

Refined thresholds for inconsistency: The effect of the graph associated with incomplete pairwise comparisons

Kolos Csaba Ágoston ^a, László Csató ^{a,b,*}

^a Corvinus University of Budapest (BCE), Institute of Operations and Decision Sciences, Department of Operations Research and Actuarial Sciences, Budapest, Hungary

^b Institute for Computer Science and Control (SZTAKI), Hungarian Research Network (HUN-REN), Laboratory on Engineering and Management Intelligence, Research Group of Operations Research and Decision Systems, Budapest, Hungary

ARTICLE INFO

JEL classification:

C44
D71

2000 MSC:

90B50
91B08

Keywords:

Decision analysis
Graph theory
Incomplete pairwise comparisons
Inconsistency
Spectral radius

ABSTRACT

The inconsistency of pairwise comparisons remains difficult to interpret in the absence of acceptability thresholds. The popular 10% cut-off rule proposed by Saaty has recently been applied to incomplete pairwise comparison matrices, which contain some unknown comparisons. This paper refines these inconsistency thresholds: we uncover that they depend not only on the size of the matrix and the number of missing entries, but also on the undirected graph whose edges represent the known pairwise comparisons. Therefore, using our exact thresholds is especially important if the filling in patterns coincide for a large number of matrices, as has been recommended in the literature. The strong association between the new threshold values and the spectral radius of the representing graph is also demonstrated. Our results can be integrated into software to continuously monitor inconsistency during the collection of pairwise comparisons and immediately detect potential errors.

“The mathematical structure known as a graph has the valuable feature of helping us to visualize, to analyze, to generalize, a situation of a problem we may encounter and, in many cases, assisting us to understand it better and possibly find a solution.”¹

(Arthur Benjamin, Gary Chartrand, Ping Zhang: *The Fascinating World of Graph Theory*)

1. Introduction

Pairwise comparison matrices play a central role in several multi-criteria decision-making methodologies such as the Analytic Hierarchy Process (AHP) (Saaty, 1977, 1980). However, while pairwise comparisons efficiently decompose complex decision-making problems into simple subproblems, this comes at a price: there is no guarantee of cardinal consistency. For example, if alternative *A* is two times better than alternative *B*, and alternative *B* is three times better than alternative *C*, alternative *A* is not necessarily six times better than alternative *C*, in contrast to their indirect comparison through alternative *B*. Indeed, both empirical (Bozóki et al., 2013) and randomly generated (Csató, 2024; Csató & Petrőczy, 2021) pairwise comparison matrices are usually

inconsistent. The level of inconsistency is measured by inconsistency indices (Brunelli, 2018).

1.1. The need for inconsistency thresholds in the incomplete case

The numerical value of inconsistency is difficult to interpret on its own, without a sharp threshold that separates acceptable and unacceptable levels of inconsistency. In the latter case, the original comparisons should be revised, for which several inconsistency reduction methods have been suggested (Mazurek et al., 2021). For instance, Bozóki et al. (2015) formulate nonlinear mixed-integer optimisation problems to determine the minimal number of matrix elements to be changed in order to reduce the inconsistency below a given value.

The first inconsistency index and threshold of acceptability have been proposed by the founder of the AHP methodology (Saaty, 1977). It is based on dividing the inconsistency of the matrix by the average inconsistency of random pairwise comparison matrices, the so-called random index. Inconsistency can be tolerated if this ratio

* Corresponding author.

E-mail addresses: kolos.agoston@uni-corvinus.hu (K.C. Ágoston), laszlo.csato@sztaki.hun-ren.hu (L. Csató).

¹ Source: Benjamin et al. (2017, p. 1).

remains below 0.1, which directly provides a threshold for inconsistency.

Pairwise comparison matrices have a natural generalisation by allowing for missing entries (Harker, 1987). Incomplete pairwise comparison matrices extend the range of potential applications to hundreds, or even millions of alternatives. For example, the pairwise comparisons of contestants in sports are usually intransitive (van Ours, 2024, 2025, 2026) and may be easily incomplete if some pairs of contestants do not play against each other. Such case studies are presented by Bozóki et al. (2016), Chao et al. (2018), Csató (2013), Petróczy and Csató (2021), Temesi et al. (2024), among others. Analogously, neither completeness nor consistency holds for bilateral remittances (Petróczy, 2021) or students' preferences (Csató & Tóth, 2020).

Even though some inconsistency indices are available for incomplete pairwise comparison matrices (Kułakowski & Talaga, 2020; Szybowski et al., 2020), determining the associated thresholds is far from trivial. Naturally, an incomplete pairwise comparison matrix can be completed (Ágoston & Csató, 2024; Bozóki et al., 2010; Csató et al., 2024; Tekile et al., 2023), but the completion methods are usually based on *minimising* inconsistency. Consequently, the thresholds derived for complete matrices become too permissive for incomplete matrices: they will accept the inconsistency of several incomplete matrices, which would be rejected unless the decision-maker provides the (almost) optimal values for all missing entries.

1.2. The research question and our contribution

According to our knowledge, there exists only one proposal of inconsistency thresholds for incomplete pairwise comparison matrices. Ágoston and Csató (2022) adopt the idea of Saaty to compute the appropriate thresholds as a function of the number of alternatives and the number of missing entries. Unsurprisingly, the random index decreases if the number of unknown comparisons is higher because the inconsistency minimisation problem contains more variables, which generally implies a lower optimum.

Incomplete pairwise comparisons are often represented by undirected graphs where the vertices are the alternatives and the edges correspond to the known comparisons. The random indices, thus, the inconsistency thresholds depend on the number of vertices and edges (Ágoston & Csató, 2022). However, is it not possible that other properties of the representing graph should also be considered?

The current paper aims to address this issue. Following the methodology of Ágoston and Csató (2022), we compute the average inconsistency of a large number of random incomplete pairwise comparison matrices where not only the number of vertices and edges, but the graph itself is fixed, up to isomorphism. The inconsistency threshold turns out to be strongly related to the spectral radius of the graph. Furthermore, the structure of the edges influences the proportion of randomly generated matrices that have an acceptable level of inconsistency.

1.3. A motivating example

Consider the following incomplete pairwise comparison matrix:

$$A = \begin{bmatrix} 1 & 2 & \star & 5 \\ 1/2 & 1 & 4 & \star \\ \star & 1/4 & 1 & 2 \\ 1/5 & \star & 1/2 & 1 \end{bmatrix}.$$

Matrix A is inconsistent as $a_{12} \cdot a_{23} \cdot a_{34} = 2 \cdot 4 \cdot 2 = 16$, but $a_{14} = 5$. The dominant eigenvalue of the optimally completed matrix is $\lambda \approx 4.084$, resulting in an inconsistency index $CI \approx 0.0284$.

According to Ágoston and Csató (2022, Table 2), the random index $RI \approx 0.306$ if there are $n = 4$ alternatives and $m = 2$ missing elements. However, Table 1 reveals that the random index is $RI \approx 0.265$ if the

positions of the two missing entries (they are placed in different columns and rows) are taken into account, too.

Therefore, the inconsistency of matrix A can be tolerated based on Ágoston and Csató (2022). However, this inconsistency is too high if the famous 10% rule of Saaty is applied in a more sophisticated way. For pairwise comparison matrices with a graph representation analogous to the above example, the novel threshold changes the decision on 1165 matrices whose entries are drawn from the Saaty scale.

1.4. Practical relevance

Recently, optimal filling in patterns have been recommended for incomplete pairwise comparison matrices that provide the closest weight vectors on average to the complete case (Szádóczi & Bozóki, 2025; Szádóczi et al., 2023, 2022). When the pairwise comparisons are asked from the decision-makers in such a sequence, a large number of incomplete pairwise comparison matrices will have the same graph representation, see the experiment of Szádóczi et al. (2025). Then it is crucial not to use the naïve thresholds given by Ágoston and Csató (2022) since they assume random positions for the missing entries in the matrix. The exact thresholds provided in the current paper apply to all incomplete pairwise comparison matrices with the same graph representation; hence, the same value remains valid for matrices collected from different decision-makers, which does not complicate the analysis of inconsistency but makes it more accurate.

