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The strong core of housing markets with partial order preferences

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ABSTRACT

We study the strong core of housing markets when agents' preferences over houses are expressed as partial orders. We provide a structural characterization of the strong core, and propose an efficient algorithm that finds an allocation in the strong core or decides that it is empty, even in the presence of forced and forbidden arcs. The algorithm satisfies the property of weak group-strategyproofness. Additionally, we show that certain results known for the strong core in the case where agents' preferences are weak orders can be extended to the setting with partial order preferences; among others, we show that the strong core in such housing markets satisfies the property of respecting improvements.

1. Introduction

A Shapley–Scarf housing market involves a set of agents, each endowed with exactly one unit of some indivisible good—a *house*—and having preferences over the houses owned by other agents in the market. Agents are assumed to trade among themselves without monetary transfers; hence, the outcome in such a market is an *allocation* that assigns to each agent exactly one house, possibly its own. Housing markets have been extensively studied in the fields of economics and computer science since the seminal work of Shapley and Scarf in 1974 (Shapley and Scarf, 1974). Their most prominent motivation comes from kidney-exchange programs (Roth et al., 2005; Biró et al., 2019, 2021), but applications include various exchange markets[1] ranging from time-sharing markets (Wang and Krishna, 2006) and time banks (Andersson et al., 2021) to on-campus housing (Abdulkadiroğlu and Sönmez, 1999).

Two prominent solution concepts widely investigated in connection to housing markets are the *core* and the *strong core*: the *core* of a housing market contains allocations from which no coalition of agents can deviate so that each of them strictly improves their situation by trading among themselves, while the *strong core* (or *strict core*) contains allocations that do not admit a coalition of agents who are able to weakly improve their situation, with at least one agent strictly improving as a result of the deviation. Shapley and Scarf described the Top Trading Cycle (TTC) mechanism, attributed to David Gale, that always finds an allocation in the core (Shapley and Scarf, 1974). Roth and Postlewaite showed that if agents' preferences are strict, then the TTC mechanism always returns the unique allocation in the strong core (Roth and Postlewaite, 1977). Roth proved that in such a model the TTC is *strategyproof*, meaning that no agent can improve their outcome under TTC by misreporting its preferences (Roth, 1982). Later, Bird (1984) showed that TTC under strict preferences is even *weakly group-strategyproof*, that is, no set of agents can improve their situation by misreporting their preferences in a coordinated fashion.

If agents can be indifferent between houses, then both the core and the strict core loose many of their appealing properties. Although the TTC mechanism always returns an allocation in the core even when agents' preferences over the houses are *weak orders* (i.e., linear orders containing ties) (Shapley and Scarf, 1974), the core in such markets may contain multiple allocations. As opposed to the core, the strong core can become empty if agents' preferences are weak orders. Quint and Wako characterized the strong core in such housing markets, and gave a polynomial-time algorithm that either finds an allocation in the strong core or concludes that the strong core is empty (Quint and Wako, 2004). Several researchers have introduced generalizations of the TTC in search of a mechanism that maintains the desirable properties of TTC even under the presence of indifference in the market; this line of research has yielded algorithms that efficiently find allocations in the core with additional properties such as Pareto-efficiency, while ensuring also strategyproofness (Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012; Aziz and de Keijzer, 2012; Plaxton, 2013; Saban and Sethuraman, 2013).

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[1] For an online exchange market for board games, see https://boardgamegeek.com/wiki/page/Math_Trades.

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Despite the significant effort to deal with weakly ordered preferences in housing markets, there is still limited knowledge about the case where agents' preferences can be *partial orders*, a generalization of weak orders. Partial orders arise in contexts where agents may regard two alternatives as *incomparable*; as opposed to indifference in weak orders, incomparability is not necessarily transitive. Despite the clear motivations for studying partial orders in the context of kidney exchange, as we explain it below, there is scant literature on housing markets that allow agents to express their preferences using partial orders. A recent study (Schlotter et al., 2024) addresses questions about the core in such a model, but we are not aware of any research concerning the strong core in a setting where preferences can be partial orders. This paper aims to address this gap in the literature.

1.1. Motivation

Partial orders naturally arise in real-life applications where agents compare alternatives based on multiple unrelated criteria. An important example of this occurs in kidney exchange, where patients exchange the incompatible organs donated by their own donors; such a situation can be modeled as a housing market where agents are incompatible patient–donor pairs. The preferences of patients over possible kidney transplants are primarily based on two factors: the age of the donor and the HLA compatibility between donor and recipient. In fact, in the UK kidney exchange program, these two factors are the only ones for which patients can set a threshold of acceptance (Biró et al., 2021). The unrelated nature of these two factors gives rise to a partial ordering: we can only assert that a certain graft is better than another if it is more advantageous regarding both of these factors, so a graft from a younger donor but with a worse HLA-matching is incomparable with a graft from an older donor but with a better HLA-matching. Such incomparability is not necessarily transitive; hence, patients' preferences over possible transplants are only partial, and not weak, orders.

Another source of partial order preferences in kidney exchange (or, in fact, in many other applications) is the fact that small differences between two alternatives may not be significant enough to influence the preferences, but such differences may add up to a more substantial contrast. For example, when the age difference between two possible donors is, say, less than a year, then a recipient might regard the corresponding grafts as equivalent; however, such incomparability is not transitive: donors A and B, as well as B and C, might have an age difference of less than a year, but A might be almost two years older than C. Such preferences can be described as so-called *semi-orders*, a special case of partial orders.

Partial orders may arise not only from preferences based on multiple criteria, but also from incomplete or coarse information, as well as from uncertainty stemming from cognitive constraints or bounded rationality. In this sense, they often provide a more realistic representation of preferences than the more restrictive assumption of weak linear orders.

For this reason, partial orders have been widely studied, e.g., in computational social choice. In ordinal elections, agents are typically required to report linear orders. In practice, however, the set of alternatives may be too large for agents to specify fully, leading to incomplete preference information (Xia and Conitzer, 2011). A similar phenomenon occurs in peer-review processes, where submissions are commonly grouped into categories such as accepted, rejected, and borderline. The subsequent deliberation over borderline papers naturally induces partial comparisons, as decision makers rarely provide a complete ranking of all submissions. Such procedures are closely related to three-way decision models (Li et al., 2020). Moreover, strategic behavior under partial preference information has been studied extensively in voting systems (Ding and Lin, 2013). See Fattore and Brüggemann (2017) for further applications where partial orders appear.[2]

These considerations motivate the use of partial orders in housing markets. They capture situations in which agents are unable or unwilling to provide complete preference information, for instance because the set of houses is large, comparisons are difficult, or relevant information is missing.

1.2. Overview of our results

We consider the strong core of housing markets where agents' preferences are expressed as partial orders, which is a setting that has not been previously studied.

After providing the preliminaries in Section 2, we explain why the Quint–Wako characterization fails when agents' preferences are partial—and not weak—orders (Section 3.1), and present an alternative characterization of the strong core in such a setting (Section 3.2). An intuitive formulation of our characterization is the following: an allocation is in the strong core of a housing market with incomplete preferences if and only if agents can be partitioned into a "peak" group of agents who each receive a house from among their best (undominated) choices, and a "residual" group whose allocation is in the strong core of the remaining submarket, with *every peak agent considering the house of each residual agent strictly worse than the house she receives*. This key difference between this characterization and the one given by Quint and Wako is this last condition, which guarantees that even though residual agents may prefer houses held by the first group, no blocking cycle arises, since no peak agent is willing to give up their allocated house for any house owned by a residual agent. Even though our characterization does not immediately yield an algorithm for finding an allocation in the strong core (if exists), in Section 4 we propose a polynomial-time algorithm for this problem. Recall that the Quint–Wako algorithm (Quint and Wako, 2004) finds an allocation in the strong core when agents' preferences are weak orders, or detects that the strong core is empty; hence, our algorithm can be thought of as a generalization of this result to the setting with partial order preferences.

In fact, our algorithm can handle forbidden and forced arcs in the graph underlying the market: it can find an allocation in the strong core that contains all forced arcs but no forbidden arcs, if such an allocation exists; see Sections 4.1–4.3. This feature is useful in applications where we need to find an allocation in the strong core subject to two types of constraints: certain agent–house pairs must be included, while others must be excluded in the allocation. Requiring a given agent to be allocated a particular house might be necessary due to priority rights or institutional rules, or to ensure that a certain vulnerable agent is assigned a favorable outcome, but it may also capture dynamic situations in which part of the allocation is already settled and needs to be extended without disrupting the already existing partial allocation. Forbidding some agent to be assigned some particular house models cases where, although an agent may prefer a given house, such an assignment is undesirable due to, e.g., ethical reasons or other external infeasibility constraints.

Furthermore, in Section 4.4 we prove that the mechanism associated with our algorithm is weakly group-strategyproof, so that no coalition of agents can (strictly) benefit from misreporting their preferences. Coalition deviations are modeled by allowing only agents within

[2] Partial orders also arise naturally in projects promoting the social inclusion of individuals with intellectual disabilities, where participants are assigned to job positions such as computer-numerical-controlled (CNC) machines (Wiesner-Steiner et al., 2014). Although this setting does not directly concern housing markets, a dynamic variant (where some participants are already assigned and new ones may enter) can be modeled as such. Preferences over positions are based on multiple criteria reflecting participants' abilities and job requirements, thereby inducing a partial order (Bruggemann et al., 2017).

Table 1
Summary of results on the problem of finding allocations in the core and strong of housing markets. If the problem is polynomial-time solvable, we write either "in P" or the name of the corresponding algorithm.

Problem	Strict orders	Weak orders	Partial orders
Core	TTC (Shapley and Scarf, 1974)	TTC (Shapley and Scarf, 1974)	TTC (Shapley and Scarf, 1974; Schlotter et al., 2024)
Core, forced/forbidden arcs	NP-hard (Schlotter et al., 2024)	NP-hard (Schlotter et al., 2024)	NP-hard (Schlotter et al., 2024)
Strong core	TTC (Roth and Postlewaite, 1977)	Quint–Wako (Quint and Wako, 2004)	**in** P (Theorem 8)
Strong core, forced/forbidden arcs	TTC (Roth and Postlewaite, 1977)	in P(Schlotter et al., 2024)	**in** P (Theorem 8)

the coalition to alter their reported preferences. Under weak group-strategyproofness, deviations that strictly benefit all coalition members are ruled out, while strong group-strategyproofness additionally excludes deviations that weakly benefit all members and strictly benefit at least one. While we guarantee weak group-strategyproofness, we also demonstrate that our algorithm is not strongly group-strategyproof.

We conclude in Section 5 with a brief summary of our results and some directions for future research.

For readers interested in additional properties of our algorithm or in the question how partial order preferences influence certain attributes of the strong core, we refer to the appendices. In Appendix A, we examine the efficiency properties of our algorithm, as well as its ability to enumerate all allocations in the strong core.

In Appendix B, we investigate certain properties of the strong core. The main result of Appendix B is that the strong core satisfies the property of *respecting improvement*: assuming that the outcome of a market is an allocation in the strong core, we show that an agent can only (weakly) benefit from an increase in the "value" of its house, i.e., from its house becoming more preferred according to the remaining agents; this observation is a generalization of the analogous result for weakly ordered preferences by Biró et al. (2023).

A preliminary version of this paper was published at AAMAS '25 (Schlotter and Mendoza-Cadena, 2025).

1.3. Related work

We have already mentioned in the introduction the series of papers containing prior work on the core and the strong core of housing markets, a line of research into which our paper fits smoothly. The closest works to our study are the papers by Quint and Wako (2004) and by Schlotter et al. (2024). The model used by Quint and Wako (2004) assumes that agents have weak preferences over the houses: in such housing markets, Quint and Wako provide a characterization of the strong core, a polynomial-time algorithm to find an allocation in the strong core, and a description of the strong core using an Integer Linear Program (ILP). As we will see, their characterization of the strong core (and thus, their algorithm for finding an allocation in it) does not extend to the model considered in this work where agents' preferences can be partial orders. By contrast, Schlotter et al. (2024) do consider partial order preferences—however, their work focuses on the concept of the core instead of the strong core. In fact, their only result that concerns the strong core is a polynomial-time algorithm for determining whether there exists a given allocation in the strong core that contains a set of forced arcs but avoids a set of forbidden arcs; however, this algorithm requires that agents' preferences be weak orders (Schlotter et al., 2024, Appendix B). See Table 1 for a comparison of the results in these works.

Besides the literature on housing markets, our paper also builds on various ideas that have been explored in the broader context of matching under preferences.

First, we emphasize that even though we are not aware of any previous work on housing markets with partial order preferences with the exception of the work by Schlotter et al. on the core (Schlotter et al., 2024), partial order preferences have been widely studied in the area of matching under preferences, appearing already in early works on stable matchings (Fleiner et al., 2007; Manlove, 2002; Irving et al., 2003; Fleiner et al., 2011; Rastegari et al., 2018, 2014).

Second, problems involving edge restrictions have also been extensively studied for stable matching problems, see e.g. Knuth (1976), Fleiner et al. (2011), Cseh and Manlove (2016), Dias et al. (2003), Cseh and Heeger (2020). In the context of housing markets, the question whether a given agent can obtain a certain house (or can avoid ending up with a certain house) has been addressed in Schlotter et al. (2024). The analogous problems have also been studied in the house allocation model where a set of objects needs to be allocated to a set of agents (without initial endowments) using serial dictatorship (Saban and Sethuraman, 2015; Aziz et al., 2013; Aziz and Mestre, 2014; Aziz et al., 2015; Cechlárová et al., 2017).

