{\gamma-1}{2}>-1,\quad G_{2n}(x):=\frac{J_n^{(\beta,\beta)}(1-\frac{x^2}{a^2})}{J_n^{(\beta,\beta)}(1)}.$$

Then (8) is applicable for any $p>0$. Recalling that $\|J_n^{(\beta,\beta)}\|_{L^\infty[-1,1]}\sim J_n^{(\beta,\beta)}(1)\sim n^\beta$ the first statement of the lemma easily follows.

In addition, we have by (8)

$$\begin{aligned}\int_{-a}^{a}|x|^\gamma|G_{2n}(x)|^p dx &= \frac{2a^{\gamma+1}}{J_n^{(\beta,\beta)}(1)^p}\int_0^1 y^\gamma|J_n^{(\beta,\beta)}(1-y^2)|^p dy\\ &= \frac{a^{\gamma+1}}{J_n^{(\beta,\beta)}(1)^p}\int_0^1(1-t)^\mu|J_n^{(\beta,\beta)}(t)|^p dt\sim\frac{a^{\gamma+1}}{n^{\beta p}}\\ &= \left(\frac{a}{n}\right)^{\gamma+1}.\end{aligned}$$

□

Now based on the univariate needle polynomials of Lemma 1 we will construct proper multivariate homogeneous needle polynomials related to an arbitrary point $z\in\partial K$ on the boundary of the convex body $K\subset\mathbb{R}^d$, $d\geq 2$ satisfying $B^d(0,r)\subset K\subset B^d(0,1)$.

For any $z\in\partial K$ denote by $\eta(z)\in S^{d-1}$ an outer normal to K at z. Clearly z can not be orthogonal to $\eta(z)$. Furthermore, let G_{2n} be the even univariate polynomials specified in Lemma 1 with $\gamma:=d-2$ and $a:=6$. Now for any given $z\in\partial K$ consider the polynomials

$$\Omega_z(x):=G_{2n}(|x-z-\langle x-z,\eta(z)\rangle\eta(z)|)\in P_{2n}^d,\tag{10}$$

where $\langle\cdot,\cdot\rangle$ stands for the usual Euclidian inner product.

Since $G_{2n}\in P_{2n}^1$ are even it is obvious that $\Omega_z(x)$ are polynomials of $x\in\mathbb{R}^d$ of degree $\leq 2n$ for every $z\in\partial K$. Note that $\Omega_z(z)=G_{2n}(0)=1$, $n\in\mathbb{N}$.

Denote by $H_z:=\{x\in\mathbb{R}^d:\langle x-z,\eta(z)\rangle=0\}$ the tangent hyperplane at $z\in\partial K$ with normal $\eta(z)$. Consider the linear homogeneous polynomials $l(x)$ defined by

$$l(x):=\frac{\langle x,\eta(z)\rangle}{\langle z,\eta(z)\rangle}\in H_1^d.$$

Clearly, $l(x)=1,\forall x\in H_z$. Polynomial $\Omega_z(x)\in P_{2n}^d$ admits a unique representation

$$\Omega_z(x)=\sum_{0\leq j\leq 2n}q_j(x),\quad q_j\in H_j^d.\tag{11}$$

Now we can introduce the homogeneous polynomials

$$Q_z(x):=\sum_{0\leq j\leq 2n}q_j(x)l^{2n-j}(x)\in H_{2n}^d.\tag{12}$$

It will be shown below in Lemma 3 that these homogeneous polynomials have the required "needle" properties. Note that on the hyperplane H_z these polynomials coincide with $\Omega_z(x)$ and thus we have

$$Q_z(x) = \Omega_z(x) = G_{2n}(|x - z|), \quad \forall x \in H_z.$$

As mentioned above for each $z \in \partial K$ we have that z can not be orthogonal to $\eta(z)$. In fact in order to make the proofs less technical we will repeatedly assume below that z and $\eta(z)$ are collinear. This can be assumed without the loss of generality based on the next geometric lemma noting that H_n^d is invariant under linear change of variables.. In what follows we will denote by $e_j \in \mathbb{R}^d, 1 \le j \le d$ the standard orthonormal basis in $\mathbb{R}^d$.

Lemma 2 *Let $K \subset \mathbb{R}^d$, $B^d(0, r) \subset K \subset B^d(0, 1), d \ge 2$ be a convex body centrally symmetric about the origin. The for any $z \in \partial K$ there exists a regular linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ so that $L(e_d) = z$, e_d is an outer normal to $D := L^{-1}(K)$ at $e_d \in \partial D$, and in addition,*

$$\|L\| \le 2, \|L^{-1}\| \le 1 + \frac{2}{r^2},$$

where $\|\cdot\|$ stands for the $l^2 \to l^2$ norm of the corresponding linear mapping.

Proof First let us observe that convexity of K and assumption $B^d(0, r) \subset K \subset B^d(0, 1)$ easily imply that the sine of the angle between $z \in \partial K$ and any tangent line to K at z is $\ge r$. Therefore it is easy to see that

$$\langle z, \eta(z)\rangle \ge r^2 \tag{13}$$

for any normal $\eta(z)$. We may assume that $\eta(z) = e_d$ (this can be achieved by a simple rotation), i.e., $|\langle z, e_d\rangle| \ge r^2$. Now let the linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $L(e_j) = e_j, 1 \le j \le d - 1, L(e_d) = z$, and set $D := L^{-1}(K)$. Then $e_d \in \partial D$, and since $H := \operatorname{span}\{e_j, 1 \le j \le d - 1\}$ is a supporting plane to K at z and $L^{-1}(e_j) = e_j, 1 \le j \le d - 1$ it follows that H is supporting to D at e_d, as well. Obviously this means that e_d is an outer normal to D at $e_d \in \partial D$.

Furthermore, for any $u = \sum_{1\le j\le d} u_j e_j \in B^d(0, 1)$ setting $u^* := \sum_{1\le j\le d-1} u_j e_j$, $|u^*| \le 1$ we have $L(u) = u^* + u_d z$ and thus

$$|L(u)| \le |u^*| + |u_d z| \le 1 + |z| \le 2.$$

Similarly, if $w = \sum_{1\le j\le d-1} w_j e_j + w_d z = w^* + w_d z \in B^d(0, 1)$ then

$$1 \ge |\langle w, e_d\rangle| = |w_d\langle z, e_d\rangle| \ge |w_d|r^2, \quad |w^*| \le |w| + |w_d z| \le 1 + |w_d| \le 1 + \frac{1}{r^2}.$$

Hence

$$|L^{-1}(w)| = |w^* + w_d e_d| \leq |w^*| + |w_d e_d| \leq 1 + \frac{1}{r^2} + |w_d| \leq 1 + \frac{2}{r^2}.$$

□

Our next lemma presents basic properties of the needle polynomials introduced above. This lemma will be essential for the proof of the main results. It is important to note the domain independent character of this lemma. Namely, given that $B^d(0,r) \subset K \subset B^d(0,1)$ the constants in the estimates of the lemma depend on r, d, p and not specific features of domain K.