Our results are also important for inconsistency monitoring (Bozóki et al., 2011). Since the pairwise comparison matrices are always filled sequentially by the decision-makers, we have a new incomplete pairwise comparison matrix after each step. By continuously checking the inconsistency of these matrices, any serious violation of inconsistency can be immediately reported to the decision-maker. Therefore, it is possible to instantly correct any error or misprint, even before all pairwise comparisons are collected. This might be especially useful due to the mental and time constraints of the experts.

1.5. Structure

The paper is organised as follows. The methodology is described in Section 2. Our main findings are presented and discussed in Section 3, while Section 4 offers concluding remarks.

2. Methodology

Inconsistency and incomplete pairwise comparison matrices are introduced in Section 2.1. Section 2.2 presents the background of computing the random index, and Section 2.3 gives an overview of our numerical analysis.

2.1. Inconsistency and incomplete pairwise comparison matrices

Let n denote the number of alternatives. The pairwise comparisons of the alternatives are collected into a *pairwise comparison matrix* $A = [a_{ij}]$, $a_{ij} > 0$ means that the i th alternative is a_{ij} times more preferred to the j th alternative. The matrix satisfies the reciprocity property: $a_{ji} = 1/a_{ij}$ for all $1 \leq i, j \leq n$.

A pairwise comparison matrix A is *consistent* if and only if $a_{ik} = a_{ij} \cdot a_{jk}$ holds for all $1 \leq i, j, k \leq n$. In other words, the pairwise comparisons are consistent if the direct comparison of any two alternatives coincides with their indirect comparison through any third alternative. Otherwise, the matrix is called *inconsistent*.

According to the Perron–Frobenius theorem (Frobenius, 1908, 1909, 1912; Perron, 1907), a pairwise comparison matrix A has a dominant real eigenvalue $\lambda_{\max}(A)$. The inconsistency index of Thomas L. Saaty (Saaty, 1977) is:

$$CI(A) = \frac{\lambda_{\max}(A) - n}{n - 1}.$$

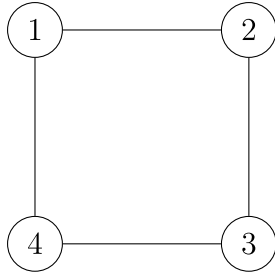


Fig. 1. The graph representation of the pairwise comparison matrix A in Example 1.

$CI(A) = 0$ if and only if the pairwise comparisons are consistent.

Saaty suggests comparing $CI(A)$ to the *random index* RI , the average CI of a large number of pairwise comparison matrices whose entries above the diagonal are drawn independently and uniformly from the following Saaty scale:

$$\{1/9, 1/8, 1/7, \dots, 1/2, 1, 2, \dots, 8, 9\}.$$

The values of the random index were computed several times as a function of the number of alternatives n (Aguarón & Moreno-Jiménez, 2003; Alonso & Lamata, 2006; Bozóki & Rapsák, 2008; Csató & Petrőczy, 2021; Ozdemir, 2005; Tummala & Ling, 1998).

The inconsistency index divided by the random index defines the inconsistency ratio:

$$CR(A) = \frac{CI(A)}{RI}.$$

Incomplete pairwise comparison matrices allow for some missing comparisons. Therefore, $A = [a_{ij}]$ is an *incomplete pairwise comparison matrix* if $a_{ij} > 0$ or $a_{ij} = \star$ such that $a_{ij} > 0$ implies $a_{ji} = 1/a_{ij}$ and $a_{ij} = \star$ implies $a_{ji} = \star$. The number of missing comparisons above the diagonal is denoted by m , hence, $|(i, j) : a_{ij} = \star| = 2m$. Incomplete pairwise comparisons have a natural graph representation. Let A be an incomplete pairwise comparison matrix. The associated undirected graph is $G = (V, E)$, where the vertices correspond to the alternatives and $e_{ij} \in E$ is an edge if and only if $i \neq j$ and the pairwise comparison a_{ij} is known.

The adjacency matrix $B = [b_{ij}]$ of graph $G = (V, E)$ is given by $b_{ij} = 1$ if $e_{ij} \in E$ and $b_{ij} = 0$ if $e_{ij} \notin E$. The *spectral radius* of a finite graph $G = (V, E)$ is the maximum of the absolute values of the eigenvalues of its adjacency matrix B .

Example 1. Take the following incomplete pairwise comparison matrix:

$$A = \begin{bmatrix} 1 & 2 & \star & 4 \\ 1/2 & 1 & 2 & \star \\ \star & 1/2 & 1 & 2 \\ 1/4 & \star & 1/2 & 1 \end{bmatrix}.$$

Its graph representation is given in Fig. 1. The spectral radius of this graph equals 2.

Understanding the properties of the spectral radius is an active research field in discrete mathematics (Stevanović, 2018). The spectral radius of a k -regular graph, where each vertex is connected to the same number of vertices k (has degree k), is equal to k . Denote the maximum and minimum degree of a graph G by $\Delta(G)$ and $\delta(G)$, respectively. Hong (1993) asked the following question more than three decades ago: If G has the smallest possible spectral radius among all graphs with n vertices and e edges, does $\Delta(G) - \delta(G) \leq 1$ hold, namely, is G almost regular? The answer seems to be positive, at least for dense graphs with $e \geq (n-1)(n-2)/2 - 2$ (Cioaba et al., 2024). Analogously, there are recent results on upper bounds of the spectral radius (Guo et al., 2019; Jin et al., 2024).

2.2. The eigenvalue minimisation problem and its solution

In order to complete an incomplete pairwise comparison matrix A , Shiraishi and Obata (2002) and Shiraishi et al. (1998) have proposed substituting the m missing entries above the diagonal with positive variables collected in the vector $x = [x_k]$, $1 \leq k \leq m$. Since inconsistency (CI) is a monotone function of the dominant eigenvalue, the following optimisation problem is worth considering:

$$\min_{x > 0} \lambda_{\max}(A(x)). \quad (1)$$

(1) can be transformed into a convex optimisation problem (Bozóki et al., 2010, Section 3). The necessary and sufficient condition for the existence of a unique solution is the connectedness of the undirected graph (Bozóki et al., 2010, Theorem 2). Therefore, we will focus on incomplete pairwise comparison matrices represented by a connected graph. Bozóki et al. (2010, Section 5) presents an algorithm to obtain the optimal solution to (1).

As discussed by Ágoston and Csató (2022), (at least) three possible approaches exist to calculate the random index for incomplete pairwise comparison matrices, depending on how the missing comparisons are related to the Saaty scale:

- Method 1: all positive variables x_k are unconstrained, the missing comparisons can be arbitrarily small or high for all $1 \leq k \leq m$;
- Method 2: all positive variables x_k are constrained by the bounds of the Saaty scale, $1/9 \leq x_k \leq 9$ for all $1 \leq k \leq m$;
- Method 3: all positive variables x_k are constrained directly by the Saaty scale, x_k should be either an integer from 1 to 9 or its reciprocal for all $1 \leq k \leq m$.

Similar to Ágoston and Csató (2022), Method 2 will be implemented to avoid numerical difficulties, as well as the problems caused by using a discrete scale for the variables in (1).

2.3. The computation approach

In contrast to Ágoston and Csató (2022), the positions of the missing entries are not chosen randomly for a given number of unknown comparisons m , but the associated graph is fixed in advance. This does not make any difference if $m = 1$ because all graphs with n vertices and $n(n-1)/2 - 1$ edges are isomorphic. However, it can be important if $m \geq 2$.

Consequently, the random index is computed as follows:

1. We choose a connected graph $G = (V, E)$ with n vertices and m missing edges compared to a complete graph;
2. We generate an incomplete pairwise comparison matrix A with graph representation G such that each element of the Saaty scale is drawn with a probability of $1/17$ for each known comparison above the diagonal;
3. We solve the minimisation problem (1) by restricting all variables in $x > 0$ between $1/9$ and 9 (Method 2 in Section 2.2), which gives a complete pairwise comparison matrix $\hat{A}$ as the optimal solution;
4. We compute and save the inconsistency index $CI(\hat{A})$;
5. We repeat Steps 2–4 to get 1 million random matrices with a given graph representation G ;
6. We compute the mean of the inconsistency indices CI from Step 4.