Third, the topic of (group-)strategyproofness has a vast literature in the area of social choice, and has been explored in connection to the TTC mechanism and its generalizations (Roth, 1982; Bird, 1984; Ma, 1994; Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012; Plaxton, 2013; Saban and Sethuraman, 2013); more recent results investigate strategyproofness in relation to different extensions of housing markets (Hong and Park, 2022; You et al., 2022), and the interesting topic of *obvious strategyproofness* in connection to TTC (Peter, 2019; Mandal and Roy, 2022).

2. Preliminaries

For a positive integer $\ell \in \mathbb{N}$, we use the notation $[\ell] = \{1, 2, \ldots, \ell\}$.

Partial orders. A partial ordering over a set S is a binary relation $\succ$ over S that is irreflexive (i.e., $s \not\succ s$ for all $s \in S$), antisymmetric (i.e., if $s \succ r$, then $r \not\succ s$), and transitive (i.e., $s \succ r$ and $r \succ q$ together imply $s \succ q$). Given a partial ordering $\succ$ over S, two elements s and r of S are *comparable* if $s \succ r$ or $r \succ s$; otherwise, they are *incomparable*, denoted by $s \sim r$. We write $s \succeq r$ for $r \not\succ s$.

Partial orders are a generalization of weak orders. A partial ordering over S is a *weak order* if and only if the incomparability relation $\sim$ is an equivalence relation: it is reflexive, symmetric, and transitive. Note that the first two conditions always hold; hence the transitivity of the incomparability relation is the key property for a partial order to be a weak order. For example, $S = \{s, r, q\}$ with $s \succ q$ but $s \sim r$ and $r \sim q$ is not a weak order, because $s \nsim q$.

Housing markets. We consider a housing market in the sense of Shapley and Scarf, where each agent initially owns one indivisible house and has preferences over houses. An allocation assigns each agent exactly one house. Since each house is initially owned by an agent, we identify houses with their owners and express preferences directly over agents. Preferences may be incomplete, reflecting acceptability constraints. Formally, a *housing market* is a pair $(N, \{\succ_a : a \in N\})$ where N is the set of *agents* and $\succ_a$ is a partial order over N for each $a \in N$, representing the preferences of agent a. Note that we identify each house with its initial owner and thus express agents' preferences over houses as partial orders over N. Recall that we write $b \sim_a c$ if $b \not\succ_a c$ and $c \not\succ_a b$ both hold, and $b \succeq_a c$ means that $c \not\succ_a b$. We say that a *prefers* c to b if $c \succ_a b$, and a *weakly prefers* c to b if $c \succeq_a b$. Agent b is *acceptable* to agent a, if $b \succeq_a a$, and we let $A(a) = \{b \in N : b \succeq_a a\}$ denote the *acceptability set* of a.

An *allocation* specifies for each agent which house they receive, with each agent receiving exactly one acceptable house (possibly its own). An allocation is *blocked* by a group of agents if they can exchange houses so that no agent is worse off and at least one agent in the group is strictly better off. If an allocation has no such blocking group of agents, then the allocation is in the *strong core*. A slightly weaker notation is the *core*. We say that an allocation is in the core of the market if there is no group of agents such that they can exchange houses in a way that all of them strictly prefer the newly obtained house over the one originally received in the allocation.

A *submarket* H' of a housing market H contains a subset of agents from H and their preferences over the agents in H'.

Housing markets represented as graphs. To represent a housing market, we use directed graphs. Below we provide a summary of graph-theoretic notions that will be useful for our study. Formally, the underlying graph of a housing market $H = (N, \{\succ_a : a \in N\})$ is a directed graph $D^H = (N, E)$ where vertices correspond to agents and arcs correspond to acceptability, that is, $E = \{(a, b) : a \in N, b \in A(a)\}$. Note that $(a, a) \in E$ for each $a \in N$, so D^H contains a loop at each agent. We say that an arc $(a, b) \in E$ is *dominated* by an arc (a', b') if $a = a'$ and $b' \succ_a b$. An arc $(a, b) \in E$ is *undominated*, if no arc in E dominates it.

Using the underlying graph representation, we can formalize the notion of an *allocation* in H as a set $X \subseteq E$ of arcs in D^H such that each agent has exactly one outgoing and exactly one incoming arc in X. For an allocation X, or more generally, for a set X of arcs in D^H such that each agent has at most one outgoing arc in X, we let $X(a)$ denote the agent b if $(a, b) \in X$; if there is no arc leaving a in X, then we set $X(a) = \emptyset$. We extend the agents' preferences over N to preferences over allocations in the straightforward way: agent a (weakly) prefers an allocation X to an allocation X' if and only if a (weakly) prefers $X(a)$ to $X'(a)$. Extending the notation, we write $X \succ_a X'$ for a preferring X to X'; we use $X \succeq_a X'$ and $X \sim_a X'$ analogously.

Remark 1. Given an arbitrary subset E' of arcs in D^H, it is possible to check in polynomial time whether H admits an allocation containing only arcs of E', by reducing this problem via a well-known, simple reduction to finding a perfect matching in a bipartite graph. This classic reduction constructs an auxiliary graph $G = (N \cup N', F)$ where $N' = \{a' : a \in N\}$ is a copy N, and the edge set is $F = \{ab' : (a, b) \in E'\}$; then allocations in H contained in E' correspond bijectively to perfect matchings in G.

Let us now formulate the strong core in terms of its graph-based representation. An allocation X is in the strong core of H, if there does not exist a *blocking cycle* for X, i.e., a cycle C in D^H such that all agents a on C weakly prefer $C(a)$ to $X(a)$, and at least one agent $a^\star$ on C (strictly) prefers $C(a^\star)$ to $X(a^\star)$. We remark that such cycles are often called *weakly* blocking cycles. We adopt the notation by Biró et al. (2023) and write $SC(H)$ for the strong core of H. An allocation X is in the *core* of H, if there does not exist a *strictly blocking cycle* for X, i.e., a cycle C in D^H such that all agents a on C strictly prefer $C(a)$ to $X(a)$.

We can also formalize the concept of a submarket as follows: given a subset $N' \subseteq N$ of agents, the *submarket* of H restricted to the agent set N' is the housing market $H_{|N'} = (N', \{\succ'_a : a \in N'\})$ where $\succ'_a$ is the restriction of $\succ_a$ to the set N' of agents.

Graph-theoretic notions. Consider a *directed graph* $D = (V, E)$ where V is the set of *vertices* and $E \subseteq V \times V$ is the set of *arcs*. An arc (u, v) leading from u to v has *tail* u and *head* v. A *loop* is an arc (v, v) for some $v \in V$. Given a set $U \subseteq V$ of vertices, we say that an arc (u, v) *leaves* U if $u \in U$ but $v \notin U$; similarly, (u, v) *enters* U if $u \notin U$ but $v \in U$.

A set $P \subseteq E$ is a *path* in D, if there exist distinct vertices $v_1, \ldots, v_\ell$ in V such that $P = \{(v_i, v_{i+1}) : i \in [\ell - 1]\}$; the vertices $v_1, \ldots, v_\ell$ *appear* on P, and P *starts* at v_1 and *ends* at v_ℓ, or equivalently, *leads* from v_1 to v_ℓ. An (u, v)-path is a path that starts at u and ends at v. A *cycle* in D is the union of an arc (u, v) and a path leading from v to u. A graph $D' = (V', E')$ is a *subgraph* of D if $V' \subseteq V$ and $E' \subseteq E$; sometimes we may treat paths or cycles as subgraphs instead of arc sets. The subgraph of D *induced by* a set $V' \subseteq V$ of vertices is obtained by deleting all vertices of $V \setminus V'$ from D together with all incident arcs. The subgraph of D *spanned by* a set $E' \subseteq E$ of arcs is the graph (V, E'). A *strongly connected component* of D is an inclusion-wise maximal set $U \subseteq V$ of vertices such that for all $u, u' \in U$ there exists a path leading from u to u', and also a path leading from u' to u. By *contracting* $U \subseteq V$, we mean the operation of adding a newly introduced vertex $u^\star$, replacing the head of each arc entering U with $u^\star$, replacing the tail of each arc leaving U with $u^\star$, and then deleting the vertices of U; note that this operation does not create loops. It is known that the strongly connected components of D yield a partition of V, and contracting each strongly connected component results in an *acyclic* digraph, i.e., a directed graph without cycles.

Given some vertex v in a directed graph $D = (V, E)$, we denote by $\delta^{\text{in}}(v) = \{(u, v) : (u, v) \in E\}$ and $\delta^{\text{out}}(v) = \{(v, u) : (v, u) \in E\}$ the set of *incoming* and *outgoing* arcs of v, respectively. Given a set $F \subseteq E$ of arcs, we extend this notation by letting $\delta^{\text{in}}_F(v) = \delta^{\text{in}}(v) \cap F$ and, similarly, $\delta^{\text{out}}_F(v) = \delta^{\text{out}}(v) \cap F$. A notion of central importance for our purposes is the following: we call a set $S \subseteq V$ an *absorbing set* in D, if S is a strongly connected component of D that is not left by any arc. In other words, when contracting S, the newly introduced vertex corresponding to S has no outgoing arcs. Note that every directed graph admits an absorbing set, because contracting all strongly connected components yields an acyclic graph which, therefore, contains a vertex with no outgoing arcs.

3. Characterizing the strong core

If agents' preferences are strict orders, then the strong core always contains a unique allocation (Roth and Postlewaite, 1977), which can be found in linear time using the Top Trading Cycle algorithm, attributed to David Gale in the seminal paper by Shapley and Scarf (1974). If preferences are weak orders, then the strong core can be empty, but Quint and Wako provided a characterization of allocations in the strong core in such a setting, and showed how to use this characterization to find an allocation in the strong core if there exists one (Quint and Wako, 2004).

In Section 3.1 we show that the characterization of the strong core by Quint and Wako fails to hold in the setting when agents' preferences can be partial orders. In Section 3.2 we generalize the results of Quint and Wako to the setting with partial order preferences by giving a characterization of the strong core.

3.1. The Quint–Wako characterization fails for partial orders

For housing markets where agents' preferences are weak orders, Quint and Wako proved the following characterization of allocations in the strong core (Quint and Wako, 2004). Recall that an absorbing set is a strongly connected component from which no arc may leave.

Theorem 2 (*Quint–Wako Characterization (Quint and Wako, 2004)*). *Suppose that $H = (N, \{\succ_a : a \in N\})$ is a housing market where each $\succ_a$ is a weak order. Let U be the set of undominated arcs in D^H, and S an absorbing set in $D^H[U]$. An allocation X for H is in the strong core of H if and only if it can be partitioned into sets X_S and $X_{N \setminus S}$ where*

- *(a) $X_S = X \cap (S \times S)$ is an allocation in $H_{|S}$ contained in U, and*
- *(b) $X_{N \setminus S} = X \cap (N \setminus S) \times (N \setminus S)$ is an allocation in the strong core of $H_{|(N \setminus S)}$.*

We prove that conditions (a) and (b) stated in Theorem 2 are *necessary* for a given allocation to be in the strong core also in the case where agents' preferences are partial orders, as stated in the following proposition.

Proposition 3. *Suppose that $H = (N, \{\succ_a : a \in N\})$ is a housing market where each $\succ_a$ is a partial order. Again, let U denote the set of undominated arcs in D^H, and let S be an absorbing set in $D^H[U]$. If X is an allocation in the strong core of H, then conditions (a) and (b) of Theorem 2 hold for X.*

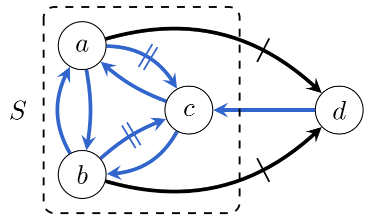


Fig. 1. The underlying graph of housing market H^1 defined in Example 1. Henceforth, loops are omitted, and undominated arcs are shown in blue, the strongly connected components they form are shown as dashed polygons. Single and double line markings | and ‖ convey domination: an arc marked with ‖ dominates an arc marked with | and leaving the same agent; e.g., (a,c) dominates (a,d) but not (a,b), and neither does (a,b) dominate (a,d). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Define $X_S = X \cap (S \times S)$ and $X_{N\setminus S} = X \setminus (S \times S)$, as in conditions (a) and (b).

Suppose for the sake of contradiction that condition (a) fails. This means that there exists some $s \in S$ such that either the arc $(s, X(s))$ leaves S, or it is dominated by some arc of D^H. Recall that S is an absorbing set in $D^H[U]$, and thus no arc of U leaves S. Thus, in either of these cases we know $(s, X(s)) \notin U$, and hence there exists an arc $e = (s,a) \in \delta^{\text{out}}_U(s)$ that dominates $(s, X(s))$, that is, $a \succ_s X(s)$. Since $e \in U$, we know that $a \in S$. Since S is a strongly connected component in $D^H[U]$, there exists a cycle C within S through e. Since s prefers C to X, and by $C \subseteq U$ no agent on C prefers X to C, we get that C is a blocking cycle for X, a contradiction. Hence, X_S is indeed an allocation in $H_{|S}$ contained in U.

To see that condition (b) holds as well, first note that by condition (a), $X \setminus X_S = X_{N\setminus S}$, and so X_S and $X_{N\setminus S}$ indeed partition X into two. Moreover, $X_{N\setminus S}$ is an allocation in the submarket $H_{|(N\setminus S)}$. Observe now that a blocking cycle for X in this submarket would be a blocking cycle for X in H as well, proving that $X_{N\setminus S}$ is indeed in the strong core of $H_{|(N\setminus S)}$. □

Example 1 shows that conditions (a) and (b) of Theorem 2 are *not sufficient* to ensure that an allocation belongs to the strong core. Thus, the characterization established by Theorem 2 for weakly ordered preferences fails to hold in the case where agents' preferences are partial orders.