Lemma 3 *Let $K \subset \mathbb{R}^d, B^d(0,r) \subset K \subset B^d(0,1), d \geq 2$ be a regular convex body centrally symmetric about the origin. Then for any $p > 0, z \in \partial K$ there exist polynomials $Q_z(x) \in H^d_{2n}$ such that with certain $c > 0$ depending only on r, d and p*

$$Q_z(z) = 1, \quad D_u Q_z(z) = 0, \quad \forall u \perp \eta(z); \tag{14}$$

$$\int_K |Q_z(x)|^p dx \leq \frac{c}{n^d}, \quad n \in \mathbb{N}. \tag{15}$$

Moreover, for every $x \in \partial K$

$$\int_{\partial K} |Q_z(x)|^p d\nu(z) \leq \frac{c}{n^{d-1}}, \quad n \in \mathbb{N}. \tag{16}$$

Proof Let G_{2n} be chosen as the even univariate polynomials of Lemma 1 with $\gamma := d-2$ and $a := 6$, and define $Q_z(x)$ by (10)-(12). Since for $x \in H_z$ we have $l(x) = 1$ it follows that

$$Q_z(x) = \Omega_z(x) = G_{2n}(|x-z|), \forall x \in H_z, \tag{17}$$

where $G_{2n}(0) = 1, G'_n(0) = 0, n \in \mathbb{N}$. These relations clearly yield (14).

Let us verify now the upper bound (15) for the given $z \in \partial K$. By the symmetry of both the underlying domain and homogeneous polynomials it suffices to verify (15) for the integral on $K^z := \{x \in K : \langle x, \eta(z)\rangle \geq 0\}$. Now for any $x \in K^z$ denote by x_z the point in H_z which is collinear with x, i.e., $x_z = \frac{\langle z,\eta(z)\rangle}{\langle x,\eta(z)\rangle} x$. In view of Lemma 2 we can assume that $z = \alpha e_d, \eta(z) = e_d, 0 < \alpha \leq 1$, and hence $x_z = \frac{\alpha}{x_d} x$. Furthermore, set

$$K_1^z := \{x \in K^z : |z - x_z| \leq 4\}, \quad K_2^z := K^z \setminus K_1^z.$$

Consider now the cone

$$\Sigma := \{x = (x_1, ...x_d) \in \mathbb{R}^d : \sqrt{x_1^2 + ... + x_{d-1}^2} \leq \frac{4x_d}{\alpha}, 0 \leq x_d \leq \alpha\}.$$

Since $x_z = \frac{\alpha}{x_d} x$ it easily follows that

$$|x_z - z| = \frac{\alpha}{x_d}\sqrt{x_1^2 + ... + x_{d-1}^2}.$$

This implies that $K_1^z \subset \Sigma$, i.e.,

$$\int_{K_1^z} |Q_z(x)|^p dx \le \int_\Sigma |Q_z(x)|^p dx$$
$$= \int_0^\alpha \int_{|y|\le \frac{4x_d}{\alpha}} |Q_z(x)|^p dy dx_d, \quad y = (x_1, ..., x_{d-1}). \tag{18}$$

Furthermore, using that $Q_z \in H_{2n}^d$ and applying (17) we have

$$Q_z(x) = \left(\frac{x_d}{\alpha}\right)^{2n} Q_z(x_z) = \left(\frac{x_d}{\alpha}\right)^{2n} G_{2n}(|x_z - z|)$$
$$= \left(\frac{x_d}{\alpha}\right)^{2n} G_{2n}\left(\frac{\alpha}{x_d}|y|\right), \quad y = (x_1, ..., x_{d-1}).$$

Applying last relation together with (18) yields

$$\int_{K_1^z} |Q_z(x)|^p dx \le \int_0^\alpha \left(\frac{x_d}{\alpha}\right)^{2np} \int_{|y|\le \frac{4x_d}{\alpha}} |G_{2n}\left(\frac{\alpha}{x_d}|y|\right)|^p dy dx_d, \quad y = (x_1, ..., x_{d-1}).$$

Evidently, the inner integral above can be estimated on the $d-1$ ball $|y| \le \frac{4x_d}{\alpha}$ using spherical transformation

$$\int_{|y|\le \frac{4x_d}{\alpha}} |G_{2n}\left(\frac{\alpha}{x_d}|y|\right)|^p dy \le c \int_{0\le\rho\le \frac{4x_d}{\alpha}} |G_{2n}\left(\frac{\alpha}{x_d}\rho\right)|^p \rho^{d-2} d\rho$$
$$= c\left(\frac{x_d}{\alpha}\right)^{d-1} \int_0^4 |G_{2n}(t)|^p t^{d-2} dt.$$

Thus combining the last two estimates above we arrive at

$$\int_{K_1^z} |Q_z(x)|^p dx \le c \int_0^\alpha \left(\frac{x_d}{\alpha}\right)^{2np+d-1} dx_d \int_0^4 |G_{2n}(t)|^p t^{d-2} dt$$
$$\le \frac{c\alpha}{n} \int_0^4 |G_{2n}(t)|^p t^{d-2} dt.$$

Now we need to recall that G_{2n} were chosen as the even univariate polynomials of Lemma 1 with $\gamma := d-2$ and $a := 6$. Hence it follows from (9) and the last upper bound above using in addition, that $\alpha = |z| \le 1$

$$\int_{K_1^z} |Q_z(x)|^p dx \le \frac{c}{n^d}, \quad n \in \mathbb{N},$$

with $c > 0$ being a constant depending only on r, p and d.

Now in order to complete the proof of the upper bound (15) we need to give a similar estimate for $\int_{K_2^z} |Q_z(x)|^p dx$. Note that for $x \in K_2^z$ we have that $|z - x_z| > 4$. Recall that by Lemma 1 for $|z - x_z| \leq a = 6$ we have

$$|Q_z(x_z)| = |G_{2n}(|x_z - z|)| \leq c, \quad |z - x_z| \leq 6, \tag{19}$$

where c depends on p and $\gamma = d - 2$.

It is well known that for any univariate polynomial $g_n \in P_n^1$ of degree $\leq n$ and any $b > 0$

$$|g_n(t)| \leq T_n(|t|/b)\|g_n\|_{L^\infty[-b,b]} \leq (2|t|/b)^n \|g_n\|_{L^\infty[-b,b]}, \quad \forall |t| > b,$$

where $T_n(t) = \frac{1}{2}((t + \sqrt{t^2 - 1})^n + (t - \sqrt{t^2 - 1})^n)$ is the classical Chebyshev polynomial of degree n.

Applying this property for polynomial $G_{2n}(t)$ with $b := 4$ and taking into account relation (19) it follows that

$$|Q_z(x_z)| = |G_{2n}(|x_z - z|)| \leq c \left(\frac{2|z - x_z|}{4} \right)^{2n}, \quad |z - x_z| > 4.$$

Furthermore, using that $Q_z \in H_{2n}^d$, $x_z = \frac{\alpha}{x_d} x$ yields

$$|Q_z(x)| = \left(\frac{|x|}{|x_z|} \right)^{2n} |Q_z(x_z)|.$$

Hence it follows from the previous estimate that

$$|Q_z(x)| \leq c \left(\frac{2|z - x_z||x|}{4|x_z|} \right)^{2n} \leq c \left(\frac{|z - x_z|}{2|x_z|} \right)^{2n}, \quad |z - x_z| > 4. \tag{20}$$

Now let us verify that $|z - x_z| \leq \frac{4}{3}|x_z|$ whenever $|z - x_z| > 4$. Indeed, if $\frac{4}{3}|x_z| < |z - x_z| \leq |z| + |x_z| \leq 1 + |x_z|$ then $|x_z| < 3$. However, $4 < |z - x_z| \leq 1 + |x_z|$, i.e., $|x_z| > 3$ when $|z - x_z| > 4$. Hence $|z - x_z| \leq \frac{4}{3}|x_z|$ and using this together with (20) yields

$$|Q_z(x)| \leq c \left(\frac{|z - x_z|}{2|x_z|} \right)^{2n} \leq c \left(\frac{2}{3} \right)^{2n}, \quad x \in K_2^z := \{x \in \partial K^z : |z - x_z| > 4\}.$$

Since $K_2^z \subset K \subset B^d(0, 1)$ it follows that

$$\int_{K_2^z} |Q_z(x)|^p dx \leq c \left(\frac{2}{3} \right)^{2n}$$

with $c > 0$ depending only on p and d. Clearly this is a stronger than required upper bound on K_2^z which completes the proof of (15).