3. Results

Section 3.1 provides a comprehensive analysis in the case of four alternatives, and Section 3.2 focuses on pairwise comparison matrices with two missing entries. Matrices of size five and six are discussed in Section 3.3. Finally, Section 3.4 presents an approximation formula based on the number of alternatives, the number of missing elements, and the value of the spectral radius that can be used without costly computations even if $n \geq 7$.

Table 1

Random indices and their implications for incomplete pairwise comparison matrices of size four.

(a) Naive random index RI that is independent of the graph (Ágoston & Csató, 2022)		
	Graph 1	Graph 2
Random index	0.3061	0.3061
Acceptable matrices ($CI \leq 0.1$)	14,789	12,095
Unacceptable matrices ($CI > 0.1$)	68,732	71,426
Ratio of acceptable matrices	17.71%	14.48%
(b) Sophisticated random index RI that depends on the graph (this paper)		
	Graph 1	Graph 2
Random index	0.2646	0.3165
Acceptable matrices ($CI \leq 0.1$)	13,633	12,343
Unacceptable matrices ($CI > 0.1$)	69,888	71,178
Ratio of acceptable matrices	16.32%	14.78%

3.1. Incomplete matrices of size four

If $n = 4$, only the case $m = 2$ unknown entries is interesting. For $m = 1$, only one graph exists since $a_{14} = \star$ can be assumed without loss of generality. For $m = 3$, a connected graph is necessarily a spanning tree, and the incomplete pairwise comparison matrix has a consistent completion if the missing entries remain unconstrained.

If $m = 2$, there are two possible graphs: (1) the two missing edges are independent, they do not share any vertex (Fig. 1); (2) the two missing edges have a common vertex. The probability of case (1) is

$$\frac{n(n-1)/2 - (n-2 + n-2 + 1)}{n(n-1)/2 - 1} = 1 - \frac{4n-8}{n(n-1)-2},$$

since there exist $n(n-1)/2 - 1$ different edges after one is missing, and the missing edge has $n-2$ neighbouring edges at both of its vertices. This value equals only $1/5 = 20\%$ for $n = 4$, but converges to 1 when n increases.

The value of the random index RI can be computed exactly: the number of possible scenarios is $17^4 = 83,521$ for both graphs due to the four known entries and the 17 elements of the Saaty scale. Hence, in contrast to what is done in Ágoston and Csató (2022), we carry out a complete enumeration instead of generating 1 million random matrices.

Table 1 reports the values of the refined random indices and their implications. The structure of the graph has a non-negligible effect; RI for Graph 1 is smaller by more than 15% compared to RI for Graph 2. Ignoring the graph misclassifies 1156 and 248 incomplete pairwise comparison matrices in the two possible cases, respectively. Even though the standard CI threshold of 0.1 becomes more (less) restrictive for Graph 1 (Graph 2) if the structure of the graph is taken into account, the number of matrices with an acceptable inconsistency is still higher for Graph 2 than for Graph 1. Considering all possible matrices allows us to exactly compute the difference between Methods 1–3 mentioned in Section 2.2, too. If the missing entries are continuous and not constrained to the interval $[1/9, 9]$ (Method 1), a closed formula exists for the optimal completion if $n = 4$: Csató et al. (2024) prove that the incomplete eigenvector and the incomplete logarithmic least squares methods (Bozóki et al., 2010; Bozóki & Tsyganok, 2019; Kwiesielewicz, 1996) imply the same optimal completion, and the optimal completion according to the incomplete logarithmic least squares method can be obtained as a solution of a linear system of equations (Bozóki et al., 2010; Kaiser & Serlin, 1978). Thus, there is no need to use the algorithm suggested by Bozóki et al. (2010, Section 5) to solve optimisation problem (1). The random indices are 0.2528 and 0.2923 for Graphs 1 and 2, respectively, which are reduced by 7.6% and 4.4% compared to the values presented in Table 1.

If the missing entries are chosen from the discrete Saaty scale (Method 3), the random indices are 0.2663 and 0.3174 for Graphs 1 and 2, respectively, which are higher by 0.31% and 0.65% compared to the values presented in Table 1. Therefore, although our continuous

relaxation clearly lowers the random index, the extent of this decrease compared to using the discrete scale seems to be minimal.

3.2. Incomplete matrices with two missing elements

Fig. 2 shows the random indices for the two different incomplete pairwise comparison matrices with two missing entries as a function of the spectral radius of the representing graph if the number of alternatives is between four and nine. The plot has two important messages. First, a higher spectral radius is always associated with a higher value of RI if $m = 2$. Second, the difference between the two cases gradually disappears when the number of alternatives grows, that is, the random indices provided in Ágoston and Csató (2022) become less distorted. This is not surprising since fixing the number of unknown comparisons means that the ratio of known elements increases together with n .

3.3. Incomplete matrices of size five and six

If $n = 5$, all cases are investigated, that is, the number of missing entries varies from 1 to 5 because $m = 6$ implies the existence of a consistent completion (if the unknown entries remain unconstrained). Only one graph exists if $m = 1$, when the threshold given in Ágoston and Csató (2022) remains valid.

Fig. 3 presents the 16 possible graphs if $n = 5$ and $2 \leq m \leq 5$. The $1 + 2 + 4 + 5 + 5 = 17$ random indices—depending on the structure of the graph—are reported in Table A.1 in the Appendix, together with their probability of occurrence if the number of missing entries m is fixed. The probabilities are estimated by simulations, contrary to the exact values derived in Section 3.1. For a fixed value of m , the spectral radius is the smallest if the graph is (almost) regular ($G_{2,2}$, $G_{3,3}$, $G_{4,1}$, $G_{5,5}$). These graphs have the smallest probability to occur except for $G_{3,3}$, and the corresponding random index RI is lower than for any other graphs having the same number of edges.

If $n = 6$, we have computed the random indices only up to $m = 7$ as the number of different graphs increases with m ; for instance, 20 scenarios exist for $n = 6$ and $m = 6$ according to Fig. 4. The random indices are provided in Table A.2 in the Appendix. Again, the regular graph G_{20} in Fig. 4 occurs with the smallest probability and has the smallest value of RI if $m = 6$, and the other regular graph G_{18} has the second smallest random index.

Fig. 5 presents the random indices for incomplete pairwise comparison matrices of size five (Fig. 5a) and six (Fig. 5b), as a function of the spectral radius of the associated graph. The monotonic relationship between the spectral radius and RI breaks first when $n = 5$ and $m = 4$ (see also Table A.1). However, a higher spectral radius usually implies a higher random index even for $n = 5$ and $n = 6$. If the number of alternatives is six, the spectral radii can overlap for different values of missing entries m , but the random indices still monotonically decrease as a function of m . In other words, the number of variables in the optimisation problem (1) is more important than their relative position in the representing graph with respect to the optimum, which is intuitive.

Table 2 shows the extreme values of the random index RI for given pairs of parameters m and n . Unsurprisingly, the influence of the graph decreases if m is fixed and the number of alternatives grows, but increases if n is fixed and the number of missing entries increases. Among the 11 scenarios, the random index can be smaller by more than 10% in five cases, and the reduction may exceed 23% if both parameters are equal to five. This reinforces that the underlying graph cannot be ignored in the calculation of inconsistency thresholds for incomplete pairwise comparison matrices.

The strong association between the spectral radius of the representing graph and the random index seems to be unexpected. Hence, we provide some theoretical intuition for this relation. Table 3 presents the number of different triads (pairwise comparison matrices of size three) for the 20 graphs with $n = 6$ and $m = 6$ that can be seen in Fig. 4. Six

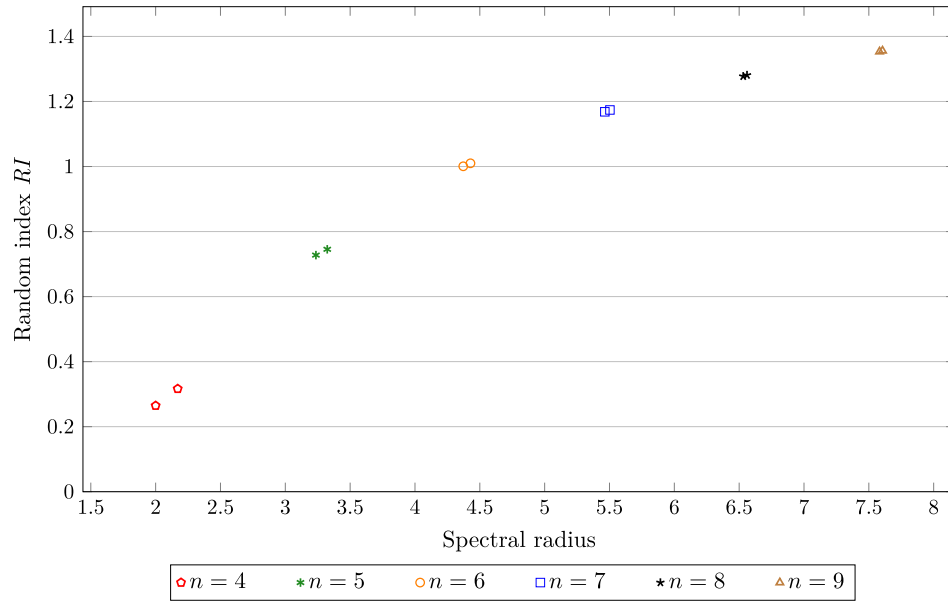


Fig. 2. Spectral radius and random index for incomplete pairwise comparison matrices with $m = 2$ missing entries.