Example 1. Consider the following market H^1 over $N = \{a,b,c,d\}$ as the set of agents; see its underlying graph in Fig. 1. Let the acceptability sets be defined as $A(a) = A(b) = N$, $A(c) = \{a,b,c\}$, and $A(d) = \{c,d\}$. For each agent x and $y \in A(x) \setminus \{x\}$, we let $y \succ_x x$; additionally, $c \succ_a d$ and $c \succ_b d$.

Note that $U = \{(a,b),(a,c),(b,c),(b,a),(c,a),(c,b),(d,c)\}$ is the set of undominated arcs, and the unique absorbing set in the underlying graph $D^{H^1}[U]$ is $S = \{a,b,c\}$. The submarket $H^1_{|S}$ admits two allocations in U: $X_1 = \{(a,b),(b,c),(c,a)\}$ and $X_2 = \{(a,c),(c,b),(b,a)\}$. The only allocation for the submarket $H^1_{|(N\setminus S)} = H_{|\{d\}}$ is $X_d = \{(d,d)\}$.

Although both X_1 and X_2 are in the strong core of $H^1_{|S}$, and X_d is in the strong core of $H_{|N\setminus S}$, neither $X_1 \cup X_d$ nor $X_2 \cup X_d$ is in the strong core of H^1: $\{(a,d),(d,c),(c,a)\}$ is a blocking cycle for $X_1 \cup X_d$, and similarly, $\{(b,d),(d,c),(c,b)\}$ is a blocking cycle for $X_2 \cup X_d$. This shows that Theorem 2 does not extend to partial order preferences.

3.2. A characterization through peak sets

Given an allocation X in H, let us define the following arc sets:

$$E_{[\sim X]} = \{(a,b) \in E : X(a) \sim_a b\},$$
$$E_{[\succ X]} = \{(a,b) \in E : b \succ_a X(a)\}, \qquad \text{and}$$
$$E_{[\succeq X]} = E_{[\sim X]} \cup E_{[\succ X]}.$$

Let $D_{[\sim X]} = (N, E_{[\sim X]})$, and let the digraphs $D_{[\succ X]}$ and $D_{[\succeq X]}$ be defined analogously. The following definition captures the central notion we need for our characterization of the strong core.

Definition 4. Given an allocation X in a housing market H, a *peak set* of X is an absorbing set in $D_{[\succeq X]}$.

In our characterization of the strong core, peak sets will play the role that absorbing sets in the subgraph of undominated arcs have in Theorem 2. The following lemma states their key property.

Lemma 5. *Suppose that X is an allocation in the strong core of H. If S a peak set of X, then $X \cap (S \times S)$ is an allocation in $H_{|S}$ consisting only of undominated arcs. Moreover, each arc leaving S is dominated by an arc of X.*

Proof. Let U denote the set of undominated arcs in D.

We first show that if S is an absorbing set in $D_{[\succeq X]}$ and $s \in S$, then the arc $(s, X(s))$ is contained in U. Assume for the sake of contradiction that $(s, X(s))$ is dominated by some arc of D^H. Then $(s, X(s))$ is also dominated by some arc $(s,s') \in U$. Now, since $(s,s') \in U$, we also know that (s,s') is in $E_{[\succeq X]}$ and thus cannot leave S, by the definition of an absorbing set, so we get $s' \in S$. Since S is a strongly connected component of $D_{[\succeq X]}$, there must exists a path Q from s' to s within $D_{[\succeq X]}[S]$. Now, since (s,s') dominates $(s, X(s))$, the cycle obtained by adding (s,s') to Q is a blocking cycle for X, a contradiction. Hence, $(s, X(s)) \in U$ for each $s \in S$.

Since $U \subseteq E_{[\succeq X]}$, and no arc of $E_{[\succeq X]}$ leaves S by the definition of an absorbing set, we obtain that $X(s) \in S$. Hence, $X \cap (S \times S)$ is indeed an allocation in $H_{|S}$ contained in U.

The last claim of the lemma follows from the fact that no arc of $E_{[\succeq X]}$ may leave S due to the definition of an absorbing set. □

Based on Lemma 5, we can now characterize the strong core. An economic interpretation of Theorem 6 is the following. Even if agents can have partial order preferences, the allocations in the strong core are not only stable but reachable via decentralized dynamics. In particular, they can be reached by agents repeatedly forming a group trading among themselves, each of them receiving an undominated, best possible choice, and then leaving the market with the same process continuing in the residual market. These dynamics are very similar to those associated with the Quint–Wako characterization of the strong core for the model with weak order preferences, but there is a crucial difference. In our characterization, every agent exiting the market at a given stage must receive a house that is not only among their most-preferred acceptable options (i.e., corresponds to an undominated arc), but is also *strictly preferred* to any house owned by agents outside the exiting group (i.e., corresponds to an arc that dominates all arcs pointing to agents in the remaining market). At the end of this process, provided that such a group and a suitable allocation are identified at each stage, the resulting allocation is guaranteed to belong to the strong core. No blocking cycle can arise, as trades within each exiting group are organized along cycles that eliminate incentives for deviations involving agents remaining in the market.

As a further observation, we also show that the exiting group of agents can be chosen as a peak set for the given allocation.

Theorem 6. *Suppose that $H = (N, \{\succ_a : a \in N\})$ is a housing market where each $\succ_a$ is a partial order, and let U be the set of undominated arcs in D^H. Then allocation X for H is in the strong core of H if and only if it can be partitioned into two sets X_1 and X_2 such that there exists a set $S \subseteq N$ of agents for which*

(a) X_1 is an allocation in $H_{|S}$ contained in U,
(b) X_2 is an allocation in the strong core of $H_{|(N\setminus S)}$, and

(c) each arc of D^H leaving S is dominated by an arc of X_1.

Moreover, S can be chosen as a peak set for X.

Proof. First suppose that X is in the strong core of H. Define S as a peak set of X, and set $X_1 := X \cap (S \times S)$ and $X_2 = X \setminus X_1$. By Lemma 5 we know that conditions (a) and (c) are fulfilled. Since X_1 is an allocation on $H_{|S}$, we necessarily have that $X_2 = X \setminus X_1$ is an allocation in $H_{|(N\setminus S)}$. To see that X_2 is in the strong core of this submarket, it suffices to note that any blocking cycle for X_2 in $H_{|(N\setminus S)}$ would also block X in H.

Second, suppose that some allocation X can be partitioned into X_1 and X_2 satisfying conditions (a)–(c) for some set S of agents; we show that X is in the strong core of H. Assume for the sake of contradiction that there is a blocking cycle C for X. First, since X_1 is an allocation for $H_{|S}$ and $X_1 \subseteq U$, we know that C must involve at least one agent not in S. Second, since X_2 is an allocation in the strong core of $H_{|(N\setminus S)}$, we know that C must also contain at least one agent not in $N \setminus S$. Thus, C involves agents from both S and $N \setminus S$, and consequently has to contain at least one arc leaving S. However, all arcs leaving S are dominated by an arc of X_1, which contradicts the assumption that C blocks X. This proves that no blocking cycle for X exists, so X is indeed in the strong core of H. □

4. Finding an allocation in the strong core

In this section, we show how Theorem 6 can be used to construct an algorithm that either finds an allocation in the strong core of a housing market where agents' preferences are partial orders, or concludes that such an allocation does not exist.

Before explaining our ideas, let us briefly describe the algorithm by Quint and Wako (2004) that, based on Theorem 2, solves this problem in the setting where agents' preferences are weak orders. Given a housing market $H = (N, \{\succ_a : a \in N\})$, the Quint–Wako algorithm first computes an absorbing set S in the subgraph $D^H[U]$ of undominated arcs in the underlying digraph. Then it checks whether $H_{|S}$ admits an allocation consisting only of undominated arcs, and if so, stores such an allocation X_S and proceeds with the submarket $H_{|(N\setminus S)}$ of the remaining agents recursively. If the recursive call returns an allocation $X_{N\setminus S}$ in the strong core of $H_{|(N\setminus S)}$, then the algorithm outputs $X_S \cup X_{N\setminus S}$; otherwise it concludes that the strong core of H is empty.

Unfortunately, the characterization of the strong core given in Theorem 6 does not readily offer an algorithm for finding an allocation in the strong core of a given housing market, or for deciding whether such an allocation exists. The reason for this is that it is not immediately clear how to find a set S of agents for which conditions (a)–(c) of Theorem 6 hold for the desired allocation X (or, more precisely, for the arcs sets X_1 and X_2 that together yield X). By Theorem 6 we know that the set S can be chosen to be a peak set of the desired allocation; however, this insight does not seem to offer a direct way to find such a set without knowing X itself.

In this section, we propose a method that resolves this issue and finds an allocation in the strong core of the given housing market, if such an allocation exists. In fact, we are going to solve the following more general computational problem:

> STRONG CORE WITH FORBIDDEN ARCS (SCFA):
> Input: A housing market H with partial orders and a set F of *forbidden* arcs in the underlying directed graph D^H.
> Task: Decide if there exists an allocation X in the strong core of H such that $X \cap F = \emptyset$, and if so, find such an allocation.

In Section 4.1 we describe a polynomial-time algorithm to solve the STRONG CORE WITH FORBIDDEN ARCS problem, and we sketch its proof of correctness in Section 4.2. We extend our algorithm to a setting with forced arcs in Section 4.3, and explain its (weak) group-strategyproofness in Section 4.4. A discussion regarding the algorithm's additional efficiency properties, showing that it guarantees pair-efficiency but not Pareto-efficiency, can be found in Appendix A.1. In Appendix A.2 we briefly compare it with the Quint–Wako algorithm (Quint and Wako, 2004) in terms of their ability to enumerate all allocations in the strong core.

4.1. Algorithm for SCFA

We now present Algorithm SCFA to solve our instance (H, F) of SCFA. An allocation in $SC(H)$ that is disjoint from F is a *solution*. A pseudocode description is shown below, followed by a detailed presentation of the algorithm.

Algorithm SCFA Solving SCFA.

Input: An instance (H, F) of SCFA where $D^H = (N, E)$.
Output: An allocation in the strong core of H disjoint from F, or $\emptyset$ if no such allocation exists.

if $|N| = 1$ **then** ▷ Dealing with a single-agent market.
  if $(a,a) \in E \setminus F$ where $N = \{a\}$ **then return** $\{(a,a)\}$.
  else return $\emptyset$.
Let U denote the set of undominated arcs in D^H.
Let $\mathcal{S}$ be the set of strongly connected components in (N, U).
Let $\mathcal{T} = \emptyset$.
for all $S \in \mathcal{S}$ **do**
  Let $E_S = \{(a,b) \in U \setminus F : a, b \in S\}$. ▷ All undominated, non-forbidden arcs in S.
  if $\exists$ an allocation for $H_{|S}$ in E_S **then** Put S into $\mathcal{T}$.
while $|\mathcal{T}| > 0$ **do**
  Let $\mathcal{R} = \emptyset$ and $T^\star = \bigcup_{S \in \mathcal{T}} S$.
  for all $S \in \mathcal{T}$ **do**
    Let $E_{S,\mathcal{T}} = \{(a,b) \in E_S : b \succ_a b' \ \forall (a,b') \in E, b' \notin T^\star\}$. ▷ All $T^\star$-valid arcs in S.
    if $\exists$ an allocation for $H_{|S}$ in $E_{S,\mathcal{T}}$ **then**
      Let X_S be such an allocation. ▷ Storing a $T^\star$-valid allocation for S.
    else Add S as an element to $\mathcal{R}$. ▷ $\mathcal{R}$ stores components with no $T^\star$-valid allocation.
  if $\mathcal{R} \neq \emptyset$ **then** Delete each $S \in \mathcal{R}$ from $\mathcal{T}$.
  else ▷ All components in $\mathcal{T}$ admit a $T^\star$-valid allocation.
    if $N = T^\star$ **then return** $\bigcup_{S \in \mathcal{T}} X_S$. ▷ Solution for the whole market.
    Let $N' = N \setminus T^\star$ and $F' = F \cap (N' \times N')$.
    Call $X' = \textbf{SCFA}(H_{|N'}, F')$. ▷ Recursion on the submarket on $N \setminus T^\star$.
    if $X' = \emptyset$ **then return** $\emptyset$ ▷ No solution for the submarket.
    else return $\bigcup_{S \in \mathcal{T}} X_S \cup X'$. ▷ Solution based on stored allocation and recursion.
return $\emptyset$.

Description of Algorithm SCFA. The main idea of our algorithm is to find a set $T^\star$ that can serve as the peak set of our desired allocation in the strong core, find an appropriate allocation on $H_{|T^\star}$, and solve the remainder recursively after deleting all agents in $T^\star$.

The algorithm starts by dealing with the case where the market only contains a single agent. Next, it computes the family $\mathcal{S}$ of strongly connected components in the digraph $D^H[U] = (N, U)$ of undominated arcs and, using the method of Remark 1, checks for each component $S \in \mathcal{S}$ whether there is an allocation for the submarket $H_{|S}$ consisting only of undominated non-forbidden arcs. Those sets that pass this test are collected in a family $\mathcal{T} \subseteq \mathcal{S}$.