Now let us turn our attention to the proof of the more delicate estimate (16). Consider any point $x \in \partial K$. Again using Lemma 2 without the loss of generality we can assume that $x = \alpha e_d, \eta(x) = e_d, \alpha > 0$.

Furthermore, let $D \subset \mathbb{R}^{d-1}$ be the orthogonal projection of K onto $\mathbb{R}^{d-1}$. Clearly, there exist convex functions $-f, g \in C^1(D)$, $\nabla f(0) = 0$ so that

$$\begin{aligned}\partial K &= \partial K^+ \cup \partial K^-, \quad \partial K^+ := \{z = (y, f(y)), y \in D\}, \ \partial K^- := \partial K \setminus \partial K^+ \\ &= \{z = (y, g(y)), y \in D\}.\end{aligned}$$

Then

$$\eta(z) = (-f_{y_1}, ..., -f_{y_{d-1}}, 1) = (-\nabla f, 1), \quad z = (y, f(y)) \in \partial K^+,$$

where ∇f stands for the gradient of f. Since $x_z = le_d$ with some $l > 0$ and $x_z - z \perp \eta(z)$ it follows by a simple calculation that we must have $l = f(y) + \langle y, \nabla f(y)\rangle$, Thus

$$x_z = (f(y) + \langle y, \nabla f(y)\rangle)e_d, \quad z - x_z = (y, -\langle y, \nabla f(y)\rangle), \quad z = (y, f(y)) \in \partial K^+.$$

Hence setting for $y \in \mathbb{R}^{d-1}$, $y := \rho u$, $\rho = |y|$, $u \in S^{d-2}$ we obtain

$$|z - x_z| = \sqrt{|y|^2 + \langle y, \nabla f(y)\rangle^2} = \rho\sqrt{1 + \langle u, \nabla f\rangle^2} = \rho\sqrt{1 + f_\rho^2}, \tag{21}$$

where f_ρ stands for the partial derivative of $f(\rho u)$ with respect to ρ. Using again the symmetry of the domain and homogeneous polynomials it suffices to estimate the integral in (16) on ∂K^+. First we will estimate this integral on

$$\partial K_x^+ := \{z \in \partial K^+ : |x_z| \le 5\}.$$

Note that

$$|z - x_z| \le |z| + |x_z| \le |z| + 5 \le 6, \quad z = (y, f(y)) \in \partial K_x^+.$$

Hence by (21)

$$\rho\sqrt{1 + f_\rho^2} = |z - x_z| \le 6, \quad \forall z \in \partial K_x^+. \tag{22}$$

Since $x_z = cx, c > 1$ and polynomial $Q_z(x)$ is homogeneous we have that $|Q_z(x)| \le |Q_z(x_z)|$, i.e.,

$$\begin{aligned}\int_{\partial K_x^+} |Q_z(x)|^p d\nu(z) &\le \int_{\partial K_x^+} |Q_z(x_z)|^p d\nu(z) = \int_{\partial K_x^+} |G_{2n}(|x_z - z|)|^p d\nu(z) \\ &= \int_{D_x} |G_{2n}(|x_z - z|)|^p \sqrt{1 + |\nabla f|^2} dy,\end{aligned} \tag{23}$$

where $D_x := \{y \in D : z = (y, f(y)) \in \partial K_x^+\}$. Note that $0 \in \mathrm{Int} D_x$.

Let us show now that $|\nabla f| \leq \frac{5}{r}$ whenever $y \in D_x$. Indeed, for any $y \in D_x, z = (y, f(y)) \in \partial K_x^+$ let L_z be the 2-dimensional plane through points $0, x_z, z$. Consider the line $l_z := \{tw + z : t \in \mathbb{R}\} \subset L_z, w = (w_1, ..., w_d) \in S^{d-1}$ which is tangent to $K \cap L_z$ at z, and set $w_z := (w_1, ..., w_{d-1}, 0)$. If $w_d = 0$, i.e., $w \perp \eta(x)$ then evidently $D_{w_z} f(z) = 0$. On the other hand if $w_d \neq 0$ then l_z intersects $\mathbb{R}^{d-1}$ at the point $w^* = -\frac{z_d}{w_d} w + z$. Note that since l_z is tangent to the convex set K at the point z it follows that $w^* \notin \mathrm{Int} K$, i.e. $|w^*| \geq r$ where r is the radius of the largest ball inscribed into K centered at the origin. In addition, $|x_z| \leq 5$ because $z = (y, f(y)) \in \partial K_x^+$. Hence in this case $|D_{w_z} f(z)| = \frac{|x_z|}{|w^*|} \leq \frac{5}{r}$. Noting that the last upper bound holds for every $w_z := (w_1, ..., w_{d-1}, 0) \in S^{d-2}$ it follows that $|\nabla f| \leq \frac{5}{r}$ whenever $y \in D_x$. Applying this bound in (23) yields

$$\int_{\partial K_x^+} |Q_z(x)|^p d\nu(z) \leq \sqrt{1 + \frac{25}{r^2}} \int_{D_x} |G_{2n}(|x_z - z|)|^p dy.$$

In addition, using the spherical transformation for $y \in \mathbb{R}^{d-1}, y := \rho u, \rho = |y|, u \in S^{d-2}$ in the last integral together with (22) we have

$$\int_{D_x} |G_{2n}(|x_z - z|)|^p dy \leq c \int_{\rho\sqrt{1+f_\rho^2} \leq 6} \rho^{d-2} |G_{2n}(\rho\sqrt{1 + f_\rho^2})|^p d\rho.$$

By the convexity of function $-f$ and relation $\nabla f(0) = 0$ it follows that $f_\rho^2(\rho u)$ is a monotone increasing function of $\rho > 0$ for each $u \in S^{d-2}$. Let us now introduce the variable $t = \phi(\rho) := \rho\sqrt{1 + f_\rho^2}, t \geq \rho$. A standard calculation yields $\phi'(\rho) \geq 1$ a.e. on S^{d-2}. Thus using this change of variables in the last integral above yields that

$$\int_{\rho\sqrt{1+f_\rho^2} \leq 6} \rho^{d-2} |G_{2n}(\rho\sqrt{1 + f_\rho^2})|^p d\rho \leq \int_0^6 t^{d-2} |G_{2n}(t)|^p dt.$$

Recalling that G_{2n} are chosen to be the even univariate polynomials of Lemma 1 with $\gamma := d - 2$ and $a := 6$ we obtain from the last three estimates that

$$\int_{\partial K_x^+} |Q_z(x)|^p d\nu(z) \leq \frac{c}{n^{d-1}}, \ n \in \mathbb{N}.$$

It remains now to estimate the needed integral on $\partial K^+ \backslash \partial K_x^+ = \{z \in \partial K^+ : |x_z| > 5\}$. Note that whenever $z \in \partial K^+ \backslash \partial K_x^+$ it follows that $|z - x_z| \geq |x_z| - |z| > 5 - 1 = 4$. Therefore recalling upper bound (20) we have

$$|Q_z(x)| \leq c \left(\frac{|z - x_z|}{2|x_z|}\right)^{2n}, \quad z \in \partial K^+ \setminus \partial K_x^+.$$

We claim, in addition that $|z - x_z| \leq \frac{3}{2}|x_z|$ when $z \in \partial K^+ \setminus \partial K_x^+$. Assuming the contrary would yield $1 + |x_z| \geq |z - x_z| > \frac{3}{2}|x_z|$, i.e., $|x_z| \leq 2$, contradicting that