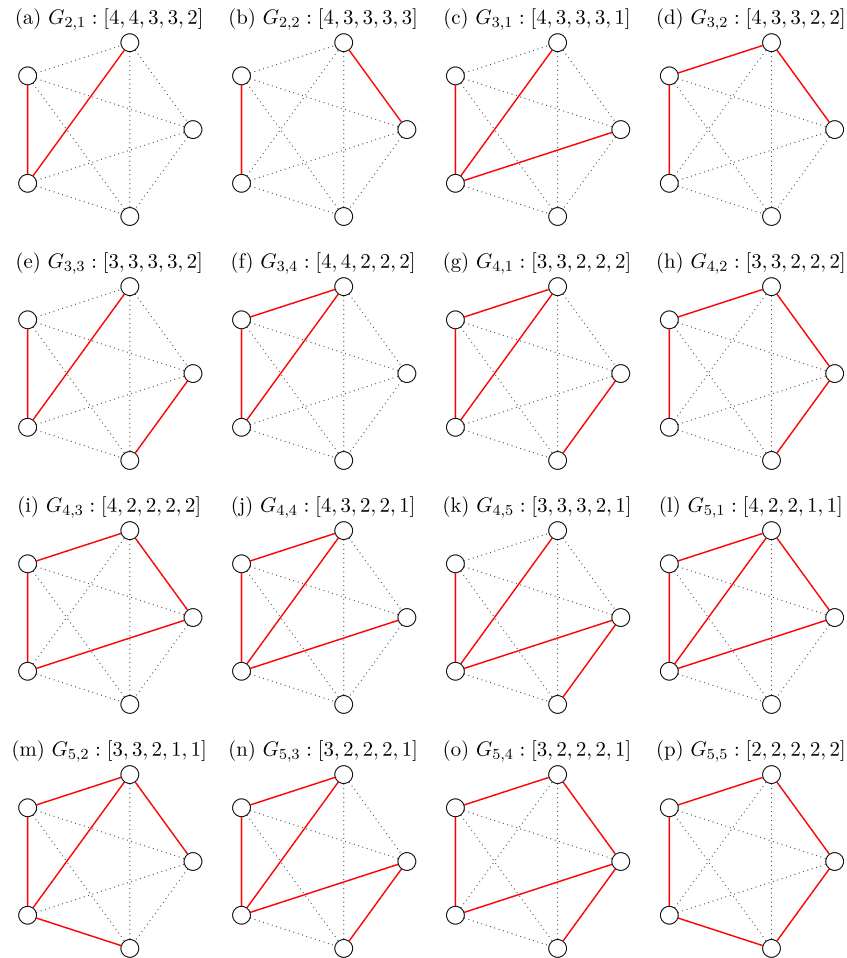


Fig. 3. Possible graph representations of incomplete pairwise comparison matrices of size five with at least two and at most five missing entries

Notes: Dotted lines indicate the known comparisons, thick red lines indicate the unknown comparisons. Graph $G_{i,j}$ has i missing edges. The vectors show the degree distribution of the associated graph. See Table A.1 for the spectral radii and random indices of these incomplete pairwise comparison matrices.

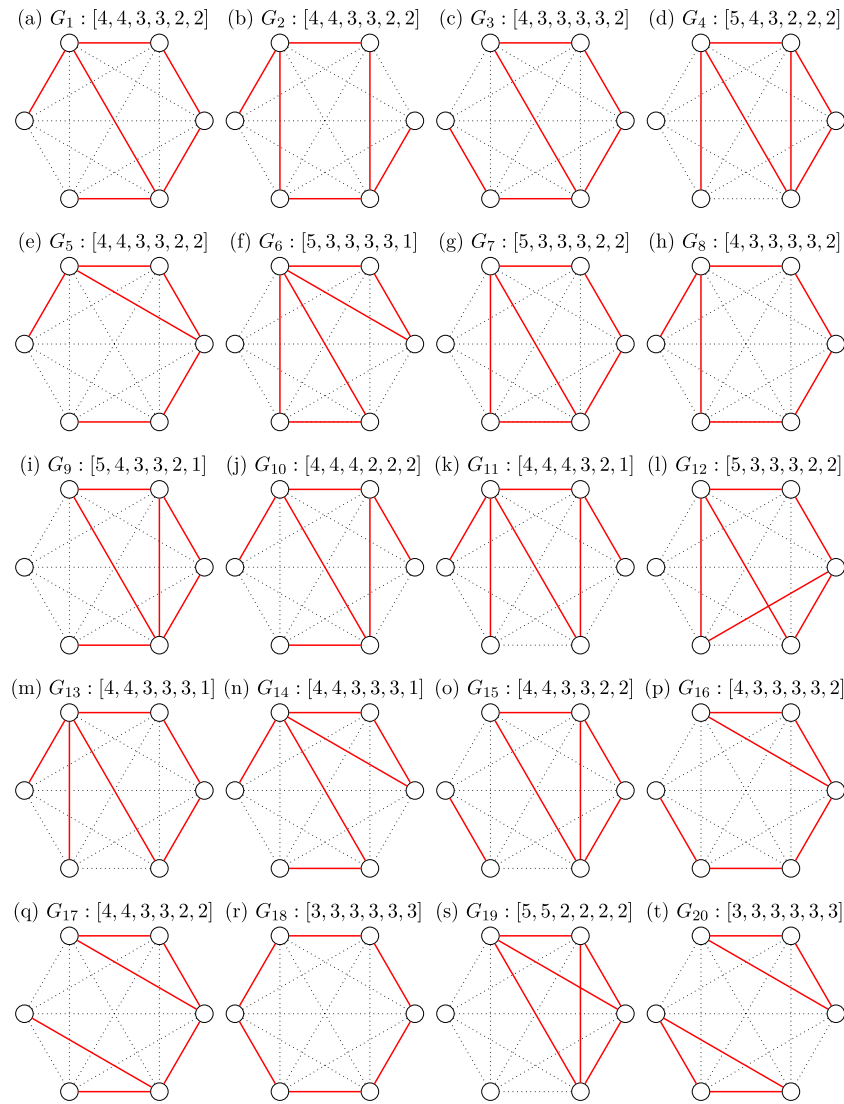


Fig. 4. Possible graph representations of incomplete pairwise comparison matrices of size six with six missing entries

Notes: Dotted lines indicate the known comparisons, thick red lines indicate the unknown comparisons. The vectors show the degree distribution of the associated graph. See Table A.2 for the spectral radii and random indices of these incomplete pairwise comparison matrices.

Table 2

Extreme random indices for incomplete pairwise comparison matrices of a fixed size n and number of missing entries m

n	m	Minimal RI	Maximal RI	Ratio
4	2	0.2646	0.3165	83.61%
5	2	0.7275	0.7454	97.60%
5	3	0.5377	0.5926	90.73%
5	4	0.3426	0.3952	86.70%
5	5	0.1717	0.2256	76.13%
6	2	1.0007	1.0099	99.09%
6	3	0.8658	0.8983	96.38%
6	4	0.7431	0.8059	92.20%
6	5	0.6126	0.6702	91.40%
6	6	0.4733	0.5402	87.61%
6	7	0.3562	0.4312	82.60%

alternatives imply 20 triads in all cases, but these triads are quite different with respect to the number of missing entries μ . For example, in graph G_1 , there are 4 triads without missing entries, and 8-8 triads with one and two missing entries, respectively. On the other hand, graph G_{20}

has 18 triads with one missing entry, and 2 triads with three missing entries.