Given this collection $\mathcal{T} \subseteq \mathcal{S}$, an iterative process is started. At each step in this iteration, Algorithm SCFA checks whether each submarket $H_{|S}$ for $S \in \mathcal{T}$ admits an allocation that contains only arcs that (i) are undominated and not forbidden, and (ii) dominate all arcs leaving $T^\star = \bigcup_{T \in \mathcal{T}} T$; the set of such arcs in denoted by $E_{S,\mathcal{T}}$ on

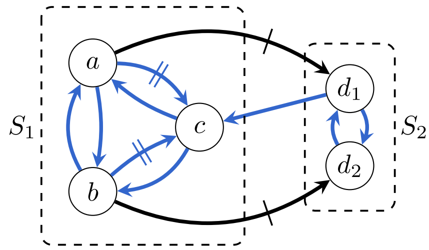


Fig. 2. The housing market H^2 defined in Example 3.

line 13. More formally, let us say that an allocation X in $H_{|S}$ is *$T^\star$-valid* for some $T^\star \subseteq N$, if

(i) X consists only of undominated and non-forbidden arcs, and
(ii) each arc leaving $T^\star$ is dominated by some arc in X.

Thus, an allocation X in $H_{|S}$ is $T^\star$-valid if and only if it only contains arcs (a,b) that are undominated, non-forbidden, and dominate every arc pointing from a to some agent not in $T^\star$. If Algorithm SCFA finds that all submarkets $H_{|S}$, $S \in \mathcal{T}$, admit a $T^\star = \bigcup_{T\in\mathcal{T}} T$-valid allocation, then it stores such an allocation for each $S \in \mathcal{T}$, deletes the agents in $T^\star$, and proceeds with the remaining market recursively—unless the remaining market is empty, in which case no recursion is necessary. If, on the contrary, $H_{|S}$ does not admit a $T^\star$-valid allocation for certain sets $S \in \mathcal{T}$, then each such set S is removed from the collection $\mathcal{T}$.

This iterative process stops either if all sets in $\mathcal{T}$ pass the test of admitting a $\bigcup_{T\in\mathcal{T}} T$-valid allocation, or if $\mathcal{T}$ becomes empty, in which case the algorithm concludes that there is no solution.

We now illustrate Algorithm SCFA on two examples, and then compare it with the TTC algorithm for the case of strict order preferences and with the Quint–Wako algorithm for the case of weak order preferences.

Example 2. Consider again the housing market H^1 defined in Example 1 and depicted in Fig. 1, to see how Algorithm SCFA runs with H^1 as its input. For simplicity, we define the set of forbidden arcs as empty. The set of strongly connected components in the subgraph of D^{H^1} spanned by all undominated arcs is $S = \{S_1, S_2\}$ where $S_1 = \{a,b,c\}$ and $S_2 = \{d\}$. Algorithm SCFA computes the arc sets $E_{S_1} = \{(a,b),(b,c),(c,a),(a,c),(c,b),(b,a)\}$ and $E_{S_2} = \emptyset$, and finds that $H^1_{|S_1}$ admits an allocation in E_{S_1}, for example the allocation $\{(a,b),(b,c),(c,a)\}$, but no allocation for $H^1_{|S_2}$ (containing only the vertex d) can be constructed from the empty set. Thus, Algorithm SCFA initializes the family $\mathcal{T}$ by setting $\mathcal{T} = \{S_1\}$.

Starting the iteration on lines 10–17, the algorithm first computes the arc set $E_{S_1,\mathcal{T}}$, obtaining the set $E_{S_1,\mathcal{T}} = \{(a,c),(c,a),(b,c),(c,b)\}$; note that neither (a,b) nor (b,a) is contained in $E_{S_1,\mathcal{T}}$, since they do not dominate the arc (a,d) or (b,d), respectively (observe that both (a,d) and (b,d) leave the set of vertices contained in $\mathcal{T}$, that is, the set $T^\star = S_1 = \{a,b,c\}$). The algorithm next observes that $E_{S_1,\mathcal{T}}$ does not contain an allocation for $H_{|S_1}$; thus, it removes S_1 from $\mathcal{T}$, leaving $\mathcal{T}$ empty. Hence, the iteration stops, and Algorithm SCFA outputs $\emptyset$ on line 24, concluding that the strong core is empty.

Example 3. Consider the following market H^2 over $N = \{a,b,c,d_1,d_2\}$ as the agent set; see its underlying graph (without loops) on Fig. 2. Let the acceptability sets be defined as $A(a) = N \setminus \{d_2\}$, $A(b) = N \setminus \{d_1\}$, $A(c) = \{a,b,c\}$, $A(d_1) = \{c,d_1,d_2\}$, and $A(d_2) = \{d_1,d_2\}$. For each agent $x \in N$ and $y \in A(x) \setminus \{x\}$, we let $y \succ_x x$; additionally, $c \succ_a d_1$ and $c \succ_b d_2$.

Consider how Algorithm SCFA runs with H^2 as its input, without forbidden arcs. All arcs are undominated except for (a,d_1) and (b,d_2). The strongly connected components in the subgraph spanned by all undominated arcs are $S_1 = \{a,b,c\}$ and $S_2 = \{d_1,d_2\}$. Algorithm SCFA computes the arc sets $E_{S_1} = \{(a,b),(b,c),(c,a),(a,c),(c,b),(b,a)\}$ and $E_{S_2} = \{(d_1,d_2),(d_2,d_1)\}$, and finds that both $H^2_{|S_1}$ and $H^2_{|S_2}$ admit an allocation in E_{S_1} and in E_{S_2}, respectively. Thus, Algorithm SCFA initializes $\mathcal{T}$ by setting $\mathcal{T} = \{S_1, S_2\}$.

Starting the iteration on lines 10–17, the algorithm first computes the arc sets $E_{S,\mathcal{T}}$ for both $S \in \mathcal{T}$, obtaining $E_{S_1,\mathcal{T}} = E_{S_1}$ and $E_{S_2,\mathcal{T}} = E_{S_2}$ and storing allocations X_{S_1} and X_{S_2} for $H_{|S_1}$ and for $H_{|S_2}$, respectively. Suppose that the algorithm finds $X_{S_1} = \{(a,b),(b,c),(c,a)\}$ and $X_{S_2} = \{(d_1,d_2),(d_2,d_1)\}$; the case where $X_{S_1} = \{(a,c),(c,b),(b,a)\}$ is similar. The algorithm finds on line 19 that $N = T^\star = S_1 \cup S_2$, and hence returns the allocation $X = X_{S_1} \cup X_{S_2}$.

Notice that $T^\star$ is a peak set for the allocation X, and the conditions of Theorem 6 are indeed fulfilled with the set S chosen to be $T^\star = N$ and the corresponding partitioning of X into $X_1 = X$ and $X_2 = \emptyset$.

If agents' preferences are strict and the set of forbidden arcs is empty, then Algorithm SCFA coincides with the variant of the TTC algorithm where in each round, instead of removing only one (arbitrarily chosen) top trading cycle, all top trading cycles are removed. To see this, observe that if preferences are strict, then each undominated arc dominates every other arc leaving the same agent. Therefore, every strongly connected component in the subgraph $D^H[U]$ of undominated arcs that contains at least one arc is a cycle. The set $\mathcal{T}$ computed by the algorithm (on lines 7–9) will contain all such cycles, and accordingly, $T^\star$ contains all agents appearing on any such cycle. Moreover, each cycle C in $\mathcal{T}$ is $T^\star$-valid, and thus none of them is removed from the set $\mathcal{T}$. Consequently, all cycles in $\mathcal{T}$, corresponding to the top trading cycles in TTC, are added to the allocation, the agents appearing on them are removed from the market, and the algorithm proceeds with the remaining submarket in a recursive manner.

If agents' preferences are weak orders and the set of forbidden arcs is empty, then Algorithm SCFA can be seen as a variant of the Quint–Wako algorithm for finding an allocation in the strong core (Quint and Wako, 2004). The algorithm by Quint and Wako is based on their characterization of the strong core as stated in Theorem 2. In fact, their notion of *minimally self-mapped set* in a digraph captures precisely the notion of an absorbing set (Quint and Wako, 2004). Their algorithm repeatedly finds an absorbing set S in the subgraph of undominated arcs, checks whether it is *supported* in the sense that it admits an allocation X_S consisting only of undominated arcs, and if so, deletes all agents in S from the market and proceeds with the remainder in a recursive manner (otherwise, it concludes that the strong core is empty). By contrast, given an instance with weak preferences, Algorithm SCFA finds a maximal family $\mathcal{T}$ of strongly connected components in the subgraph $D^H[U]$ of undominated arcs such that

- each component in $\mathcal{T}$ is supported, and
- there is no undominated arc leaving $\mathcal{T}$. This condition is equivalent to the $T^\star$-validity of the allocations X_T, $T \in \mathcal{T}$, where $T^\star = (\bigcup_{T\in\mathcal{T}} T)$, due to agents' preferences being weak orders: indeed, $T^\star$-validity requires that each arc f leaving $T^\star$ is dominated by some arc in X_T, $T \in \mathcal{T}$, which happens if and only if f itself is *not* undominated.

Thus, instead of repeatedly finding an absorbing set in $D^H[U]$ and removing it from the market, our algorithm removes a maximal family $\mathcal{T}$ of strongly connected components in $D^H[U]$ that are supported and such that no undominated arc leaves $\mathcal{T}$. This can be achieved, alternatively, by running the Quint–Wako algorithm for several rounds, removing an absorbing set in each round. Hence, Algorithm SCFA on instances with weak order preferences can be interpreted as a variant of the Quint–Wako algorithm that performs several rounds in one iteration.

4.2. Correctness of Algorithm SCFA

The correctness of Algorithm SCFA relies on the following key fact.

Lemma 7. *If X is an allocation in the strong core of H disjoint from F, and P is a peak set of X, then $P \subseteq \bigcup_{T\in\mathcal{T}} T$ always holds on lines 10–17 during the execution of Algorithm SCFA on (H,F).*

Proof. First, recall that if P is a peak set for X, then no arc of $E_{[\geq X]}$ leaves P by definition. This implies that no undominated arc leaves P, and therefore, each strongly connected component of $D^H[U]$ is either entirely contained in P, or is disjoint from it. Hence, if $\mathcal{S}$ denotes the family of strongly connected components in $D^H[U]$, then $P = \bigcup_{S\in\mathcal{S}_P} S$ for some $\mathcal{S}_P \subseteq \mathcal{S}$.

By Lemma 5, the arc $(p, X(p))$ is undominated for each $p \in P$. Since no undominated arc leaves P, this implies that X contains a collection of pairwise vertex-disjoint cycles in U covering all agents in P. By the definition of strongly connected components, each such cycle must be entirely contained in some strongly connected component of $D^H[U]$. Summarizing this, we obtain that for each $S \in \mathcal{S}_P$ the arc set $X_S = \{(s, X(s)) : s \in S\}$ is an allocation in $H_{|S}$ that is contained in U and disjoint from F. Consequently, each set $S \in \mathcal{S}_P$ is added to $\mathcal{T}$ on line 9 of Algorithm SCFA. Hence, at the beginning of the iteration on lines 10–17, $P \subseteq \bigcup_{T\in\mathcal{T}} T$ holds.

We prove that this remains true throughout the run of the algorithm. For this, it suffices to see that whenever some set $S \in \mathcal{S}$ is removed from T, then $S \notin \mathcal{S}_P$. So assume that $S \in \mathcal{S}_P$ is removed from $\mathcal{T}$ on line 17, and before the removal, $P \subseteq T^\star$ for $T^\star = \bigcup_{T\in\mathcal{T}} T$. Then $H_{|S}$ does not admit an allocation in $E_{S,\mathcal{T}}$, that is, $H_{|S}$ does not admit a $T^\star$-valid allocation.

Recall now that allocation X_S for $H_{|S}$ is P-valid due to Lemma 5. However, since $P \subseteq T^\star$, all arcs leaving $T^\star$ necessarily leave P as well, and hence must be dominated by an arc of X_S. It follows that X_S is $T^\star$-valid. This contradicts our assumption that $H_{|S}$ does not admit a $T^\star$-valid allocation, proving that no set in $\mathcal{S}_P$ is ever removed by the algorithm from $\mathcal{T}$. □

The correctness of Algorithm SCFA is stated below.

Theorem 8. *Algorithm SCFA correctly solves each instance of SCFA, and runs in $O(|N|^2 \cdot |E|^{1+o(1)})$ time where (N,E) is the underlying graph of the housing market H of the instance.*

Proof. Let $I = (H,F)$ denote our input instance. We use induction on $|N|$, the number of agents in I, to prove the theorem. Notice that if $|N| = 1$, then the algorithm is clearly correct, since any solution must consist of a single loop that is adjacent to the unique agent in N and is not a forbidden arc—exactly the property checked on line 2. Henceforth, let us assume that the algorithm correctly solves all instances with less than $|N|$ agents.

Claim 9. *If I admits a solution, then Algorithm SCFA does not reject I by outputting $\emptyset$.*

Claim proof. Let Y be a solution for I, and let P be a peak set for Y. By Lemma 7, $P \subseteq \bigcup_{T\in\mathcal{T}} T$ holds throughout lines 10–17 of Algorithm SCFA when executed on instance I; therefore, the algorithm never reaches line 24, and hence cannot output $\emptyset$ on line 24.

Assume now for the sake of contradiction, that the algorithm outputs $\emptyset$ on line 22, which happens only if the recursive call on line 21 returns $\emptyset$. Observe that since $|\mathcal{T}| > 0$ for the collection $\mathcal{T}$ at this point, we know $|N'| < |N|$, and thus our inductive hypothesis implies that the instance $I' = (H_{|N'}, F')$ does not admit a solution.

Consider now the collection $\mathcal{T}$ and the agent set $T^\star = \bigcup_{T\in\mathcal{T}} T$. Observe that since each $S \in \mathcal{T}$ admits a $T^\star$-valid allocation X_S, no undominated arc leaves $T^\star$.