$|x_z| > 5$ for $x \in \partial K^+ \setminus \partial K_x^+$. Thus

$$|z - x_z| \leq \frac{3}{2}|x_z|, \quad z \in \partial K^+ \setminus \partial K_x^+.$$

Applying this in the last estimate above we arrive at

$$|Q_z(x)| \leq c\left(\frac{3}{4}\right)^{2n}, \quad z \in \partial K^+ \setminus \partial K_x^+$$

which completes the proof of Lemma 3. □

Let us introduce now the $L^p, 0 < p \leq \infty$ Markov Factors for homogeneous polynomials $h \in H_n^d$ measuring the L^p norms of Dh (i.e., the Euclidian norms of the gradient of h) on ∂K:

$$M_n(K, p) := \sup_{h \in H_n^d} \frac{\|Dh\|_{L^p(\partial K)}}{\|h\|_{L^p(\partial K)}}.$$

As it was mentioned in the Introduction in case of L^∞ norm Harris [8] showed that for any 0-symmetric convex body $K \subset \mathbb{R}^d$ we have that $M_n(K, \infty) = O(n \log n)$. In addition, it was verified in [9] that under the C^{1+} smoothness condition on the convex domain $M_n(K, \infty) = O(n)$. As usual, L^∞ Markov type results can be used to derive Nikolskii type estimates providing lower bounds for L^p norms via L^∞ norms.

Lemma 4 *Let $K \subset \mathbb{R}^d$, $B^d(0, r) \subset K \subset B^d(0, 1)$, $d \geq 2$ be a convex body centrally symmetric about the origin. Then for any $x \in \partial K$ and $p > 0$*

$$\lambda_n^*(x, K) \geq \frac{c}{M_n(K, \infty)^d}. \tag{24}$$

Moreover, for every $h \in H_n^d$

$$\|h\|_{L^\infty(\partial K)} \leq c M_n(K, \infty)^{\frac{d-1}{p}} \|h\|_{L^p(\partial K)}. \tag{25}$$

Proof First let us observe that convex body K evidently satisfies the next property:

$$\mu(K \cap B^d(y, \delta)) \geq c\delta^d; \quad \nu(\partial K \cap B^d(y, \delta)) \geq c\delta^{d-1}, \quad \forall y \in \partial K, \quad \forall \delta > 0, \tag{26}$$

where μ, ν are the d dimensional Lebesgue measure and $d-1$ dimensional Lebesgue surface measure on ∂K, respectively. The first estimate in (26) is related to the fact that $K \cap B^d(y, \delta)$ contains a spherical sector centered at y of radius δ and central angle with tangent $\geq \frac{r}{2}$. The second lower bound follows by observing that $\partial K \cap B^d(y, \delta)$ can be viewed as the graph of a convex function $f : \Upsilon \to \partial K \cap B^d(y, \delta)$, $B^{d-1}(0, r) \subset \Upsilon \subset B^{d-1}(0, 1)$ with partial derivatives of f being $\leq 1 + \frac{2}{r}$ on $B^{d-1}(0, r/2)$. This easily implies that $f^{-1}(\partial K \cap B^d(y,\delta))$ contains a ball $B^{d-1}(0, c\delta)$ with a proper $c > 0$ depending only on r, i.e., the second needed estimate follows, as well.

Consider now any $x \in \partial K$ and an abitrary $h \in H_n^d$ satisfying $h(x) = 1$. Then for some $y \in \partial K$ we have that $|h(y)| = \|h\|_{L^\infty(\partial K)} \geq 1$. Set

$$\delta_n := \frac{1}{2M_n(K,\infty)};\ \Omega_1 := K \cap B^d(y,\delta_n),\quad \Omega_2 := \partial K \cap B^d(y,\delta_n).$$

Then it follows by the convexity of K and the mean value theorem that for every $z \in \Omega_1$

$$|h(y) - h(z)| \leq |y - z| \|Dh\|_{L^\infty(K)} \leq \delta_n M_n(K,\infty)\|h\|_{L^\infty(\partial K)} \leq \frac{\|h\|_{L^\infty(\partial K)}}{2}.$$

Hence

$$|h(z)| \geq |h(y)| - |h(y) - h(z)| \geq \frac{\|h\|_{L^\infty(\partial K)}}{2} \geq \frac{1}{2},\quad \forall z \in \Omega_1.$$

This together with (26) easily yields

$$\int_K |h|^p \geq \int_{\Omega_1} |h|^p \geq 2^{-p}\mu(\Omega_1) \geq c\delta_n^d.$$

Since $\delta_n := \frac{1}{2M_n(K,\infty)}$ the last lower bound evidently yields (24).

Similarly,

$$\begin{aligned}\|h\|_{L^p(\partial K)} &= \left(\int_{\partial K} |h|^p d\nu\right)^{\frac{1}{p}} \geq \left(\int_{\Omega_2} |h|^p d\nu\right)^{\frac{1}{p}} \\ &\geq \frac{\|h\|_{L^\infty(\partial K)}}{2}\nu^{\frac{1}{p}}(\Omega_2) \geq c\|h\|_{L^\infty(\partial K)}\delta_n^{\frac{d-1}{p}}\end{aligned}$$

which obviously implies (25). □

As it was mentioned earlier for multivariate polynomials in general it is considerably harder to derive Markov type estimates in L^p norm than in L^∞ norm. The next fundamental Lemma 6 based on application of needle polynomials of Lemma 3 establishes a relation between L^p and L^∞ Markov Factors for homogeneous polynomials. But prior to verifying this lemma we will need an auxiliary statement concerning *tangential gradients* of homogeneous polynomials. For a regular convex body $K \subset \mathbb{R}^d$ the tangential gradient of a function f at any $z \in \partial K$ is defined by

$$D_T f(z) := \max\{|D_u f(z)| : u \in S^{d-1},\ u \perp \eta(z)\}.$$

Lemma 5 *If $K \subset \mathbb{R}^d$, $B^d(0,r) \subset K \subset B^d(0,1)$, $d \geq 2$ is a regular convex body centrally symmetric about the origin then for every $h \in H_n^d$ and every $z \in \partial K$*

$$Dh(z) \leq (1 + r^{-2})D_T h(z) + nr^{-2}|h(z)|. \tag{27}$$

Proof Given an arbitrary $z \in \partial K$ any $u \in S^{d-1}$ can be written as $u = \alpha w + \beta z, \alpha, \beta \in \mathbb{R}$ where $w \in S^{d-1}$, $w \perp \eta(z)$. Moreover by (13) we must have $|\langle z, \eta(z)\rangle| \geq r^2$, i.e.

$$1 \geq |\langle u, \eta(z)\rangle| = |\beta\langle z, \eta(z)\rangle| \geq \beta r^2.$$

Therefore $\beta \leq r^{-2}$ and in addition, $|\alpha| = |u - \beta z| \leq 1 + r^{-2}$. Thus recalling that $D_z h(z) = nh(z), \forall h \in H_n^d$ we obtain

$$\begin{aligned}|D_u h(z)| &= |\alpha D_w h(z) + \beta D_z h(z)| \leq |\alpha| D_T h(z) + |\beta| n |h(z)| \\ &\leq (1 + r^{-2}) D_T h(z) + r^{-2} n |h(z)|.\end{aligned}$$

□

Lemma 6 *Let $K \subset \mathbb{R}^d$, $B^d(0, r) \subset K \subset B^d(0, 1)$, $d \geq 2$ be a convex body centrally symmetric about the origin. Then for any $p > 0$*

$$M_n(K, p) \leq cn^{\frac{1-d}{p}} M_{3n}(K, \infty)^{\frac{d-1}{p}+1} \tag{28}$$

with some $c > 0$ depending only on r, d and p.