Inconsistency is closely connected to triads since the definition of consistency involves three alternatives. Indeed, several inconsistency indices simply aggregate the inconsistencies of all triads (Brunelli, 2018). Even though the inconsistency index CI is not such a triad-based index, Cavallo (2020, Table 2c) uncovers an extremely strong monotonic increasing correlation between CI and the inconsistency index suggested by Peláez and Lamata (2003), which is the average determinant of all sub-matrices of order three, for $4 \leq n \leq 7$. Furthermore, there exists a unique inconsistency index (up to multiplication) if $n = 3$ (Csató, 2019); hence, CI is functionally related to any reasonable inconsistency index in the case of three alternatives.

Thus, reducing CI (the objective function of problem (1)) is essentially equivalent to minimising the inconsistency of the triads. However, the inconsistency of a triad cannot be decreased if it has no missing entries ($\mu = 0$), and this is difficult due to the appearance of the same pairwise comparison in $n - 2$ different triads if it has one missing entry ($\mu = 1$). Consequently, the optimum of optimisation problem (1) is expected to be lower if there are few triads with $\mu = 1$ and, especially, $\mu = 0$ missing entries.

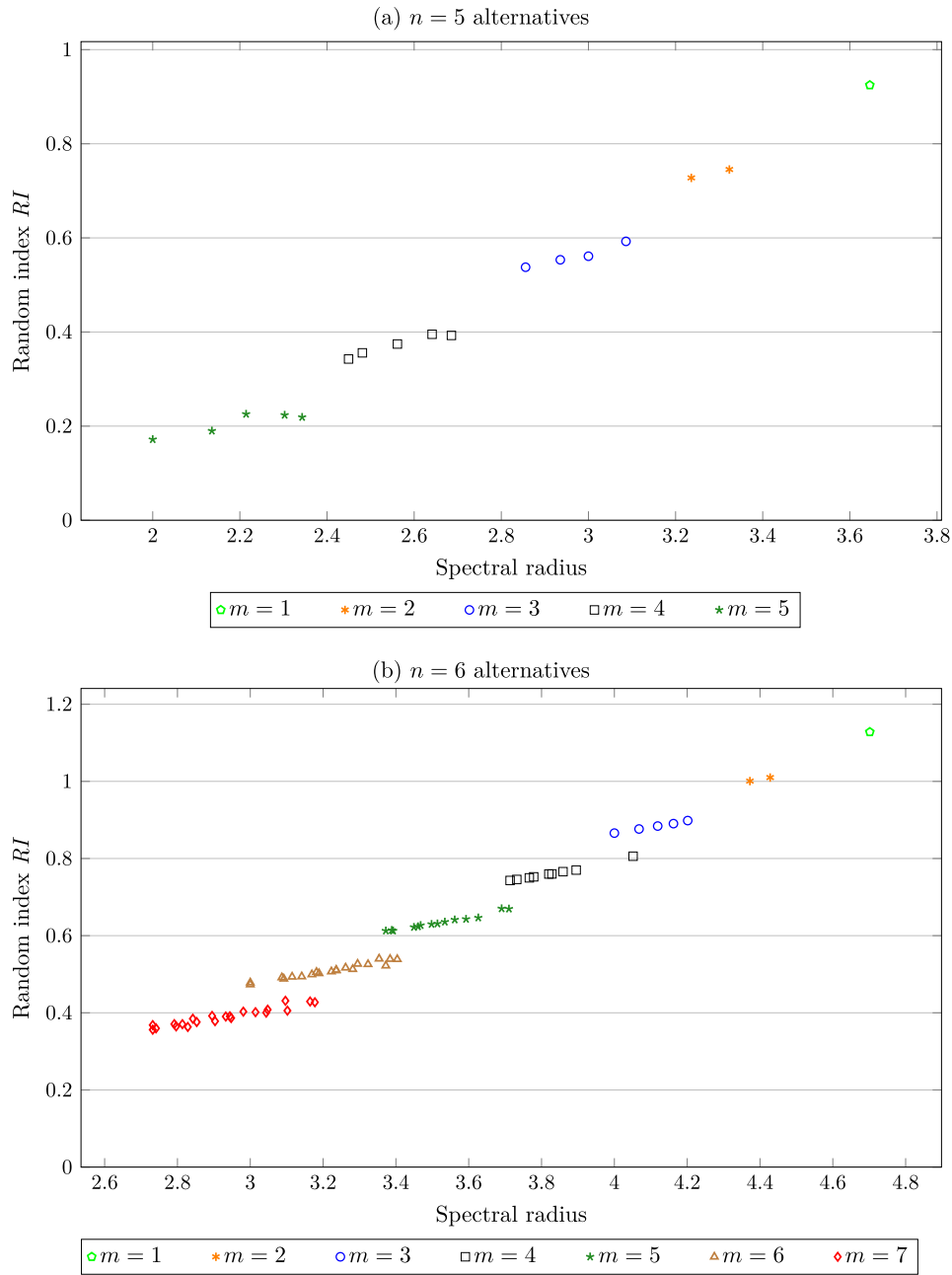


Fig. 5. Spectral radius and random index for incomplete matrices of size five and six.

In order to check the connection between the spectral radius and the distribution of triads, we have estimated two linear regressions by ordinary least squares (OLS). The dependent variable is the spectral radius, while the independent variables are the constant and the number of triads with $\mu = 0$ and $\mu = 1$ in both cases. The two samples are provided by graphs with $m = 6$ (see Table 3) and $m = 7$ when the number of alternatives is $n = 6$. In both regressions, all coefficients remain significantly positive even at the 0.1% level, and the value of adjusted R^2 exceeds 0.9. Hence, the spectral radius basically depends on the number of these two types of triads.

To summarise, the value of the spectral radius is probably related to the minimisation of the dominant eigenvalue in (1) through the distribution of triads with respect to the number of missing entries μ . A

higher spectral radius is associated with more triads that have at most one missing entry, when decreasing the average inconsistency of triads becomes more challenging. In addition, the optimal value of CI is almost functionally related to the average inconsistency of triads, as shown by Cavallo (2020).

Finally, Fig. 6 presents the relationship between the random index and the proportion of randomly generated incomplete pairwise comparison matrices with the same graph representation that have an acceptable inconsistency. The effect of the graph becomes more important if the number of missing entries m is higher. Interestingly, a smaller (more restrictive) random index may lead to a higher probability of acceptable inconsistency, similar to what can be seen in Table 1 for $n = 4$ alternatives.