We claim that no arc of Y leaves $T^\star$ either. To see this, assume that some arc $(a,b) \in Y$ leaves $T^\star$; then (a,b) is dominated by the arc $f = (a, X_S(a)) \in X_S$. Since X_S contains only arcs within $H_{|S}$ and S is a strongly connected component of $D^H[U]$, we know that there is a path $Q \subseteq U$ from the head of f to the tail of f. The cycle C formed by Q and the arc f is therefore contained in $E_{[\geq Y]}$, with agent a strictly preferring C to Y, i.e., C is a blocking cycle for Y, a contradiction. This proves that no arc of Y leaves $T^\star$. However, then the restriction of Y to the set $N' = N \setminus T^\star$ of remaining agents, i.e., the arc set $Y' = Y \cap (N' \times N')$ is an allocation in $H_{|N'}$. It is also clear that Y' is in the strong core of $H_{|N'}$, as a blocking cycle in $H_{|N'}$ for Y' would also block Y in H. Since $Y' \subseteq Y$ is disjoint from F, we obtain that Y' is a solution for the instance I', a contradiction proving the claim. ◁

Next, let us show the counterpart of Claim 9.

Claim 10. *If Algorithm SCFA outputs a set $X \subseteq E$, then X is solution for I.*

Claim proof. Suppose that Algorithm SCFA outputs a set X on line 19. In this case, by $N = \bigcup_{S\in\mathcal{T}} S$ we know that X is an allocation for H. Moreover, we also have $X \subseteq \bigcup_{S\in\mathcal{T}} E_{S,\mathcal{T}} \subseteq U \setminus F$. Hence, it is clear that no blocking cycle for X exists, and therefore, X is a solution for the input instance.

Suppose now that Algorithm SCFA outputs a set X on line 23, and let us consider the values of $\mathcal{T}$, N' and F' at this point. Due to our induction hypothesis, we know that X' is a solution for the instance $I' = (H_{|N'}, F')$. By line 13, we also know that the allocation X_S for $H_{|S}$ determined on line 15 is $T^\star$-valid for each $S \in \mathcal{T}$ where $T^\star = \bigcup_{T\in\mathcal{T}} T$.

First, it is clear that the set $X = X' \cup \bigcup_{S\in\mathcal{T}} X_S$ is an allocation in H, as X' is an allocation in $H_{N'}$, and $\bigcup_{S\in\mathcal{T}} X_S$ is an allocation in $H_{|T^\star}$. It is also obvious that X does not contain any forbidden arcs, due to our induction hypothesis and the definition of $T^\star$-validity. It remains to show that X is in the strong core of H.

To show $X \in \mathcal{SC}(H)$, we apply Theorem 6 with $X_1 := \bigcup_{S\in\mathcal{T}} X_S$ and $X_2 := X'$. First, since X_1 is an allocation in $H_{|T^\star}$ that only contains undominated arcs, it is immediate that $X_1 \in \mathcal{SC}(H_{|T^\star})$. Second, recall that $X_2 \in \mathcal{SC}(H_{N'}) = \mathcal{SC}(H_{|N\setminus T^\star})$ by induction. Third, since the allocation X_S is $T^\star$-valid for each $S \in \mathcal{T}$, we get that each arc leaving $T^\star$ is dominated by an arc of X_1. Hence, conditions (a)–(c) of Theorem 6 are satisfied, proving that X is indeed in the strong core of H, and hence, it is a solution for I. ◁

Due to Claims 9 and 10, to show the correctness of Algorithm SCFA it suffices to prove that the algorithm always produces an output in the claimed running time.

To see this, observe that $|\mathcal{T}| \leq |N|$, and thus lines 10–17 are performed at most $|N|$ times, since Algorithm SCFA either produces an output or decreases $|\mathcal{T}|$ during each iteration. The bottleneck in each iteration—ignoring the time spent in the recursive call on line 21—is to compute a perfect matching in a bipartite graph on $2|N|$ vertices and at most $|E|$ edges, which takes $O(|E|^{1+o(1)})$ time (Chen et al., 2023) using the recent algorithm for maximum flow by Chen et al. (2023). Therefore, the algorithm's running time without the recursive call on line 21 is $O(|N| \cdot |E|^{1+o(1)})$. The recursion introduces an additional factor of $|N|$, because the number of agents decreases in each recursive call. This implies a total running time of $O(|N|^2 \cdot |E|^{1+o(1)})$. □

Corollary 11. *SCFA with n agents and m arcs in the underlying graph can be solved in $O(n^2 \cdot m^{1+o(1)})$ time.*

4.3. Extension to forbidden and forced arcs

We remark that Algorithm SCFA can also be used to solve the extension of SCFA with forced arcs.

In this setting, the aim is to find an allocation in the strong core where, besides the set of forbidden arcs specified for our instance of SCFA, we are additionally given a set of *forced* arcs that must be included in the allocation we are looking for. Such constraints arise naturally in many applications where it may be necessary to respect already existing partial allocations, or to ensure that certain vulnerable agents are guaranteed a high-quality outcome; recall the motivation provided in Section 1.2.

> Strong Core with Forbidden and Forced Arcs (SCFFA):
> Input: A housing market H with partial orders, a set F^+ of *forced* arcs and a set F^- of *forbidden* arcs in the underlying directed graph D^H.
> Task: Decide if there exists an allocation X in the strong core of H such that $F^+ \subseteq X$ and $X \cap F^- = \emptyset$, and if so, find such an allocation.

To solve SCFFA in a housing market H, it is sufficient to solve the variant that only involves forbidden arcs. Indeed, since an allocation contains *exactly one* outgoing arc for each agent, we know that an allocation in H contains an arc (a,b) of the underlying graph D^H if and only if it does not contain any of the arcs $\delta^{\text{out}}(a) \setminus \{(a,b)\}$ in D^H. Therefore, we can reduce an instance of (H, F^+, F^-) of SCFFA to an equivalent instance (H,F) of SCFA where $F = F^- \cup \bigcup_{(a,b)\in F^+} \left(\delta_A^{\text{out}}(a) \setminus \{(a,b)\}\right)$. Thus, we obtain the following consequence of Corollary 11.

Corollary 12. *SCFFA with n agents and m arcs in the underlying graph can be solved in $O(n^2 \cdot m^{1+o(1)})$ time.*

4.4. Strategyproofness

A desirable property of the mechanism associated with Algorithm SCFA besides its efficiency is that it is *weakly group-strategyproof*, i.e., it is not possible for a coalition of agents to strictly improve their situation via misrepresenting their preferences. We are also interested in the notion of *strong group-strategyproofness*, where the coalition of agents misrepresent their preferences in order to help at least one agent to strictly improve its situation.

To formalize this definition, let $f_{\mathsf{SCFA}}(H,F)$ denote the set of allocations that can be obtained via Algorithm SCFA on input (H,F); note that there can be several such allocations due to line 15, where the algorithm may pick an arbitrary one from a set of possible allocations for a given submarket $H_{|S}$. Given a housing market $H = (N, \{\succ_a : a \in N\})$ and a coalition $C \subseteq N$ of agents, a *C-deviation* from H is a housing market $H' = (N, \{\succ'_a : a \in N\})$ where $\succ_a = \succ'_a$ for all $a \in N \setminus C$.

Definition 13. Let f be a mechanism that to each instance (H,F) of the Strong Core with Forbidden Arcs problem assigns a set $f(H,F)$ of solutions for (H,F) where $f(H,F) \subseteq SC(H)$. We say that f is *weakly group-strategyproof* (with respect to SCFA) if there does not exist an instance (H,F) with $H = (N, \{\succ_a : a \in N\})$, a coalition $C \subseteq N$ and a C-deviation H' from H such that there are allocations $X \in f(H,F)$ and $X' \in f(H',F)$ for which $X'(c) \succ_c X(c)$ for each $c \in C$. We say that f is *strongly group-strategyproof* if there does not exist an instance (H,F) with $H = (N, \{\succ_a : a \in N\})$, a coalition $C \subseteq N$ and a C-deviation H' from H such that there are allocations $X \in f(H,F)$ and $X' \in f(H',F)$ for which $X'(c) \succeq_c X(c)$ for each $c \in C$ and, moreover, $X'(c) \succ_c X(c)$ for at least one agent $c \in C$.

We remark that Definition 13 takes into account deviations where agents may only misreport their preferences but have no possibility to change the set of forbidden arcs.

We show that the mechanism associated with Algorithm SCFA—which, given the preferences submitted by the agents in the market, applies Algorithm SCFA with some (arbitrary) tie-breaking method for selecting the returned allocation—is weakly group-strategyproof. The proof relies on the observation that, roughly speaking, whenever an agent p is removed from the market during Algorithm SCFA, the allocation fixed for p is the best p can hope for, as long as no agent removed earlier than (or together with) p is contained in the deviating coalition C. Using this claim for the first agent c of C removed from the market, one can show that there is at least one agent removed before c for which the allocation obtained as a result of the deviation is disadvantageous; this gives rise to a blocking cycle, leading to a contradiction that proves the result.

Theorem 14. *The mechanism associated with Algorithm SCFA is weakly group-strategyproof.*

Proof. Suppose for the sake of contradiction that there exist an instance (H,F) of SCFA with $H = (N, \{\succ_a : a \in N\})$, a coalition $C \subseteq N$ of agents, a C-deviation $H' = (N, \{\succ'_a : a \in N\})$ from H, together with allocations $X \in f_{\mathsf{SCFA}}(H,F)$ and $X' \in f_{\mathsf{SCFA}}(H',F)$ for which $X'(c) \succ_c X(c)$ for each agent $c \in C$ in the coalition.

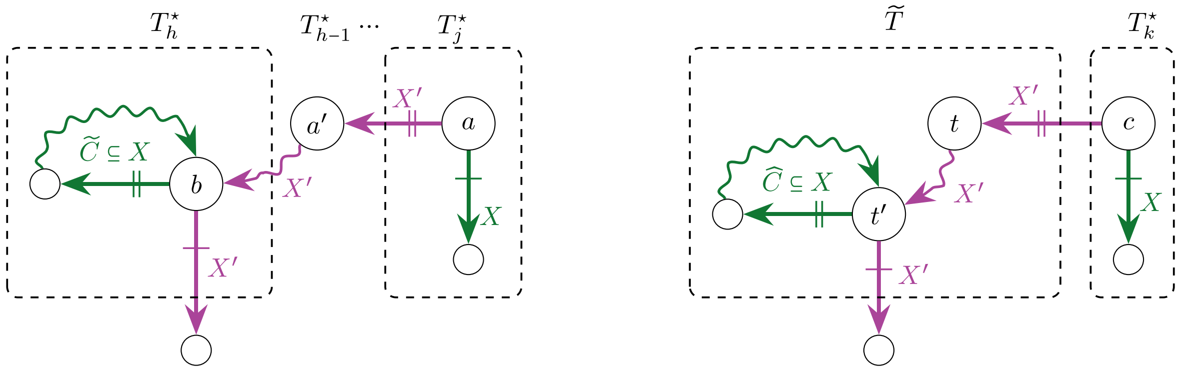
Recall that Algorithm SCFA run on input (H,F) repeatedly removes agents from H, and performs a recursive call on the remaining submarket on line 21. Consider the run of the algorithm that produces X as its output. Let $I_i = (H_{|N_i}, F_i)$ be the input of the ith call of the algorithm during this run, with I_1 being the original input instance, and with each further call I_i, $i \geq 2$, being initiated at line 21 of the previous call. Let $T_i^\star$ denote the set of agents contained in $T^\star$ (defined in line 11) at the point when the algorithm reaches line 19 in the ith call. Then either the algorithm stops in the ith call, or deletes the agents of $T_i^\star$ from the market on line 20, right before the recursive call on I_{i+1}, in which case we have $N_{i+1} = N_i \setminus T_i^\star$.

Consider the smallest k for which $T_k^\star \cap C \neq \emptyset$, and let $c \in C \cap T_k^\star$ be an agent in coalition C removed from the market in the kth call. Note that the arc $(c, X(c))$, which the algorithm will put into the returned allocation X, is undominated within the submarket $H_{|N_k}$, due to the definitions of the arc sets E_S and $E_{S,\mathcal{T}}$ on lines 8 and 13. By $X'(c) \succ_c X(c)$, we get that $X'(c)$ must have been removed earlier from the market, so $X'(c) = t$ for some agent $t \in \widetilde{T} = T_1^\star \cup \dots \cup T_{k-1}^\star$. By the definition of k, we know $C \cap \widetilde{T} = \emptyset$. We prove the following.