Proof First let us observe that the domain independent character of the statement of the lemma yields that we can assume without the loss of generality that K is regular. Indeed, any 0-symmetric convex body K satisfying $B^d(0, r) \subset K \subset B^d(0, 1)$ can be approximated by regular 0-symmetric convex domains $D \subset K$ satisfying $B^d(0, \frac{r}{2}) \subset D \subset B^d(0, 1)$. Hence if Lemma 5 holds for regular convex domains then a standard compactness argument (applied with n being fixed) provides the needed statement for any convex K, as well.

Consider any homogeneous polynomial $h \in H_n^d$ such that $\|h\|_{L^p(\partial K)} = 1$. By Lemma 3 for any $z \in \partial K$ there exist polynomials $Q_z(x) \in H_{2n}^d$ such that relations (14)-(16) hold. Set

$$q_z(x) := h(x) Q_z(x) \in H_{3n}^d.$$

Since by (14) we have $Q_z(z) = 1$, $D_u Q_z(z) = 0$, $\forall u \perp \eta(z), u \in S^{d-1}$ it easily follows that

$$D_u q_z(z) = h(z) D_u Q_z(z) + D_u h(z) Q_z(z) = D_u h(z) Q_z(z) = D_u h(z), \quad \forall u \perp \eta(z).$$

Therefore applying in addition estimate (25) it follows that for every $u \perp \eta(z), u \in S^{d-1}$

$$\begin{aligned}|D_u h(z)|^p &\leq \|D_u q_z\|_{L^\infty(\partial K)}^p \leq M_{3n}^p(K, \infty) \|q_z\|_{L^\infty(\partial K)}^p \\ &\leq c M_{3n}(K, \infty)^{p+d-1} \int_{\partial K} |q_z(x)|^p d\nu(x),\end{aligned}$$

i.e.,

$$D_T^p h(z) \leq c M_{3n}(K, \infty)^{p+d-1} \int_{\partial K} |q_z(x)|^p d\nu(x).$$

Furthermore, integrating the last upper bound over $z \in \partial K$ and using Fubini theorem we obtain

$$\begin{aligned}\int_{\partial K} D_T^p h(z) d\nu(z) &\leq c M_{3n}(K,\infty)^{d-1+p} \int_{\partial K}\int_{\partial K} |q_z(x)|^p d\nu(x) d\nu(z)\\ &= c M_{3n}(K,\infty)^{d-1+p} \int_{\partial K} |h(x)|^p \int_{\partial K} |Q_z(x)|^p d\nu(z) d\nu(x).\end{aligned}$$

Recall that by (16) we have

$$\int_{\partial K} |Q_z(x)|^p d\nu(z) \leq \frac{c}{n^{d-1}}, \quad \forall x \in \partial K.$$

Hence applying this estimate in the last upper bound above yields

$$\|D_T h\|_{L^p(\partial K)} \leq c n^{\frac{1-d}{p}} M_{3n}(K,\infty)^{\frac{d-1}{p}+1} \|h\|_{L^p(\partial K)}.$$

Combining this upper bound with (27) we finally arrive at

$$\begin{aligned}\|Dh\|_{L^p(\partial K)} &\leq (1+r^{-2})\|D_T h\|_{L^p(\partial K)} + n r^{-2}\|h\|_{L^p(\partial K)}\\ &\leq c(n^{\frac{1-d}{p}} M_{3n}(K,\infty)^{\frac{d-1}{p}+1} + n)\|h\|_{L^p(\partial K)}.\end{aligned}$$

It remains to note that since $M_n(K,\infty) \geq n, \forall n \in \mathbb{N}$ the last estimate yields the statement of Lemma 6. □

Proof of Theorems 1-3 For even $n \in \mathbb{N}$ Theorem 1 is an immediate consequence of Lemma 3 since any convex body $K \subset \mathbb{R}^d$ satisfying $B^d(0,r) \subset K \subset B^d(0,1)$ can be easily approximated by regular convex bodies containing $B^d(0, \frac{r}{2})$ (and also contained in $B^d(0,1)$). Using that estimates of Lemma 3 are domain independent a standard compactness argument (with $n \in \mathbb{N}$ being fixed) yields then the statement of the theorem in case of even n. If $n = 2m+1, m \in \mathbb{N}$ is odd and $z \in \partial K$ we can consider homogeneous needle polynomial $Q_z(x) \in H_{2m}^d$ of degree $2m$ satisfying all requirements of Theorem 1. Then setting

$$Q_z^*(x) := |z|^{-2}\langle z, x\rangle Q_z(x) \in H_n^d$$

yields a homogeneous polynomial $Q_z^*(x) \in H_n^d$ of degree n which evidently satisfies

$$Q_z^*(z) = Q_z(z) = 1, \quad |Q_z^*(x)| \leq \frac{|x|}{|z|}|Q_z(x)| \leq \frac{1}{r}|Q_z(x)|, \quad \forall x \in K.$$

Since $Q_z(x)$ satisfies conditions (3), (4) it follows that for $Q_z^*(x)$ these estimates hold, as well. This completes the proof of Theorem 1.

The upper bound (6) in Theorem 2 evidently follows from estimate (3) of Theorem 1. The lower bound in (6) can be obtained from Lemma 4 combined with $M_n(K,\infty) \leq$

$cn \log n$ proved by Harris [8]. Similarly, when K is C^{1+} Lemma 4 and the estimate $M_n(K, \infty) \le c_K n$ verified in [9], Corollary 1, p. 489 yields the lower bound in (7). This completes the proof of Theorem 2.

Likewise, Theorem 3 is a consequence of upper bound (28) of Lemma 6 and estimates $M_n(K, \infty) \le cn \log n$ (any convex body) and $M_n(K, \infty) \le c_K n$ (C^{1+} convex body). □

In order to verify Theorem 4 we will need some additional auxiliary statements. Let $\partial^k f_{x_{j_1},\dots x_{j_k}}$ denote the k-th partial derivative of f with respect to variables $x_{j_1}, \dots x_{j_k}$ and set

$$\nabla_d^k f(x) := \sum_{1 \le j_1 < \dots < j_k \le d} |\partial^k f_{x_{j_1},\dots x_{j_k}}|, \quad x = (x_1, \dots x_d), \quad 1 \le k \le d.$$

The next lemma is a multivariate version of the mean value inequality which we will need in the sequel.

Lemma 7 *For any $d \ge 1, r > 0, 1 \le k \le d$ and any C^d function f on $[0, r]^d$, we have*

$$|f(x) - r^{-d} \int_{[0,r]^d} f(z)dz| \le \sum_{1 \le k \le d} r^{k-d} \int_{[0,r]^d} \nabla_d^k f(z)dz, \quad \forall x \in [0, r]^d.$$

Proof We prove the above statement by induction on dimension $d \ge 1$. For $d = 1$ we clearly have for every $x \in [0, r]$

$$|f(x) - r^{-1} \int_{[0,r]} f(z)dz| = r^{-1} |\int_{[0,r]} \int_{[x,z]} f'(t)dtdz| \le \int_{[0,r]} |f'(x)|dx$$

which is the needed upper bound for $d = 1$.