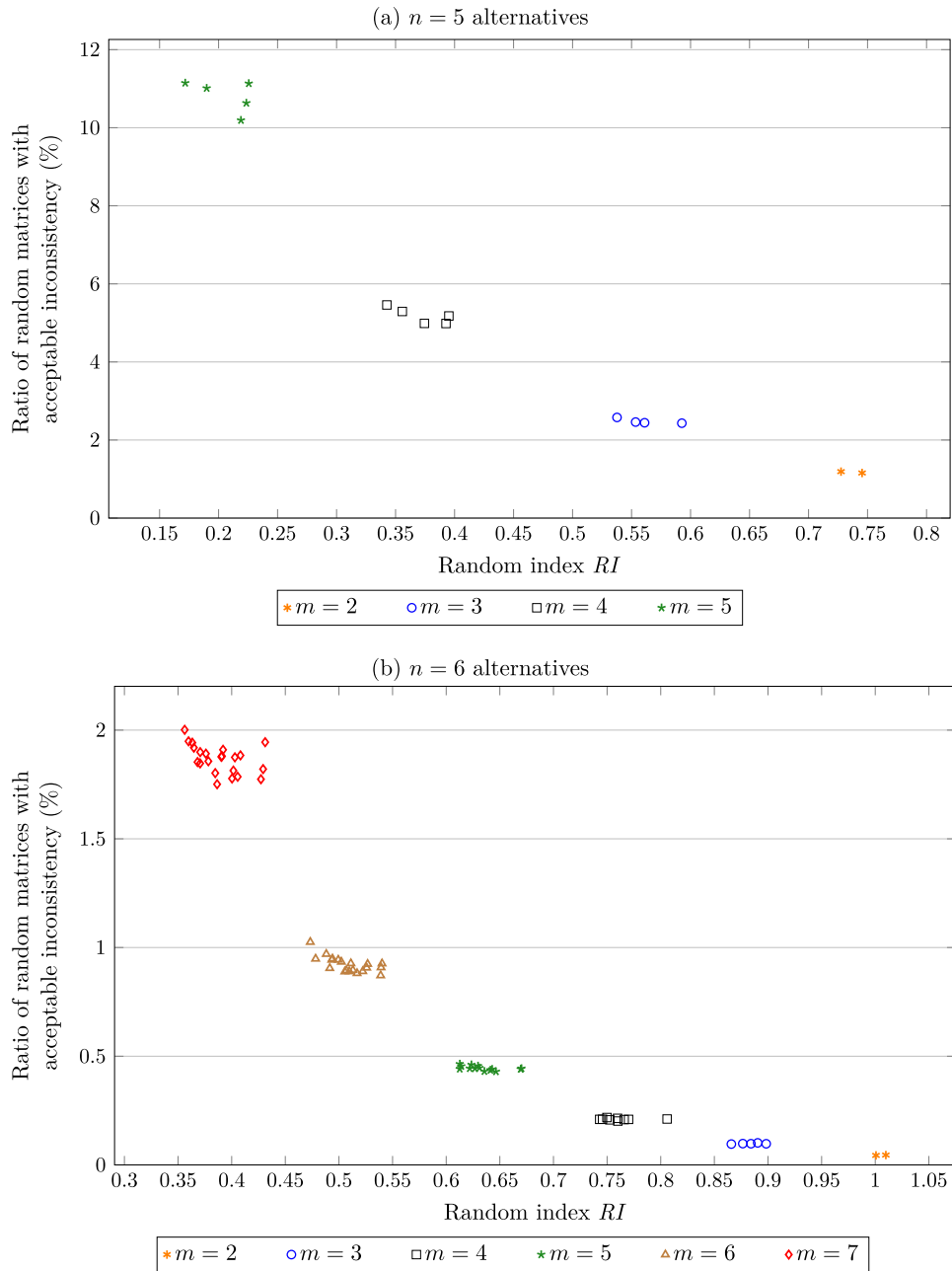


Fig. 6. Random index and the probability of acceptable inconsistency for incomplete matrices of size five and six.

3.4. An approximation formula for the random index

The computations in Section 3.3 are quite complex and time-consuming, which calls for a heuristic predictor of the random value based on the spectral radius ρ .

Let $\mathcal{G}(n, m)$ be the set of graphs for a given number of alternatives n and missing comparisons m . Let $\rho_{n,m}$ and $RI_{n,m}$ be the expected values of the spectral radius and random index for a fixed pair of n and m , respectively. For any graph $G_i \in \mathcal{G}(n, m)$, define

$$\Delta\rho_i = \rho(G_i) - \rho_{n,m}, \text{ and}$$

$$\Delta RI_i = RI(G_i) - RI_{n,m}.$$

We estimate the linear regression $\Delta RI_i = \beta \Delta\rho_i$ by ordinary least squares (OLS) for the 73 observations provided by $n = 6$ and $1 \leq m \leq 7$ in Table A.2 in the Appendix. This gives $\beta = 0.16119$. However, it is not guaranteed that $\beta = 0.16119$ remains valid if the number of alternatives is

$n \geq 7$. For example, $\beta = 0.19219$ if the sample is the set of 5×5 matrices, namely, Table A.1. Indeed, the regression line is apparently steeper in Fig. 5a than in Fig. 5b, but the number of observations is only 17 in the first case. Thus, $\beta = 0.16119$ is assumed to hold for any $n \geq 7$ in the following.

Then, for any graph $G_i \in \mathcal{G}(n, m)$, the random index RI_i is approximated by:

$$RI_i \approx RI_{n,m} + \beta \Delta\rho_i. \quad (2)$$

In formula (2), $\Delta\rho_i$ depends on ρ_i (the spectral radius of graph G_i) and $\rho_{n,m}$. The latter is estimated by generating 100 thousand random pairwise comparison matrices for each pair of parameters n and m , and computing the average spectral radius if the corresponding graph is connected, which is not guaranteed for a high number of missing comparisons m . These values are presented in Table A.3 in the Appendix for

Table 3

Spectral radius and the distribution of triads for graphs representing incomplete pairwise comparison matrices of size $n = 6$ with $m = 6$ missing entries.

Graph	Spectral radius	Number of triads with missing entries μ			
		$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
G_1	3.1819	4	8	8	0
G_2	3.2361	4	8	8	0
G_3	3.0868	3	10	7	0
G_4	3.2814	4	10	4	2
G_5	3.1692	3	11	5	1
G_6	3.3234	4	10	4	2
G_7	3.2227	4	9	6	1
G_8	3.1149	3	10	7	0
G_9	3.4037	5	8	5	2
G_{10}	3.2361	4	9	6	1
G_{11}	3.3839	5	7	7	1
G_{12}	3.2618	5	6	9	0
G_{13}	3.3539	5	6	9	0
G_{14}	3.2948	4	9	6	1
G_{15}	3.1413	2	14	2	2
G_{16}	3.0922	2	13	4	1
G_{17}	3.1888	3	11	5	1
G_{18}	3	2	12	6	0
G_{19}	3.3723	4	12	0	4
G_{20}	3	0	18	0	2

Notes: Each incomplete pairwise comparison matrix has 20 triads. The graphs are depicted in Fig. 4.

$6 \leq n \leq 9$. Note that this calculation does not require any specific algorithm or solving any optimisation problem.

On the other hand, if $n \geq 7$, the values of $RI_{n,m}$ are neither provided here, nor in Ágoston and Csató (2022) for all m . However, Ágoston and Csató (2022) present an approximation based on RI_n that is available for all plausible number of alternatives n (Aguarón & Moreno-Jiménez, 2003, Table 1):

$$RI_{n,m} \approx \left[1 - \frac{2m}{(n-1)(n-2)} \right] RI_n. \quad (3)$$

We have checked formula (2) for a pairwise comparison matrix with $n = 8$ alternatives, $m = 5$ missing entries, and a spectral radius of $\rho_i = 6.097$. The expected value of the spectral radius is $\rho_{8,5} = 5.862$ (see Table A.3 in the Appendix), which leads to $\Delta\rho_i = 0.235$. The value of $RI_{n,m}$ is estimated from (3) as 1.07 since $RI_n = 1.404$. Hence, our heuristic predictor results in $RI_i = 1.07 + 0.16119 \cdot 0.235 = 1.1076$. The correct random index, given by the methodology described in Section 2.3, equals 1.117. Consequently, the error of the approximation is within 1% in this particular case.

4. Conclusions

Our paper provides refined random indices for incomplete pairwise comparison matrices in order to determine the threshold of acceptability for their inconsistency. The values of Ágoston and Csató (2022) are made more accurate by considering not only the number of alternatives and missing entries, but also the graph representing the known comparisons. We show that the graph structure has a non-negligible effect and the random index is in a strict—albeit not perfect—relationship with the spectral radius of the corresponding graph. Furthermore, the underlying graph influences the ratio of randomly generated pairwise comparison matrices whose inconsistency can be accepted, which might question the validity of the 10% rule of Saaty.

The research is far from closed. First, the random indices are fully computed only up to five alternatives due to the complexity of the numerical calculations. Second, even though the association with the spectral radius is surprisingly strong, other measures of graph “robustness” may provide an even better fit.

CRedit authorship contribution statement

Kolos Csaba Ágoston: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – review & editing, Project administration; **László Csató:** Conceptualization, Methodology, Validation, Investigation, Writing – original draft, Writing – review & editing, Visualization, Supervision, Project administration, Funding acquisition.

Data Availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We acknowledge the Digital Government Development and Project Management Ltd. for awarding us access to the Komondor HPC facility based in Hungary.

We are grateful to Zsombor Szádóczi for useful advice.

Two reviewers and three anonymous colleagues provided valuable comments and suggestions on earlier drafts.