Claim 15. *For each agent $a \in \widetilde{T}$, we have $X \succeq_a X'$.*

Claim proof. For a contradiction, assume that $X' \succ_a X$ for some $a \in \widetilde{T}$, and let us choose $a \in T_j^\star$ so as to minimize the index $j \in [k-1]$. Since the arc $(a, X(a))$ is undominated in $H_{|N_j}$, we know that $X'(a) = a'$ for some $a' \in T_1^\star \cup \dots \cup T_{j-1}^\star$. Since X' is an allocation, it contains a cycle through the arc $(a, X'(a))$ entering the set $T_1^\star \cup \dots \cup T_{j-1}^\star$; hence, it also contains an arc leaving the set $T_1^\star \cup \dots \cup T_{j-1}^\star$. See Fig. 3(a) for an illustration. Let $(b, X'(b))$ be such an arc, and let $h \in [j-1]$ be the index for which $b \in T_h^\star$. Then $\{(v, X(v)) : v \in T_h^\star\}$ contains a cycle $\widetilde{C}$ through b. Note that $X \succeq_v X'$ for each agent $v \in V(\widetilde{C}) \subseteq T_h^\star$, due to our choice of a and $h < j$. Moreover, since $(b, X'(b))$ leaves $T_h^\star$, the arc $(b, X(b)) \in \widetilde{C} \subseteq X$ dominates the arc $(b, X'(b))$ due to the definition of $E_{S,\mathcal{T}}$ on line 13, i.e., $X(b) \succ_b X'(b)$. Therefore, $\widetilde{C}$ is a blocking cycle for X' in H. Recall also that there is no deviating agent in $\widetilde{T}$, i.e., $C \cap \widetilde{T} = \emptyset$, and thus $\succ_v = \succ'_v$ for each agent $v \in V(\widetilde{C}) \subseteq T_h^\star \subseteq \widetilde{T}$. This implies that $\widetilde{C}$ is a blocking cycle for X' in H' as well, which contradicts the correctness of Algorithm SCFA established in Theorem 8. ◁

We will apply again the ideas used in the proof of Claim 15; see Fig. 3(b) for an illustration. Recall that $(c, X'(c)) = (c, t)$ is an arc that enters $\widetilde{T}$, so we know that X' must also contain an arc that leaves $\widetilde{T}$; let $(t', X'(t'))$ be such an arc for some $t' \in \widetilde{T}$. Recall that $(t', X(t'))$ dominates all arcs leaving $\widetilde{T}$, so we have $X \succ_{t'} X'$. By $t' \in \widetilde{T}$, we also know that the cycle $\widehat{C}$ of X covering t' only contains agents within $\widetilde{T}$. By Claim 15, this implies that $X \succeq_a X'$ for each agent a on $\widehat{C}$, and $X \succ_{t'} X'$ for agent t' on $\widehat{C}$. This means that $\widehat{C}$ is a blocking cycle for X' in H.



(a) Illustration for Claim 15.

(b) Illustration for Theorem 14.

Fig. 3. Illustrations for Theorem 14. Paths are shown as zigzag lines. Arcs and paths belonging to allocation X (or X') are shown in **green** (in **purple**, respectively). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Recall that since there is no deviating agent in $\widetilde{T}$, i.e., $C \cap \widetilde{T} = \emptyset$, we further have $\succ_a = \succ'_a$ for each agent $a \in V(\widehat{C}) \subseteq \widetilde{T}$. Hence, $\widehat{C}$ is a blocking cycle for X' in H' as well, which contradicts the correctness of Algorithm SCFA established in Theorem 8. □

Unfortunately, the mechanism associated with our algorithm is not strongly group-strategyproof as shown by the following example.

Example 4. Consider the following market H^3 over $N = \{a, b, c, d, e\}$ as the agent set; see Fig. 4(a). The acceptability sets are $A(a) = \{b\}$, $A(b) = \{c\}$, $A(c) = \{a, b, d\}$, $A(d) = \{e\}$, and $A(e) = \{a, d\}$. For each agent x and $y \in A(x) \setminus \{x\}$, we let $y \succ_x x$; additionally, we set $b \succ_c d$ and $a \succ_e d$.

Consider the output of Algorithm SCFA with H^3 as its input, without forbidden arcs. All arcs are undominated except for (c, d) and (e, d). The strongly connected components in the subgraph spanned by all undominated arcs are $S_1 = \{a, b, c\}$, $S_2 = \{d\}$, and $S_3 = \{e\}$. Thus, the algorithm first stores the allocation $\{(a, b), (b, c), (c, a)\}$ for S_1 and deletes agents a, b, and c. In the remaining market, arcs (d, e) and (e, d) are undominated, and the 2-cycle they form is a strongly connected component covering all remaining agents. Hence, the algorithm returns the allocation $X = \{(a, b), (b, c), (c, a), (d, e), (e, d)\}$.

Let H^4 be the $\{c\}$-deviation of H^3 where preferences are defined the same way as in H^3 with the only exception that c is indifferent between b and d in H^4; see Fig. 4(b). Then the set of undominated arcs forms a unique strongly connected component containing all agents, and the algorithm returns the allocation $X' = \{(a, b), (b, c), (c, d), (d, e), (e, a)\}$. Notice that c is indifferent between X and X', but e strictly prefers X' to X. Hence, this shows that Algorithm SCFA is not strongly group-strategyproof.

5. Discussion

We investigated the strong core of housing markets in a model where agents' preferences are partial orders, filling a gap in the literature. Our main result is a polynomial-time algorithm for obtaining an allocation in the strong core whenever the strong core is not empty; the mechanism associated to our algorithm is weakly group-strategyproof and can accommodate forbidden and forced arcs.

As our algorithm is neither strongly group-strategyproof nor Pareto-efficient, the question arises whether there is an algorithm with these properties that finds an allocation in the strong core if there exists one.

To gain further insight from an algorithmic perspective, it is important to understand which properties of the strong core remain valid in our generalized model with partial order preferences. In fact, Appendix B offers a collection of such observations. Among others, we establish that the strong core in such housing markets *respects improvement*, meaning that an improvement of some agents' endowment is bound to be beneficial for that agent in terms of the allocations in the strong core; see Appendix B.2 for more details. We also prove that the ILP proposed by Biró et al. (2023) for the strong core remains sound even when agents' preferences are partial orders. Their ILP formulation is developed incrementally, beginning with the core, then incorporating constraints for competitive allocations, and finally characterizing the strong core. Since our focus is exclusively on the strong core, we show that certain constraints—namely, those enforcing competitiveness—can be omitted when preferences are partial orders. See Afacan et al. (2024) and Morrill and Roth (2024) for a discussion on the relationship between the strong core, the weak core, and competitive equilibria.

As the strong core can be empty in housing markets with indifferences, it would be interesting to study allocations that are "nearly in the strong core". E.g., can we find allocations for which a few agents cover all blocking cycles? Such an allocation could be stabilized by compensating these agents, thus preventing deviations. Or can we find an allocation from which no deviation yields a strict improvement for at least k agents for some constant $k \in \mathbb{N}$?

CRediT authorship contribution statement

Ildikó Schlotter: Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Mirabel Mendoza-Cadena:** Writing – review & editing, Writing – original draft, Investigation.

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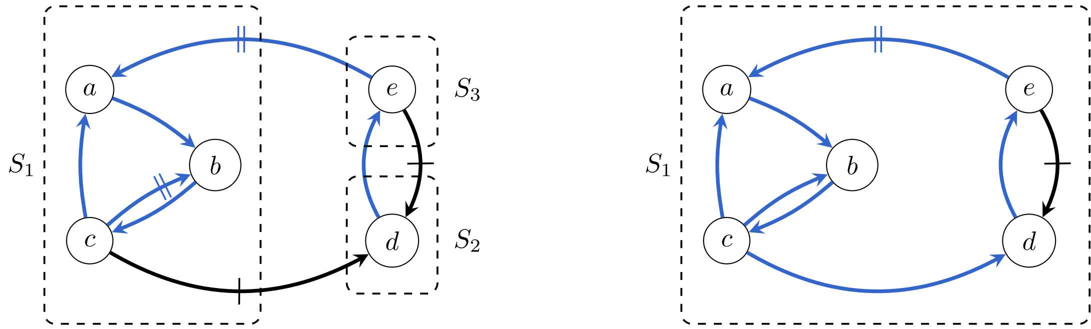

(a) The underlying graph of housing market H^3.
(b) The underlying graph of housing market H^4.

Fig. 4. The underlying graphs of the housing markets defined in Example 4.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: The authors declare that they have no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Additional properties of Algorithm SCFA

In this section, we discuss certain properties of Algorithm SCFA. In Appendix A.1 we briefly examine its efficiency properties, while in Appendix A.2 we turn to the question whether the algorithm can be used to find all allocations in the strong core.

A.1. Efficiency properties of Algorithm SCFA

We consider two properties for efficiency. An allocation X in a housing market $H = (N, \{\succ_a\}_{a\in N})$ is *Pareto-efficient* if there is no other allocation X' in H such that $X' \succeq_a X$ for all agents $a \in N$ and, moreover, $X' \succ_a X$ for at least one agent $a \in N$. A relaxation of this property called pair-efficiency was introduced by Ekici (2024): an allocation X is *pair-efficient* if there are no two agents a and b such that they prefer each other's allocation over their own in X, that is, $X(b) \succ_a X(a)$ and $X(a) \succ_b X(b)$. It is not hard to see that a Pareto-efficient allocation is also pair-efficient, but the inverse is not true.

Roughly speaking, a mechanism is Pareto- or pair-efficient if it always outputs such an allocation. More formally, let f be a mechanism that assigns to each housing market H a set $f(H)$ of allocations. We say that f is *Pareto-efficient* or *pair-efficient* if all allocations in $f(H)$ are Pareto-efficient or pair-efficient, respectively, for each housing market H.

By the seminal result of Ma (1994), we know that the unique mechanism that is strategyproof and Pareto-efficient is the TTC.[3] Recently, Ekici (2024) showed that TTC can also be characterized as the unique mechanism that is strategyproof and pair-efficient. In view of these results, it is interesting to examine whether Algorithm SCFA, which can be viewed as a generalization of TTC, satisfies these properties.

[3] Ma also lists *individual rationality* among the properties used to characterize TTC; however, our definition of an allocation already assumes this property, as we require an allocation to assign each agent an acceptable house.

Proposition 16. *Algorithm SCFA is not Pareto-efficient.*

Proof. Consider the housing market H^3 as defined in Example 4 (see Fig. 4(b) for an illustration). Here, the algorithm returns the allocation $X = \{(a,b),(b,c),(c,a),(d,e),(e,d)\}$. However, allocation $X' = \{(a,b),(b,c),(c,d),(d,e),(e,a)\}$ is a so-called Pareto-improvement over X: all agents are indifferent between X and X' except for agent e, who prefers X' over X. □

On the other hand, our algorithm satisfied the more relaxed property of pair-efficiency.

Proposition 17. *Algorithm SCFA is pair-efficient.*

Proof. Suppose for the sake of contradiction that there exists an instance (H, F) with $H = (N, \{\succ_a : a \in N\})$ such that it has an allocation $X \in f(H, F)$ returned by Algorithm SCFA for which there are two agents a and b such that $X(b) \succ_a X(a)$ and $X(a) \succ_b X(b)$. Consider the moment when Algorithm SCFA adds the arc $(a, X(a))$ to the allocation; at this point, the arc $(a, X(a))$ must be undominated in the current market. By $X(b) \succ_a X(a)$, this implies that b must have been removed from the market earlier than this moment. However, by symmetry, we obtain analogously that a must have been removed from the market earlier than b. This contradiction shows the pair-efficiency of Algorithm SCFA. □

A.2. Finding all allocations in the strong core

It is known that the Quint–Wako algorithm can find all allocations in the strong core of a housing market $H = (N, \{\succ_a\}_{a\in N})$ in which agents' preferences are described by weak orders. More precisely, if S is an absorbing set of the graph D^H underlying H, then by choosing an allocation (consisting of undominated arcs) for the submarket $H_{|S}$ appropriately, and using also appropriate choices within the recursive call on $H_{|(N\setminus S)}$, the Quint–Wako algorithm can return any fixed allocation X in the strong core of H. Interestingly, this also holds for Algorithm SCFA if agents' preferences are weak orders; hence, Algorithm SCFA is as powerful as the Quint–Wako algorithm when used for enumerating the strong core, as stated in Proposition 18. By contrast, if agents' preferences are partial orders, then the strong core may contain allocations that can never be returned by Algorithm SCFA, as shown below in Example 5.

Proposition 18. *Suppose that H is a housing market where agents' preferences are weak orders. When run on the instance $(H, \emptyset)$, Algorithm SCFA can find each allocation in the strong core of H, by appropriately choosing between possible allocations on line 15.*

Proof. Suppose that X is an allocation in the strong core of some housing market H. Let $\mathcal{S}$ denote the set of strongly connected components in the subgraph $D^H[U]$ spanned by the set U of undominated arcs. For each $S \in \mathcal{S}$, define the arc set $X_{|S} = X \cap (S \times S)$. The following facts follow from Theorem 2:

(i) $X_{|S}$ is an allocation in the submarket $H_{|S}$ for each set $S \in \mathcal{S}$;
(ii) if $H_{|S}$ admits an allocation contained in U, then $X_{|S} \subseteq U$;
(iii) if S is an absorbing set in $D^H[U]$, then $X_{|S} \subseteq U$.

Next, we are going to show that Algorithm SCFA will reach line 19 at some point where the set family $\mathcal{T}$ contains all absorbing sets in $D^H[U]$ and possibly some other sets from $\mathcal{S}$.

To see this, first notice that each absorbing set of $D^H[U]$ is added to the family $\mathcal{T}$ on line 9, due to fact (iii) above. Second, by definition, no arc leaving an absorbing set S of $D^H[U]$ can be undominated. Since agents' preferences are weak orders, all arcs leaving a given agent $a \in S$ are dominated by all undominated arcs leaving a. Hence, $E_S = E_{S,\mathcal{T}}$ for every possible set family $\mathcal{T}$ that contains S. This means that no absorbing set is ever added to $\mathcal{R}$ on line 16. Hence, at some point the algorithm reaches line 19, and when this happens, the set family $\mathcal{T}$ contains all absorbing sets of $D^H[U]$, and possibly some other sets from $\mathcal{S}$ as well.