Now assume that lemma holds for $d - 1$. Then applying the induction hypothesis to $f(y, t)$, $y \in [0, r]^{d-1}$ with any fixed $t \in [0, r]$ yields

$$|f(x) - r^{-d+1} \int_{[0,r]^{d-1}} f(y, t)dy| \le \sum_{1 \le k \le d-1} r^{k-d+1} \int_{[0,r]^{d-1}} \nabla_{d-1}^k f(y, t)dy := A. \tag{29}$$

In addition, set

$$B := |r^{-d+1} \int_{[0,r]^{d-1}} f(y, t)dy - r^{-d} \int_{[0,r]^d} f(z)dz|.$$

Then we have by (29)

$$|f(x) - r^{-d} \int_{[0,r]^d} f(z)dz| \le A + B. \tag{30}$$

For the quantity B we easily obtain the upper bound

$$B = r^{-d} \Big| \int_{[0,r]^d} (f(y,t) - f(z))dz \Big| \leq r^{1-d} \int_{[0,r]^d} |\partial^1 f_{x_d}|dz. \tag{31}$$

In order to estimate quantity A consider any $1 \leq j_1 < \ldots < j_k \leq d-1$ and a fixed $t \in [0,r]$. Then

$$\begin{aligned}
\int_{[0,r]^{d-1}} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(y,t)|dy &= \frac{1}{r} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(y,t)|dz \\
&\leq \frac{1}{r} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(z)|dz + \frac{1}{r} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(y,t) - \partial^k f_{x_{j_1},\ldots,x_{j_k}}(z)|dz \\
&\leq \frac{1}{r} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(z)|dz + \int_{[0,r]^d} |\partial^{k+1} f_{x_{j_1},\ldots,x_{j_k},x_d}(z)|dz.
\end{aligned}$$

Hence

$$\begin{aligned}
A &\leq \sum_{1 \leq k \leq d-1} r^{k-d} \sum_{1 \leq j_1 < \ldots < j_k \leq d-1} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_k}}(z)|dz \\
&\quad + \sum_{2 \leq k \leq d} r^{k-d} \sum_{1 \leq j_1 < \ldots < j_{k-1} \leq d-1} \int_{[0,r]^d} |\partial^k f_{x_{j_1},\ldots,x_{j_{k-1}},x_d}(z)|dz \\
&= \sum_{1 \leq k \leq d} r^{k-d} \int_{[0,r]^d} \nabla_d^k f(z)dz - r^{1-d} \int_{[0,r]^d} |\partial^1 f_{x_d}|dz.
\end{aligned}$$

Finally, the last estimate together with (30) and (31) easily yields the statement of the lemma for d-dimensional case. □

Next lemma provides several auxiliary statements regarding growth properties of homogeneous polynomials on convex bodies. We will need in the sequel some additional notions. Given a convex body $K \subset \mathbb{R}^d$ and any $x \in \mathbb{R}^d$ let $x + K := \{x + z : z \in K\}$ be the shift of the set K by x. In addition, denote by $aK := \{az : z \in K\}$, $a > 0$ the dilation of K by a.

Lemma 8 *Let $K \subset \mathbb{R}^d$ be a centrally symmetric convex body satisfying $B^d(0,r) \subset K \subset B^d(0,1)$. Then for any $p > 0$, and any $h_n \in H_n^d$*

$$\int_{aK} |h_n(z)|^p dz = a^{pn+d} \int_K |h_n(z)|^p dz, \quad a > 0; \tag{32}$$

$$\int_{x+K} |h_n(z)|^p dz \leq \left(1 + \frac{|x|}{r}\right)^{pn+d} \int_K |h_n(z)|^p dz, \quad \forall x \in \mathbb{R}^d; \tag{33}$$

$$r^2 \int_{\partial K} |h_n(z)|^p d\nu(z) \leq (pn+d) \int_K |h_n(z)|^p dz \leq \int_{\partial K} |h_n(z)|^p d\nu(z). \tag{34}$$

Proof First relation easily follows by substitution $z = ay$, $y \in K$

$$\int_{aK} |h_n(z)|^p dz = \int_K |h_n(ay)|^p a^d dy = \int_K a^{pn+d} |h_n(y)|^p dy.$$

Now let us verify that $x + K \subset \left(1 + \frac{|x|}{r}\right) K$ which together with (32) immediately yields (33). Indeed, for any $x \in \mathbb{R}^d$ we have $\frac{rx}{|x|} \in B^d(0, r) \subset K$ and therefore

$$\left(1 + \frac{|x|}{r}\right)^{-1} (x + y) = \left(1 + \frac{|x|}{r}\right)^{-1} \left(\frac{|x|}{r}\left(\frac{rx}{|x|}\right) + y\right) \in K$$

as a convex combination of $\frac{rx}{|x|}$, $y \in K$. Evidently, this means that $x+y \in \left(1 + \frac{|x|}{r}\right) K$ completing the proof of (33).

The proof of (34) relies on the Gauss-Ostrogradskij formula

$$\int_K \mathrm{div} F(z) dz = \int_{\partial K} \langle F(z), \eta(z)\rangle d\nu(z)$$

applied to the vector field $F(z) := z|h_n(z)|^p$, $z \in \mathbb{R}^d$.

Using that for every $h_n \in H_n^d$ we have $\langle \nabla h_n(z), z\rangle = nh_n(z)$, $z \in \mathbb{R}^d$ straightforward differentiation yields

$$\mathrm{div} F(z) = d|h_n(z)|^p + p|h_n(z)|^{p-1} \mathrm{sgn} h_n(z) \langle \nabla h_n(z), z\rangle = (d + pn)|h_n(z)|^p.$$

Thus applying the Gauss-Ostrogradskij formula

$$\int_K (d + pn)|h_n(z)|^p dz = \int_{\partial K} |h_n(z)|^p \langle \eta(z), z\rangle d\nu(z).$$

Note that for the outer unit normal $\eta(z)$ at $z \in \partial K \subset B^d(0, 1)$ we have by (13)

$$r^2 \le \langle \eta(z), z\rangle \le 1.$$

This clearly yields estimates (34). □

It should be noted that relations (34) essentially mean that for every $h_n \in H_n^d$

$$\int_K |h_n(z)|^p dz \sim \frac{1}{n} \int_{\partial K} |h_n(z)|^p d\nu(z).$$

This in particular yields that the statement of Theorem 3 remains true with all $\|...\|_{L^p(\partial K)}$ norms being replaced by $\|...\|_{L^p(K)}$ norms.

Now the Markov type order n upper bound of Theorem 3 given for C^{1+} domains will be applied for estimating oscillations of homogeneous polynomials via their L^p

norms. Such estimates are known to play a crucial role in verifying Marcinkiewicz-Zygmund type discretization results, see e.g., [4] and [5]. As usual, oscillation of function f on a set D is defined as

$$\operatorname{osc}(f, D) := \sup_{x,y\in D} |f(x) - f(y)|.$$

Lemma 9 *Let $n \in \mathbb{N}, d \geq 2, p \geq 1$ and consider a C^{1+} centrally symmetric convex body $K \subset \mathbb{R}^d$ satisfying $B^d(0, r) \subset K \subset B^d(0, 1)$. Then for every $0 < \delta < 1$ there exists a partition $\partial K = \cup_{1\leq j\leq N} G_j$ with $N \leq \left(\frac{cn}{\delta}\right)^{d-1}$, $\operatorname{diam}(G_j) \leq \frac{c\delta}{n}$ such that for every $h_n \in H_n^d$ we have*

$$\sum_{j=1}^{N} \nu(G_j)\operatorname{osc}^p(h_n, G_j) \leq c_K \delta^p \int_{\partial K} |h_n|^p dx.$$