The research was supported by the National Research, Development and Innovation Office under Grants Advanced 152220 and FK 145838, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

Appendix A.

Table A.1

Random indices for incomplete pairwise comparison matrices of size $n = 5$.

Value of m	Graph	Probability	Spectral radius	Random index RI	Ratio of acc.
1	1	100.00%	3.6458	0.9246	—
2	1	67.05%	3.3234	0.7454	1.16%
2	2	32.95%	3.2361	0.7275	1.19%
3	1	16.92%	3.0861	0.5926	2.43%
3	2	49.49%	2.9354	0.5535	2.46%
3	3	25.28%	2.8558	0.5377	2.58%
3	4	8.31%	3	0.5611	2.44%
4	1	4.62%	2.4495	0.3426	5.46%
4	2	29.59%	2.4812	0.3557	5.29%
4	3	6.89%	2.5616	0.3745	4.99%
4	4	29.05%	2.6855	0.3927	4.98%
4	5	29.85%	2.6412	0.3952	5.18%
5	1	12.92%	2.3429	0.2190	10.19%
5	2	27.28%	2.3028	0.2235	10.63%
5	3	27.31%	2.1358	0.1899	11.01%
5	4	26.76%	2.2143	0.2256	11.13%
5	5	5.72%	2	0.1717	11.15%

Notes: The column Probability shows the probability that the given graph occurs if all missing comparisons are placed randomly for a fixed pair of parameters m and n . The column Ratio of acc. shows the probability that an incomplete pairwise comparison matrix with the given graph representation, whose known entries are drawn uniformly from the Saaty scale, has an acceptable level of inconsistency, for a fixed pair of parameters m and n . If the spectral radius is equal to an integer k , the graph is k -regular.

Table A.2Random indices for incomplete pairwise comparison matrices of size $n = 6$.

Value of m	Graph	Probability	Spectral radius	Random index RI	Ratio of acc.
1	1	100%	4.7016	1.1280	—
2	1	56.87%	4.4279	1.0099	0.05%
2	2	43.13%	4.3723	1.0007	0.04%
3	1	13.16%	4.2015	0.8983	0.10%
3	2	39.95%	4.1190	0.8841	0.10%
3	3	3.34%	4	0.8658	0.10%
3	4	4.39%	4.1623	0.8904	0.10%
3	5	39.16%	4.0678	0.8765	0.10%
4	1	2.10%	4.0514	0.8059	0.21%
4	2	6.60%	3.7321	0.7457	0.21%
4	3	4.50%	3.7664	0.7498	0.22%
4	4	26.75%	3.8590	0.7659	0.21%
4	5	25.76%	3.7785	0.7525	0.21%
4	6	13.58%	3.7136	0.7431	0.21%
4	7	4.31%	3.8201	0.7597	0.21%
4	8	13.12%	3.8951	0.7700	0.21%
4	9	3.28%	3.8284	0.7600	0.20%
5	1	12.48%	3.4679	0.6267	0.44%
5	2	12.35%	3.5344	0.6356	0.43%
5	3	11.74%	3.5926	0.6429	0.44%
5	4	12.15%	3.4979	0.6297	0.46%
5	5	6.00%	3.7105	0.6694	0.44%
5	6	11.81%	3.5141	0.6310	0.45%
5	7	2.07%	3.4495	0.6219	0.44%
5	8	11.80%	3.3885	0.6141	0.45%
5	9	3.17%	3.6262	0.6464	0.43%
5	10	6.05%	3.4609	0.6235	0.46%
5	11	3.92%	3.6903	0.6702	0.44%
5	12	1.51%	3.3723	0.6126	0.44%
5	13	3.00%	3.5616	0.6412	0.44%
5	14	1.95%	3.3923	0.6127	0.46%
6	1	7.38%	3.1819	0.5055	0.89%
6	2	3.92%	3.2361	0.5111	0.93%
6	3	7.48%	3.0868	0.4915	0.90%
6	4	7.24%	3.2814	0.5130	0.90%
6	5	14.59%	3.1692	0.4994	0.94%
6	6	2.01%	3.3234	0.5259	0.91%
6	7	6.87%	3.2227	0.5069	0.89%
6	8	6.92%	3.1149	0.4933	0.94%
6	9	7.18%	3.4037	0.5389	0.87%
6	10	2.36%	3.2361	0.5097	0.89%
6	11	7.14%	3.3839	0.5393	0.91%
6	12	1.44%	3.2618	0.5169	0.88%
6	13	3.28%	3.3539	0.5402	0.93%
6	14	7.54%	3.2948	0.5267	0.92%
6	15	1.93%	3.1413	0.4940	0.95%
6	16	7.34%	3.0922	0.4882	0.97%
6	17	3.64%	3.1888	0.5023	0.93%
6	18	1.23%	3	0.4782	0.95%
6	19	0.32%	3.3723	0.5222	0.89%
6	20	0.16%	3	0.4733	1.03%

Table A.2
Continued.

Value of m	Graph	Probability	Spectral radius	Random index RI	Ratio of acc.
7	1	10.93%	2.9809	0.4030	1.87%
7	2	11.70%	3.0143	0.4016	1.81%
7	3	2.87%	3.1642	0.4293	1.82%
7	4	2.92%	2.8951	0.3920	1.91%
7	5	6.32%	2.9439	0.3909	1.88%
7	6	11.43%	2.8529	0.3760	1.89%
7	7	2.95%	2.7913	0.3707	1.90%
7	8	5.84%	2.8136	0.3706	1.85%
7	9	2.10%	3.1020	0.4056	1.79%
7	10	5.95%	2.9327	0.3903	1.88%
7	11	3.07%	2.9032	0.3783	1.86%
7	12	1.88%	3.0965	0.4312	1.94%
7	13	2.87%	3.0478	0.4082	1.88%
7	14	5.59%	3.0437	0.4003	1.78%
7	15	2.83%	2.8422	0.3847	1.80%
7	16	6.34%	2.7964	0.3648	1.92%
7	17	2.82%	2.7321	0.3682	1.85%
7	18	0.87%	3.1774	0.4273	1.77%
7	19	5.76%	2.7411	0.3598	1.95%
7	20	1.62%	2.7321	0.3562	2.00%
7	21	0.28%	2.8284	0.3633	1.94%
7	22	3.07%	2.9474	0.3864	1.75%

Notes: The column Probability shows the probability that the given graph occurs if all missing comparisons are placed randomly for a fixed pair of parameters m and n . The column Ratio of acc. shows the probability that an incomplete pairwise comparison matrix with the given graph representation, whose known entries are drawn uniformly from the Saaty scale, has an acceptable level of inconsistency, for a fixed pair of parameters m and n . If the spectral radius is equal to an integer k , the graph is k -regular.

Table A.3

Expected value of the spectral radius for graphs representing incomplete pairwise comparison matrices of size $6 \leq n \leq 9$.

Alternatives (n)	Missing comparisons (m)	Average spectral radius ($\rho_{n,m}$)	Connected graphs
6	1	4.7016	100,000
6	2	4.4040	100,000
6	3	4.1074	100,000
6	4	3.8124	100,000
6	5	3.5177	99,792
6	6	3.2217	98,801
6	7	2.9213	95,783
6	8	2.6127	88,763
6	9	2.2896	75,105
7	1	5.7417	100,000
7	2	5.4837	100,000
7	3	5.2263	100,000
7	4	4.9694	100,000
7	5	4.7129	100,000
7	6	4.4571	99,990
7	7	4.2014	99,896
7	8	3.9463	99,608
7	9	3.6899	98,838
7	10	3.4315	97,164
7	11	3.1712	93,953
7	12	2.9050	88,140
7	13	2.6308	78,316
7	14	2.3457	63,384
8	1	6.7720	100,000
8	2	6.5442	100,000
8	3	6.3166	100,000
8	4	6.0891	100,000
8	5	5.8621	100,000
8	6	5.6351	100,000
8	7	5.4085	100,000
8	8	5.1821	99,998
8	9	4.9564	99,976
8	10	4.7312	99,916
8	11	4.5057	99,776
8	12	4.2802	99,460
8	13	4.0543	98,783
8	14	3.8271	97,623
8	15	3.5985	95,556
8	16	3.3669	92,188
8	17	3.1330	87,003
8	18	2.8932	79,219
8	19	2.6464	67,968
8	20	2.3937	53,336

Table A.3

Continued.