Consider the value of the set family $\mathcal{T}$ at this point and the agent set $T^\star = \bigcup_{T \in \mathcal{T}} T$. Notice that $T^\star$ must be *closed* in the sense that there cannot be an undominated arc leaving $T^\star$: indeed, the existence of such an arc (a, b) would imply that no arc leaving a would be contained in $E_{S,\mathcal{T}}$ for the set $S \in \mathcal{T}$ containing a, and hence, S would have been added to $\mathcal{R}$ on line 16, resulting in the deletion of S from $\mathcal{T}$.

Using again that agents' preferences are weak orders, the observation that $T^\star$ is closed, i.e., no undominated arc leaves $T^\star$, implies that all arcs leaving $T^\star$ are dominated by any arc in U. Therefore, we know that $E_{S,\mathcal{T}} = E_S \subseteq U$ for each $S \in \mathcal{T}$. Moreover, due to line 9, the submarket $H_{|S}$ admits an allocation consisting only of undominated arcs. By facts (i) and (ii) above, this means that $X_{|S}$ is also an allocation consisting only of undominated arcs. Since there are no forbidden arcs, this means that $X_{|S} \subseteq E_{S,\mathcal{T}}$. Hence, there exists an appropriate choice for the algorithm to store the allocation $X_{|S}$ on line 15 for each $S \in \mathcal{T}$; assume henceforth that this happens.

In the case $N = T^\star$, this implies that the allocation $\bigcup_{S \in \mathcal{T}} X_{|S}$ returned on line 19 is exactly the allocation X. If $N' = N \setminus T^\star \neq \emptyset$, then $X \cap (N' \times N')$ is clearly in the strong core of $H_{|N'}$. Hence, by an inductive argument we may assume that the recursive call on line 21 returns the allocation $X' = X \cap (N' \times N')$ for the submarket $H_{|N'}$, and thus, the algorithm outputs the allocation $\bigcup_{S \in \mathcal{T}} X_{|S} \cup X' = X$ on line 23. □

Contrasting Proposition 18, we now show that if agents' preferences are partial orders, then there may exist allocations in the strong core that can never be returned by Algorithm SCFA. Such an instance is presented in Example 5.

Example 5. Consider the following market H^5 over $N = \{a, b, c, d_1, d_2\}$ as the agent set; see its underlying graph without loops on Fig. 5. Let the acceptability sets be defined as $A(a) = A(b) = A(c) = N \setminus \{d_2\}$ and $A(d_1) = A(d_2) = \{d_1, d_2\}$. For each agent $x \in N$ and $y \in A(x) \setminus \{x\}$, we let $y \succ_x x$; additionally, we set $d_1 \succ_a b$, $d_1 \succ_b c$ and $d_1 \succ_c a$.

Consider how Algorithm SCFA runs with H^5 as its input without forbidden arcs. The set of undominated arcs in the graph underlying the market is $U = \{(b, a), (a, c), (c, a), (d_1, d_2)\} \cup \{(x, d_1) : x \in N \setminus \{d_1\}\}$. The set of strongly connected components in the subgraph of D^{H^5} spanned by all undominated arcs is $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{a, b, c\}$ and $S_2 = \{d_1, d_2\}$. Algorithm SCFA computes on line 8 the arc sets $E_{S_1} = \{(a, c), (c, b), (b, a)\}$ and $E_{S_2} = \{(d_1, d_2), (d_2, d_1)\}$, and finds that both $H^5_{|S_1}$ and $H^5_{|S_2}$ admit an allocation in E_{S_1} and in E_{S_2}, respectively. Thus, Algorithm SCFA initializes the family $\mathcal{T}$ by setting $\mathcal{T} = \{S_1, S_2\}$.

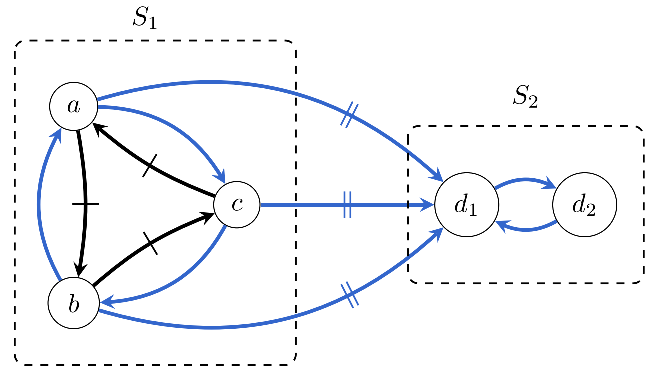


Fig. 5. The underlying graph of housing market H^5 defined in Example 5.

Starting the iteration on lines 10–17, the algorithm first computes the arc sets $E_{S,\mathcal{T}}$ for both sets $S \in \mathcal{T}$, obtaining $E_{S_1,\mathcal{T}} = E_{S_1}$ and $E_{S_2,\mathcal{T}} = E_{S_2}$ and storing the allocations $X_{S_1} = \{(a, c), (c, b), (b, a)\}$ and $X_{S_2} = \{(d_1, d_2), (d_2, d_1)\}$, which are the unique allocations for $H^5_{|S_1}$ in $E_{S_1,\mathcal{T}}$ and for $H^5_{|S_2}$ in $E_{S_2,\mathcal{T}}$, respectively; note that no other allocations can be stored at this point. The algorithm finds on line 19 that $N = T^\star = S_1 \cup S_2$, and hence returns the allocation $X_{S_1} \cup X_{S_2}$.

Even though Algorithm SCFA can only output the single allocation described above, the strong core of H^5 contains another allocation: namely, the allocation $\{(a, b), (b, c), (c, a)\} \cup X_{S_2}$.

Appendix B. Properties of the strong core

In this section, we establish some of the most relevant characteristics of the strong core.

B.1. Incomparability of strong core allocations

We start with the remarkable fact that all agents find any two allocations X and Y in the strong core incomparable. This is stated in Lemma 19, whose proof relies on structural observations about the arc set obtained, roughly speaking, by letting each agent a choose between $(a, X(a))$ and $(a, Y(a))$ according to its preferences.

Lemma 19. *Given a housing market $H = (N, \{\succ_a : a \in N\})$ with partial order preferences and allocations X and Y in the strong core of H, it must hold that $X \sim_a Y$ for each agent $a \in N$.*

Proof. Suppose for the sake of contradiction that $Y \succ_a X$ for some agent $a \in N$. We build a walk W starting at agent a using an iterative procedure as follows. The procedure distinguishes between two states, *X-preferring* and *Y-preferring*, starting initially in a Y-preferring state and with W containing zero arcs. At each step, we take the agent b at the end of W and choose the next arc of W among the arcs leaving b:

(a) if $Y \succ_b X$, we pick $(b, Y(b))$ and we switch to (or remain in) the Y-preferring state;
(b) if $X \succ_b Y$, we pick $(b, X(b))$ and we switch to (or remain in) the X-preferring state;
(c) if $X \sim_b Y$, and the procedure is currently in Z-preferring mode for some $Z \in \{X, Y\}$, then we pick $(b, Z(b))$.

We stop this procedure at the point where W ceases to be a directed path, because the last arc added to W points to some vertex already contained in W, thus creating a cycle C.

Let u be the last agent on W for whom $W \succ_u X$ or $W \succ_u Y$; note that $u = a$ is possible, so u is well-defined. This means that u is the last agent at which step (a) or (b) above was applied. Observe that if u is on C, then C blocks X or Y, because $W(c) \succeq_c X(c)$ and $W(c) \succeq_c Y(c)$

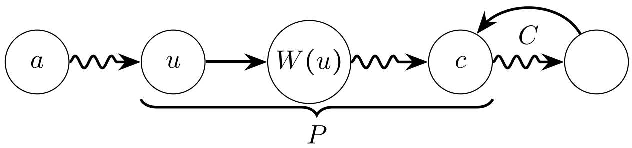


Fig. 6. Illustration of walk W for Lemma 19. Paths are shown as zigzag lines.

hold for each c on C by the definition of W, and u prefers C to X or to Y. Hence, this contradicts $X, Y \in \mathcal{SC}(H)$.

Thus, we get that u is not on C, as shown in Fig. 6. Then the arcs put into W after and including $(u, W(u))$ form the union of the cycle C and a path P leading from u to some agent c on C, with $V(P) \cap V(C) = \{c\}$. Suppose that $W(u) = X(u)$ (the case where $W(u) = Y(u)$ is symmetrical), so the procedure entered (or remained in) the X-preferring state at u. By the definition of u, after picking the arc $(u, X(u))$ at u, the procedure always applied step (c) above, and thus remained in X-preferring mode. Consequently, $W(z) = X(z)$ for all agents z that follow u on W. However, this means that c is entered by two arcs of X, one on P and one on C (the last arc put into W). This contradicts the assumption that X is an allocation, finishing our proof. □

An important consequence of Lemma 19 is that even though there might be allocations in the strong core that can never be found by Algorithm SCFA, this is not really relevant from the viewpoint of the agents, as all agents find any two allocations in the strong core incomparable. Lemma 19 also plays a crucial role in proving that the strong core respects improvement.

B.2. The effect of improvements on the strong core

We next aim to examine the effect of a change in the market when the house of some agent p improves—is such a change necessarily (weakly) advantageous for p in terms of the strong core? Such questions are essential for understanding the incentives of agents to improve their endowments. In the context of a kidney exchange program, does bringing a "better" (e.g., younger) donor always benefit the patient, assuming that the program offers a strong core allocation?

Similar questions have been already studied in the context of housing markets (Roth et al., 2005; Schlotter et al., 2024), but this line of research has its roots in a study by Balinski and Sönmez (1999) on college admission. The authors investigated how an improvement in a student's test scores may have undesirable effects. Their results inspired further studies of the same flavor about school choice and other allocation problems (Hatfield et al., 2016; Sönmez and Switzer, 2013; Klaus and Klijn, 2021).

To formalize the concept of improvement, consider two housing markets $H = (N, \{\succ_a : a \in N\})$ and $H' = (N, \{\succ'_a : a \in N\})$ over the same agent set N. For two agents $p, q \in N$, we say that H' is a *(p,q)-improvement* of H if H' is obtained from H by shifting the position of p "upward" in the preferences of agent q; see also Example 6. Formally, we require that

1. for each $a \in N \setminus \{q\}$, $\succ_a = \succ'_a$, that is, we only allow changes in the preferences of agent q;
2. for each $a \in N \setminus \{p\}$, $p \succ_q a$ implies $p \succ'_q a$, and $p \succeq_q a$ implies $p \succeq'_q a$, that is, agent p may only become more preferred for agent q in $\succ'$, and not less preferred;
3. for each $a, b \in N \setminus \{p\}$, we have $a \succ_q b$ if and only if $a \succ'_q b$, that is, the preferences for agent q remain unchanged when comparing two agents not involving p.

Now, a *p-improvement* of H is the result of a sequence of (p, q_i)-improvements of H for some series of agents $q_1, \dots, q_k \in N$.

Example 6. Consider the following market H^6 over $N = \{a, b, c, d\}$ as the agent set; see Fig. 7(a). The acceptability sets are defined as $A(a) = A(b) = \{a, b\}$, $A(c) = \{a, c, d\}$, and $A(d) = \{c, d\}$. For each agent x and $y \in A(x) \setminus \{x\}$, we let $y \succ_x x$; additionally, we set $a \succ_c d$. Then $\mathcal{SC}(H^6) = \{X_{\text{orig}}\}$ where $X_{\text{orig}} = \{(a, b), (b, a), (c, c), (d, d)\}$.

Let H^7 be the c-improvement of H^6 where the preferences are defined in the same way as in H^6 but b now prefers c to its own house, so its acceptability set becomes $A(b) = N \setminus \{d\}$; see Fig. 7(b). Then $\mathcal{SC}(H^7)$ contains the unique allocation $\{(a, b), (b, c), (c, a), (d, d)\}$, which c prefers to X_{orig}.

Now let H^8 be the c-improvement of H^6 where the preferences are defined the same way as in H^6 but a now prefers c to its own house; see Fig. 7(c). The strong core of H^8 then becomes empty, since the strongly connected component $\{a, b, c\}$ within the subgraph of undominated arcs does not contain an allocation.

To study the effects of change in a housing market, we investigate how the strong core changes when a p-improvement occurs for some agent $p \in N$. In particular, we aim to understand whether a p-improvement is always beneficial (or, at least, not detrimental) to agent p; hence, we aim to check the following property.

Definition 20. The strong core of housing markets *respects improvement*, if for all housing markets H and H' such that H' is a p-improvement of H for some agent p, we have $X' \succeq_p X$ for all allocations $X \in \mathcal{SC}(H, p)$ and $X' \in \mathcal{SC}(H', p)$.

Note that Definition 20 allows the strong core of H' to become empty as a result of an improvement, which may indeed happen if agents' preferences are weak or partial orders, as we have already seen in Example 6.

Remark 21. Notably, Lemma 19 does not hold for the core of a housing market with indifferences. Hence, Biró et al. (2023) introduced the following notion: the core of housing markets *respects improvement for the best available house* if for all housing markets H and H' such that H' is a p-improvement of H for some agent p, $X' \succeq_p X$ holds whenever X and X' are among the most-preferred allocations by p within the core of H and of H', respectively; the respecting improvement property for the *worst* available house is defined analogously. These two notions coincide for the strong core with the property expressed in Definition 20. Schlotter et al. (2024) proved that the core of housing markets respects improvement for the best available house, but not for the worst available house.