Proof We will construct the proper partition considering first the $d - 1$ dimensional cube

$$I^{d-1} := \left[-\frac{r}{2\sqrt{d}}, \frac{r}{2\sqrt{d}}\right]^{d-1} \subset B^{d-1}(0, r/2)$$

and the corresponding cylinder $F^d(r) := \{(y, t) : y \in I^{d-1},\ t \in \mathbb{R}\}$. Evidently, we can partition the $d - 1$ dimensional cube I^{d-1} with edge $\frac{r}{\sqrt{d}}$ into the union of m^{d-1} cubes $I^{d-1} = \cup_{1\leq j\leq m^{d-1}} I_j^{d-1}$ where each $d - 1$ dimensional cube I_j^{d-1} has edge $r_m := \frac{r}{m\sqrt{d}}$, and the integer m will be specified below. Set $K_0 := K \cap F^d(r)$. Then evidently with some positive convex C^1 function f we have

$$K \cap F^d(r) = \{(y, t) :\ y \in I^{d-1},\ -f(-y) \leq t \leq f(y)\}.$$

By symmetry it suffices to construct a proper partition for the "upper" part of the boundary of $K \cap F^d(r)$ given by

$$\partial K_0 := \{(y, f(y)) : y \in I^{d-1}\}.$$

In addition convexity of the domain $K \subset \mathbb{R}^d$ satisfying $B^d(0, r) \subset K \subset B^d(0, 1)$ easily yields that

$$|\nabla f(y)| \leq \frac{2}{r},\ \ y \in I^{d-1}. \tag{35}$$

Now setting $G_j := \{(y, f(y)) : y \in I_j^{d-1}\}$ evidently yields a partition $\partial K_0 = \cup_{1\leq j\leq m^{d-1}} G_j$ for the upper part of intersection of ∂K with the cylinder $F^d(r)$. Clearly γ_d such cylinders (rotations of $F^d(r)$) will cover all of ∂K where γ_d depends only on d. This will yield a similar partition of cardinality $N := \gamma_d m^{d-1}$ for all of the boundary ∂K. In addition, relation (35) easily implies that

$$\operatorname{diam}(G_j) \leq c r_m \leq \frac{c}{m},\ \ 1 \leq j \leq m^{d-1}.$$

Given arbitrary $z_j = (y_j, f(y_j)) \in G_j$, $y_j \in I_j^{d-1}$, $1 \le j \le m^{d-1}$ let the linear function $L_j(y)$, $1 \le j \le m^{d-1}$ be the tangent planes to ∂K at $z_j \in G_j$. Then

$$L_j(y_j) = f(y_j), \quad |\nabla L_j| = |\nabla f(y_j)| \le \frac{2}{r}, \quad 1 \le j \le m^{d-1}.$$

Furthermore, consider an arbitrary $h_n \in H_n^d$ and set $g_j(y) := h_n(y, L_j(y)),\ 1 \le j \le m^{d-1}$. Then obviously

$$\begin{aligned}&g_j(y_j) = h_n(z_j), \quad \nabla_{d-1}^k g_j(y) \le c\nabla_{d-1}^k h_n(y, L_j(y)),\\ &\quad y \in I_j^{d-1},\ 1 \le j \le m^{d-1},\ 1 \le k \le d-1.\end{aligned}$$

Now we will apply Lemma 7 to g_j on the $d-1$ dimensional cube I_j^{d-1} with edge $r_m := \frac{r}{m\sqrt{d}}$ clearly yielding

$$\begin{aligned}&\operatorname{osc}(h_n, G_j) = \operatorname{osc}(g_j, I_j^{d-1}) \le 2 \sum_{1\le k\le d-1} r_m^{k-d+1} \int_{I_j^{d-1}} \nabla_{d-1}^k g_j(y)dy \le\\ &c \sum_{1\le k\le d-1} r_m^{k-d+1} \int_{I_j^{d-1}} \nabla_{d-1}^k h_n(y, L_j(y))dy \le c \sum_{1\le k\le d-1} r_m^{k-d+1}(A_{k,j} + B_{k,j})\end{aligned} \tag{36}$$

with

$$\begin{aligned}A_{k,j} &:= \int_{I_j^{d-1}} \nabla_{d-1}^k h_n(y, f(y))dy \le \int_{I_j^{d-1}} \nabla_{d-1}^k h_n(y, f(y))\sqrt{1+\nabla f}dy\\ &= \int_{G_j} \nabla_{d-1}^k h_n(z)d\nu(z)\end{aligned}$$

and

$$B_{k,j} := \int_{I_j^{d-1}} |\nabla_{d-1}^k h_n(y, L_j(y)) - \nabla_{d-1}^k h_n(y, f(y))|dy.$$

Now we need to estimate quantities $B_{k,j}$. Note that for every $y \in I_j^{d-1}$

$$a_j := \min_{y\in I_j^{d-1}} f(y) \le f(y) \le L_j(y) \le b_j := \max_{y\in I_j^{d-1}} f(y) + \max_{y\in I_j^{d-1}} |f(y) - L_j(y)|.$$

Since $d-1$ dimensional cubes I_j^{d-1} have edge r_m, $f(y)$ is a C^1 function satisfying (35), and $L_j(y)$ is its tangent plane at $z_j = (y_j, f(y_j)) \in G_j$, $y_j \in I_j^{d-1}$ it follows that

$$a_j \le f(y) \le L_j(y) \le b_j \le a_j + cr_m. \tag{37}$$

Thus setting $I_j^d := I_j^{d-1} \times [a_j, b_j]$, $1 \le j \le m^{d-1}$ we obtain a d-dimensional disjoint boxes with edge lengthes $\le cr_m \le \frac{c}{m}$ such that

$$J_m^d := \cup_{1\le j\le m^{d-1}} I_j^d \subset cr_m e_d + K, \quad e_d := (0, ..., 0, 1). \tag{38}$$

In addition, we have by (37)

$$\begin{aligned} B_{k,j} &= \int_{I_j^{d-1}} |\nabla_{d-1}^k h_n(y, L_j(y)) - \nabla_{d-1}^k h_n(y, f(y))| dy \\ &\le \int_{I_j^{d-1}} \int_{a_j}^{b_j} \nabla_d^{k+1} h_n(y, t) dt dy \\ &= \int_{I_j^d} \nabla_d^{k+1} h_n(z) dz. \end{aligned}$$

Evidently, it follows by (36) that

$$\mathrm{osc}^p(h_n, G_j) \le c \sum_{1\le k\le d-1} r_m^{p(k-d+1)} (A_{k,j}^p + B_{k,j}^p) \le c \sum_{1\le k\le d-1} \frac{A_{k,j}^p + B_{k,j}^p}{m^{p(k-d+1)}}.$$

Furthermore applying above upper bounds for $A_{k,j}$, $B_{k,j}$ together with the Hölder inequality yields

$$\begin{aligned} A_{k,j}^p &\le (\int_{G_j} \nabla_{d-1}^k h_n(z) d\nu(z))^p \le \nu(G_j)^{p-1} \int_{G_j} (\nabla_{d-1}^k h_n(z))^p d\nu(z) \\ &\le \frac{c}{m^{(p-1)(d-1)}} \\ &\quad \int_{G_j} (\nabla_{d-1}^k h_n(z))^p d\nu(z); \\ B_{k,j}^p &\le \mu(I_j^d)^{p-1} \int_{I_j^d} (\nabla_d^{k+1} h_n(z))^p dz \le \frac{c}{m^{d(p-1)}} \int_{I_j^d} (\nabla_d^{k+1} h_n(z))^p dz. \end{aligned}$$

Using the last three upper bounds and recalling that $\nu(G_j) \le \frac{c}{m^{d-1}}$ imply

$$\begin{aligned} \nu(G_j)\mathrm{osc}^p(h_n, G_j) \le c &\sum_{1\le k\le d-1} \frac{1}{m^{pk}} \int_{G_j} (\nabla_{d-1}^k h_n(z))^p d\nu(z) \\ &+ c \sum_{1\le k\le d-1} \frac{1}{m^{p(k+1)-1}} \int_{I_j^d} (\nabla_d^{k+1} h_n(z))^p dz. \end{aligned}$$