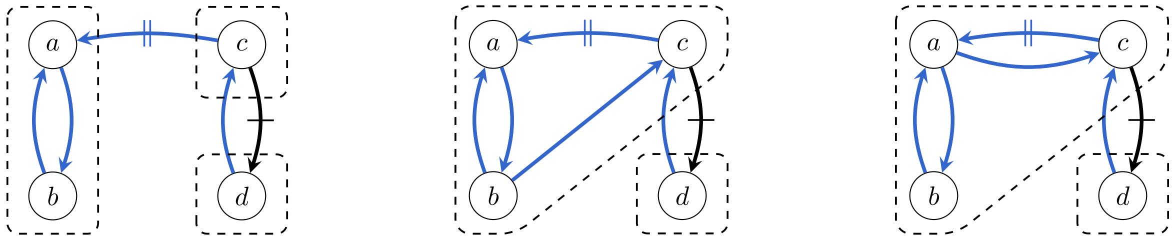
Table 2 summarizes the results regarding the property of respecting improvements for the strong core and the core, including Theorem 22, the main result of this section. Although the proof of Theorem 22 is elementary in its technique, it requires certain insights and uses ideas from the proof of Lemma 19.

Theorem 22. *Let $H = (N, \{\succ_a : a \in N\})$ be a housing market in which agents' preferences are partial orders, and let H' be a p-improvement of H for some agent p. Then for all allocations X and X' in the strong core of H and H', respectively, it holds that $X' \succeq_p X$.*

Proof. Let $H' = (N, \{\succ'_a : a \in N\})$ be obtained from H as the result of a series of (p, q_i)-improvements for some agents $q_1, \dots, q_k$, and let $Q = \{q_1, \dots, q_k\}$.

Assume for the sake of contradiction that $X \succ_p X'$.

We will re-use the procedure described in the proof of Lemma 19, applied to allocations X and X' with starting point p and starting in the X-preferring state; we use the preferences $\succ'$ as given for H'. Let W be the resulting walk. Note that if W does not include an arc pointing to p, then the arguments presented in the proof of Lemma 19 either yield a contradiction to the assumption that X or X' is an allocation, or prove the presence of a cycle C that blocks X or X' in H'. By $X' \in \mathcal{SC}(H')$,



(a) Original housing market H^6. (b) Market H^7: a (c,b)-improvement of H^6. (c) Market H^8: a (c,a)-improvement of H^6.

Fig. 7. Illustration for Example 6.

Table 2
Summary of results on the effect of improvements. RI stands for "respects improvement", while RI-best/RI-worst stands for "respects improvement for the best/worst available house", respectively. We use ✔ to indicate that the given property holds; ✔∃ additionally signals that the strong core (or core) cannot become empty as the result of a p-improvement; ✗ means that the given property fails to hold.

Preferences	Strong core RI	Core RI-best	Core RI-worst
Strict orders	✔∃ (Biró et al., 2023)	✔∃ (Schlotter et al., 2024)	✗ (Schlotter et al., 2024)
Weak orders	✔ (Biró et al., 2023)	✔∃ (Schlotter et al., 2024)	✗ (Schlotter et al., 2024)
Partial orders	✔ Theorem 22	✔∃ (Schlotter et al., 2024)	✗ (Schlotter et al., 2024)

the only possibility is that C blocks X in H'. Given that $X \in \mathcal{SC}(H)$, we also know that C does not block X in H. By the definition of a (p,q)-improvement, we know that this can happen only if C contains some of the arcs $\{(q_i,p) : q_i \in Q\}$, because the only agents whose preferences differ in H and H' are the agents in Q, and moreover, the only agent that can become more preferred by some $q_i \in Q$ when compared to some other agent is agent p.

Notice now that $(q_i,p) \in C$ for some $q_i \in Q$ implies $C = W$. Recall that $X \succ_p X'$; hence $C(p) = W(p) = X(p)$, and therefore C is a cycle that blocks X' in H': all agents on C weakly prefer C to X' by the definition of W, and p strictly prefers C to X. This contradiction proves the theorem. □

B.3. Formulation as an integer linear program

Quint and Wako were the first to provide an ILP for the strong core of housing markets (Quint and Wako, 2004). Subsequently, Biró et al. (2023) provided an improved ILP formulation for the strong core where the number of constraints is only linear in the size of the market. Biró et al. (2023) assumed weakly ordered preferences; hence it is not immediately clear whether their formulation remains correct for partial order preferences. We prove that the ILP formulation for the strong core by Biró et al. remains sound even when agents' preferences are partial orders.

In their paper, Biró et al. (2023) gave an incremental ILP formulation: they first gave an ILP for the core, then added a set of constraints to obtain an ILP for the set of so-called competitive allocations, and finally added another set of constraints for an ILP describing the strong core. Since our focus here is solely on the strong core, we show that we can omit certain constraints from this incremental description (namely, those that ensure that the allocation is competitive) and still obtain an ILP that defines the strong core, even when agents' preferences are partial orders. The relationship between the strong core, the weak core, and competitive equilibrium is discussed by Afacan et al. (2024) and Morrill and Roth (2024).

Recall our notation for the arc sets $E_{[\succ X]}$ and $E_{[\succeq X]}$ defined in Section 3.2.

Let $H = (N, \{\succ_a : a \in N\})$ be a housing market with partial orders with underlying graph $D^H = (N,E)$. Let $|N| = n$. The ILP defining the strong core of H is shown in ILP_{sc}. For each $(i,j) \in E$, it contains a variable y_{ij} with the following interpretation:

$$y_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is an arc in the allocation,} \\ 0 & \text{otherwise.} \end{cases}$$

A further set of variables, p_i for each $i \in N$, can be thought of as the *price* of i.

(ILP_{sc})

$$\sum_{j:(i,j)\in E} y_{ij} = 1 \qquad \forall i \in N, \qquad (1)$$

$$\sum_{j:(j,i)\in E} y_{ji} = 1 \qquad \forall i \in N, \qquad (2)$$

$$p_i + 1 \le p_j + n \cdot \sum_{k:j \preceq_i k} y_{ik} \qquad \forall (i,j) \in E, \qquad (3)$$

$$p_i \le p_j + n \cdot \sum_{k:j \prec_i k} y_{ik} \qquad \forall (i,j) \in E, \qquad (4)$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in N, \qquad (5)$$

$$p_i \in [n] \qquad \forall i \in N. \qquad (6)$$

Constraints (1) and (2) together with (5) determine an allocation X: defining $X(i) = j$ if and only if $y_{ij} = 1$, constraints (1) and (2) guarantee that X contains exactly one arc leaving and entering each agent $i \in N$, respectively. Constraints (3) and (6) ensure that there is no strictly blocking cycle for X, that is, X is in the core. To see this, recall that X is in the core of H if and only if arcs of $E_{[\succ X]}$ form an acyclic subgraph of D^H, which in turn happens if and only if there is a topological order of the agents in the subgraph of D^H spanned by $E_{[\succ X]}$. The existence of such a topological order is equivalent to the existence of values p_i, $i \in N$, such that $p_i < p_j$ for each arc $(i,j) \in E_{[\succ X]}$, which is in turn guaranteed by constraints (3) and (6). Finally, adding constraints (4) narrows down the set of allocations that correspond to the feasible solutions of our ILP to the strong core.

The correctness of ILP_{sc} is expressed by Proposition 24 whose proof relies on Lemma 23 and follows ideas from Biró et al. (2023). We remark that the ILP formulations of Biró et al. (2023) that describe the core and the set of competitive allocations also remain sound under the generalization that agents' preferences are partial orders.

To show the correctness of ILP_{sc}, we need the following lemma.

Lemma 23. *An allocation X is in the strong core of H if and only if there exist prices $p_i \in [n]$ for each $i \in N$ such that*

- *$p_i < p_j$ for each $(i,j) \in E_{[\succ X]}$, and*
- *$p_i \le p_j$ for each $(i,j) \in E_{[\succeq X]}$.*

Proof. First, if $X \in \mathcal{SC}(H)$, then there is no cycle consisting of arcs in $E_{[\succeq X]}$ that also contains an arc in $E_{[\succ X]}$, as such a cycle would be a blocking cycle. Hence, no strongly connected component of the graph $D_{[\succeq X]} = (N, E_{[\succeq X]})$ may contain a cycle in $E_{[\succ X]}$, as that would give rise to a blocking cycle for X.

Let D' be the digraph obtained by contracting each strongly connected component of $D_{[\succeq X]}$. Let $\varphi(i)$ denote the vertex of D' corresponding to some $i \in N$. On the one hand, we know that D' is acyclic due to its definition. On the other hand, we know that for each arc (i,j) in $E_{[\succ X]}$ we have $\varphi(i) \neq \varphi(j)$, and so $(\varphi(i), \varphi(j))$ is an arc in D'. Notice also that D' may further contain arcs that correspond to arcs in $E_{[\succeq X]}$ that run between different strongly connected components of $D_{[\succeq X]}$; such arcs may exist because agents' preferences are partial orders.

Since D' is acyclic, it admits a topological ordering, so we can assign a value $p_v \in [n]$ for each vertex v of D' such that $p_v < p_{v'}$ holds for each arc (v, v') of D'. This gives rise to prices p_i for $i \in N$ by setting $p_i = p_{\varphi(i)}$. In this way, we have $p_i < p_j$ for each $(i,j) \in E_{[\succ X]}$, because for such arcs, $(\varphi(i), \varphi(j))$ is an arc of D'. We also know that $p_i \le p_j$ for each $(i,j) \in E_{[\succeq X]}$, because either $\varphi(i) = \varphi(j)$, which implies $p_i = p_j$, or again $(\varphi(i), \varphi(j))$ is an arc of D', implying $p_i < p_j$.

For the other direction of the proof, assume that there are prices p_i for each $i \in N$ fulfilling the conditions of the lemma. We show that X cannot admit a blocking cycle. Assume for the sake of contradiction that C is a cycle that blocks X. Then $C \subseteq E_{[\succeq X]}$ implies that $p_i \le p_j$ for each $(i,j) \in C$, which can only happen if $p_i = p_j$ for each $(i,j) \in C$, since C is a cycle. However, by the definition of a blocking cycle, C contains an arc (i', j') of $E_{[\succ X]}$, for which $p_{i'} < p_{j'}$ follows from the condition on the prices as stated by the lemma, yielding a contradiction. Thus, X is indeed in the strong core of H. □

Proposition 24. *There exists a feasible solution for* $\mathrm{ILP}_{\mathrm{sc}}$ *if and only if the housing market* H *admits an allocation in the strong core.*

Proof. Let X be an allocation in the strong core of H. Set y_{ij} as 1 whenever each $(i,j) \in X$, and 0 otherwise. Then (1), (2), and (5) clearly hold. Define prices p_i for each $i \in N$ so that they satisfy the conditions of Lemma 23. Since $p_i \in [n]$ for each $i \in N$, constraint (6) is satisfied.

We show that inequality (3) holds for each $(i,j) \in E$. On the one hand, if $(i,j) \in E_{[\succ X]}$, then this is clear by our assumption on the prices, as we have $p_i + 1 \le p_j$. On the other hand, if $(i,j) \notin E_{[\succ X]}$, that is, $X(i) \succeq_i j$, then $X(i)$ is contained in the set $\{k : k \succeq_i j\}$. Therefore, we get $\sum_{k:k\succeq_i j} y_{ik} \ge 1$. Since both p_i and p_j are in $[n]$, this implies

$$p_i + 1 \le n + 1 \le p_j + n \le p_j + n \cdot \sum_{k:k\succeq_i j} y_{ik}$$

as desired, proving (3).

Next, we show that (4) holds for each $(i,j) \in E$. On the one hand, if $(i,j) \in E_{[\succeq X]}$, then $p_i \le p_j$ holds by our choice of the prices, which implies (4). On the other hand, if $(i,j) \notin E_{[\succeq X]}$, that is, $X(i) \succ_i j$, then $X(i)$ is contained in the set $\{k : k \succ_i j\}$, and therefore we get $\sum_{k:k\succ_i j} y_{ik} \ge 1$, which implies

$$p_i \le n \le n \cdot \sum_{k:k\succ_i j} y_{ik} \le p_j + n \cdot \sum_{k:k\succ_i j} y_{ik}$$

proving (4).

Suppose now that values y_{ij} for each $(i,j) \in E$ and values p_i for each $i \in N$ yield a feasible solution to $\mathrm{ILP}_{\mathrm{sc}}$. Constraints (1), (2), and (5) determine an allocation X where an arc $(i,j) \in E$ is contained in X if and only if $y_{ij} = 1$.

We show that the values p_i, $i \in N$, fulfill the conditions of Lemma 23. By constraint (6), we know $p_i \in [n]$ for each $i \in N$.

To show the first condition, assume that $(i,j) \in E_{[\succ X]}$. This means that $j \succ_i X(i)$, or in other words, $X(i) \not\succeq_i j$. Thus, $X(i)$ is not contained in $\{k : k \succeq_i j\}$. Therefore, $\sum_{k:k\succeq_i j} y_{ik} = 0$. Hence by (3) we know

$$p_i + 1 \le p_j + n \cdot \sum_{k:k\succeq_i j} y_{ik} = p_j,$$

that is, $p_i < p_j$. This proves that the first condition of Lemma 23 is satisfied.

Second, assume that $(i,j) \in E_{[\succeq X]}$. Then $j \succeq_i X(i)$, or in other words, $X(i) \not\succ_i j$. Thus, $X(i)$ is not contained in $\{k : k \succ_i j\}$. Therefore $\sum_{k:k\succ_i j} y_{ik} = 0$. Hence by (4) we know

$$p_i \le p_j + n \cdot \sum_{k:k\succ_i j} y_{ik} = p_j,$$

that is, $p_i \le p_j$. This proves that the second condition of Lemma 23 is satisfied. Hence, by Lemma 23, X is indeed in the strong core of H. □

Data availability

No data was used for the research described in the article.

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