Thus taking the sum for $1 \le j \le m^{d-1}$ yields

$$\sum_{1\le j\le m^{d-1}} \nu(G_j)\mathrm{osc}^p(h_n, G_j) \le \sum_{1\le k\le d-1} \frac{c}{m^{pk}} \int_{\partial K} (\nabla^k_{d-1} h_n(z))^p d\nu(z)$$
$$+ \sum_{1\le k\le d-1} \frac{c}{m^{p(k+1)-1}} \int_{J^d_m} (\nabla^{k+1}_d h_n(z))^p dz := \Sigma_1 + \Sigma_2.$$

Now setting $m = \left[\frac{n}{\delta}\right] + 1$ and using the Markov type estimate of Theorem 3 we obtain for the sum Σ_1

$$\Sigma_1 = \sum_{1\le k\le d-1} \frac{c}{m^{pk}} \int_{\partial K} (\nabla^k_{d-1} h_n(z))^p d\nu(z) \le \sum_{1\le k\le d-1} \frac{c_K}{m^{pk}} n^{kp}$$
$$\int_{\partial K} |h_n(z)|^p d\nu(z) \le c_K \delta^p \int_{\partial K} |h_n(z)|^p d\nu(z).$$

In order to estimate Σ_2 we need to recall first that by (38) $J^d_m \subset \frac{ce_d}{n} + K$. Hence using (33), (34) and the Markov type estimate of Theorem 3

$$\int_{J^d_m} (\nabla^{k+1}_d h_n(z))^p dz \le \left(1 + \frac{c}{n}\right)^{pn+d} \int_K (\nabla^{k+1}_d h_n(z))^p dz \le c \int_K (\nabla^{k+1}_d h_n(z))^p dz$$
$$\le \frac{c}{n} \int_{\partial K} (\nabla^{k+1}_d h_n(z))^p d\nu(z) \le c_K n^{(k+1)p-1} \int_{\partial K} |h_n(z)|^p d\nu(z).$$

Therefore with $m = \left[\frac{n}{\delta}\right] + 1$ we have

$$\Sigma_2 = \sum_{1\le k\le d-1} \frac{c}{m^{p(k+1)-1}} \int_{J^d_m} (\nabla^{k+1}_d h_n(z))^p dz \le c_K \delta^{2p-1} \int_{\partial K} |h_n(z)|^p d\nu(z).$$

Finally, this leads to

$$\sum_{1\le j\le m^{d-1}} \nu(G_j)\mathrm{osc}^p(h_n, G_j) \le \Sigma_1 + \Sigma_2 \le c_K \delta^p \int_{\partial K} |h_n(z)|^p d\nu(z).$$

Last upper bound provides the needed partition for the intersection of ∂K with the cylinder $F^d(r)$. Since γ_d such cylinders (rotations of $F^d(r)$) cover all of ∂K this evidently leads to a partition of the whole boundary ∂K, as well, after we delete overlapping parts of the elements of partitions. Of course, the asymptotic relation $N \sim m^{d-1} \sim \left(\frac{n}{\delta}\right)^{d-1}$ for cardinality the partition will be preserved. Likewise, the asymptotic upper bounds $\mathrm{diam}(G_j) \le cr_m \le \frac{c}{m} \le \frac{c\delta}{n}$ will still hold after deleting the overlapping parts. □

Proof of Theorem 4 The statement of Theorem 4 easily follows from Lemma 9. Indeed, using Lemma 9 and the L^p triangle inequality it follows that for any $x_j \in G_j$, $1 \leq j \leq N$

$$\left| \left(\sum_{1\leq j\leq N} \nu(G_j)|h_n(x_j)|^p \right)^{\frac{1}{p}} - \|h_n\|_{L^p(\partial K)} \right| \leq \left(\sum_{1\leq j\leq N} \nu(G_j)\mathrm{osc}^p(h_n, G_j) \right)^{\frac{1}{p}}$$
$$\leq c_K\delta\|h_n\|_{L^p(\partial K)}, \tag{39}$$

i.e.,

$$(1-c_K\delta) \sum_{1\leq j\leq N} \nu(G_j)|h_n(x_j)|^p \leq \int_{\partial K} |h_n(z)|^p d\nu(z)$$
$$\leq (1+c_K\delta) \sum_{1\leq j\leq N} \nu(G_j)|h_n(x_j)|^p.$$

with proper $c_K > 0$ and every $0 < \delta < \frac{1}{c_K}$. Now setting $\epsilon := c_K\delta$ evidently leads to the statement of Theorem 4. □

Proof of Theorem 5 First note that setting $p = 1$, $\delta := \frac{1}{3c_K}$ in (39) it follows that with a proper partition $\partial K = \cup_{1\leq j\leq N} G_j$, $N \sim n^{d-1}$, $\mathrm{diam}(G_j) \leq \frac{c_K}{n}$ and arbitrary $x_j \in G_j$ we have for any homogeneous polynomial $h_n \in H_n^d$ satisfying $h_n(x_j) \geq 0$, $1 \leq j \leq N$

$$\sum_{1\leq j\leq N} \nu(G_j)\mathrm{osc}(h_n, G_j) \leq \frac{1}{3}\int_{\partial K} |h_n(z)|d\nu(z)$$

and

$$\int_{\partial K} |h_n(z)|d\nu(z) \leq \frac{3}{2} \sum_{1\leq j\leq N} \nu(G_j)h_n(x_j).$$

Therefore,

$$|\sum_{1\leq j\leq N} \nu(G_j)h_n(x_j) - \int_{\partial K} h_n(z)d\nu(z)| \leq \sum_{1\leq j\leq N} \nu(G_j)\mathrm{osc}(h_n, G_j)$$

$$\leq \frac{1}{3}\int_{\partial K} |h_n(z)|d\nu(z) \leq \frac{1}{2} \sum_{1\leq j\leq N} \nu(G_j)h_n(x_j).$$

Hence for any $h_n \in H_n^d$ satisfying $h_n(x_j) \geq 0$, $1 \leq j \leq N$

$$\frac{1}{2} \sum_{1\leq j\leq N} \nu(G_j)h_n(x_j) \leq \int_{\partial K} h_n(z)d\nu(z) \leq \frac{3}{2} \sum_{1\leq j\leq N} \nu(G_j)h_n(x_j).$$

Then it follows by a standard application of the convex separation theorem (see [6], pp. 1965-66 for details) that there exist $\lambda_j \geq c\nu(G_j)$ so that

$$\int_{\partial K} h_n(z) d\nu(z) = \sum_{1 \leq j \leq N} \lambda_j h(x_j), \quad \forall h \in H_n^d. \tag{40}$$

Thus we obtain the desired cubature formula with $\lambda_j \geq c\nu(G_j)$, $N \sim n^{d-1}$. It remains to verify the needed asymptotic upper bound for λ_j in case when $n = 2m$ is even. For any $x_j \in G_j \subset \partial K$ consider the homogeneous needle polynomial $Q_{x_j} \in H_m^d$, $Q_{x_j}(x_j) = 1$ of degree m which using Theorem 1 with $p = 2$ satisfies

$$\int_{\partial K} Q_{x_j}(x)^2 d\nu(z) \leq \frac{c}{n^{d-1}}.$$

Using the last upper bound together with (40) applied for $Q_{x_j}^2 \in H_n^d$ leads to

$$\frac{c}{n^{d-1}} \geq \int_{\partial K} Q_{x_j}^2(z) d\nu(z) = \sum_{1 \leq k \leq N} \lambda_j Q_{x_j}^2(x_k) \geq \lambda_j Q_{x_j}^2(x_j) = \lambda_j.$$

This completes the proof of Theorem 5. □

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