Alternatives (n)	Missing comparisons (m)	Average spectral radius ($\rho_{n,m}$)	Connected graphs
9	1	7.7958	100,000
9	2	7.5918	100,000
9	3	7.3879	100,000
9	4	7.1841	100,000
9	5	6.9803	100,000
9	6	6.7768	100,000
9	7	6.5732	100,000
9	8	6.3699	100,000
9	9	6.1669	100,000
9	10	5.9641	99,999
9	11	5.7615	99,995
9	12	5.5589	99,991
9	13	5.3568	99,970
9	14	5.1547	99,919
9	15	4.9525	99,814
9	16	4.7507	99,639
9	17	4.5486	99,272
9	18	4.3461	98,727
9	19	4.1433	97,796
9	20	3.9390	96,356
9	21	3.7332	94,115
9	22	3.5260	90,745
9	23	3.3160	85,917
9	24	3.1026	79,167
9	25	2.8848	69,962
9	26	2.6613	58,357
9	27	2.4324	44,760

Notes: 100 thousand random pairwise comparison matrices are generated for each pair of parameters m and n , but the average spectral radius is computed only from connected graphs. The column Connected graphs shows the number of connected graphs for a fixed pair of parameters m and n .

References

- Ágoston, K. Cs., & Csató, L. (2022). Inconsistency thresholds for incomplete pairwise comparison matrices. *Omega*, 108, 102576.
- Ágoston, K. Cs., & Csató, L. (2024). A lexicographically optimal completion for pairwise comparison matrices with missing entries. *European Journal of Operational Research*, 314(3), 1078–1086.
- Aguarón, J., & Moreno-Jiménez, J. M. (2003). The geometric consistency index: Approximated thresholds. *European Journal of Operational Research*, 147(1), 137–145.
- Alonso, J. A., & Lamata, M. T. (2006). Consistency in the analytic hierarchy process: A new approach. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 14(4), 445–459.
- Benjamin, A., Chartrand, G., & Zhang, P. (2017). *The fascinating world of graph theory*. Princeton, New Jersey, USA: Princeton University Press.
- Bozóki, S., Csató, L., & Temesi, J. (2016). An application of incomplete pairwise comparison matrices for ranking top tennis players. *European Journal of Operational Research*, 248(1), 211–218.
- Bozóki, S., Dezső, L., Poesz, A., & Temesi, J. (2013). Analysis of pairwise comparison matrices: An empirical research. *Annals of Operations Research*, 211(1), 511–528.
- Bozóki, S., Fülöp, J., & Koczkodaj, W. W. (2011). An LP-based inconsistency monitoring of pairwise comparison matrices. *Mathematical and Computer Modelling*, 54(1–2), 789–793.
- Bozóki, S., Fülöp, J., & Poesz, A. (2015). On reducing inconsistency of pairwise comparison matrices below an acceptance threshold. *Central European Journal of Operations Research*, 23(4), 849–866.
- Bozóki, S., Fülöp, J., & Rónyai, L. (2010). On optimal completion of incomplete pairwise comparison matrices. *Mathematical and Computer Modelling*, 52(1–2), 318–333.
- Bozóki, S., & Rapcsák, T. (2008). On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices. *Journal of Global Optimization*, 42(2), 157–175.
- Bozóki, S., & Tsyganok, V. (2019). The (logarithmic) least squares optimality of the arithmetic (geometric) mean of weight vectors calculated from all spanning trees for incomplete additive (multiplicative) pairwise comparison matrices. *International Journal of General Systems*, 48(4), 362–381.
- Brunelli, M. (2018). A survey of inconsistency indices for pairwise comparisons. *International Journal of General Systems*, 47(8), 751–771.
- Cavallo, B. (2020). Functional relations and Spearman correlation between consistency indices. *Journal of the Operational Research Society*, 71(2), 301–311.
- Chao, X., Kou, G., Li, T., & Peng, Y. (2018). Jie Ke versus AlphaGo: A ranking approach using decision making method for large-scale data with incomplete information. *European Journal of Operational Research*, 265(1), 239–247.
- Cioaba, S. M., Gupta, V., & Marques, C. (2024). On the minimum spectral radius of connected graphs of given order and size. *Special Matrices*, 12(1), 20240027.
- Csató, L. (2013). Ranking by pairwise comparisons for Swiss-system tournaments. *Central European Journal of Operations Research*, 21(4), 783–803.
- Csató, L. (2019). Axiomatizations of inconsistency indices for triads. *Annals of Operations Research*, 280(1–2), 99–110.
- Csató, L. (2024). Right-left asymmetry of the eigenvector method: A simulation study. *European Journal of Operational Research*, 313(2), 708–717.
- Csató, L., Ágoston, K. Cs., & Bozóki, S. (2024). On the coincidence of optimal completions for small pairwise comparison matrices with missing entries. *Annals of Operations Research*, 331(1), 239–247.
- Csató, L., & Petróczy, D. G. (2021). On the monotonicity of the eigenvector method. *European Journal of Operational Research*, 292(1), 230–237.
- Csató, L., & Tóth, C. (2020). University rankings from the revealed preferences of the applicants. *European Journal of Operational Research*, 286(1), 309–320.
- Frobenius, G. (1908). Über Matrizen aus positiven Elementen I. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, (pp. 471–476).
- Frobenius, G. (1909). Über Matrizen aus positiven Elementen II. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, (pp. 514–518).
- Frobenius, G. (1912). Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, (pp. 456–477).
- Guo, J.-M., Wang, Z.-W., & Li, X. (2019). Sharp upper bounds of the spectral radius of a graph. *Discrete Mathematics*, 342(9), 2559–2563.
- Harker, P. T. (1987). Alternative modes of questioning in the analytic hierarchy process. *Mathematical Modelling*, 9(3–5), 353–360.
- Hong, Y. (1993). Bounds of eigenvalues of graphs. *Discrete Mathematics*, 123(1–3), 65–74.
- Jin, Y.-L., Zhang, J., & Zhang, X. D. (2024). Upper bounds of spectral radius of symmetric matrices and graphs. *Linear Algebra and its Applications*, 682, 152–163.
- Kaiser, H. F., & Serlin, R. C. (1978). Contributions to the method of paired comparisons. *Applied Psychological Measurement*, 2(3), 423–432.
- Kulakowski, K., & Talaga, D. (2020). Inconsistency indices for incomplete pairwise comparisons matrices. *International Journal of General Systems*, 49(2), 174–200.
- Kwiesielewicz, M. (1996). The logarithmic least squares and the generalized pseudoinverse in estimating ratios. *European Journal of Operational Research*, 93(3), 611–619.
- Mazurek, J., Perzina, R., Strzałka, D., Kowal, B., & Kuraś, P. (2021). A numerical comparison of iterative algorithms for inconsistency reduction in pairwise comparisons. *IEEE Access*, 9, 62553–62561.
- Ozdemir, M. S. (2005). Validity and inconsistency in the analytic hierarchy process. *Applied Mathematics and Computation*, 161(3), 707–720.
- Peláez, J. I., & Lamata, M. T. (2003). A new measure of consistency for positive reciprocal matrices. *Computers & Mathematics with Applications*, 46(12), 1839–1845.
- Perron, O. (1907). Zur theorie der matrices. *Mathematische Annalen*, 64(2), 248–263.
- Petróczy, D. G. (2021). An alternative quality of life ranking on the basis of remittances. *Socio-Economic Planning Sciences*, 78, 101042.
- Petróczy, D. G., & Csató, L. (2021). Revenue allocation in formula one: A pairwise comparison approach. *International Journal of General Systems*, 50(3), 243–261.
- Saaty, T. L. (1977). A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15(3), 234–281.
- Saaty, T. L. (1980). *The analytic hierarchy process: Planning, priority setting, resource allocation*. New York: McGraw-Hill.
- Shiraishi, S., & Obata, T. (2002). On a maximization problem arising from a positive reciprocal matrix in AHP. *Bulletin of Informatics and Cybernetics*, 34(2), 91–96.
- Shiraishi, S., Obata, T., & Daigo, M. (1998). Properties of a positive reciprocal matrix and their application to AHP. *Journal of the Operations Research Society of Japan*, 41(3), 404–414.
- Stevanović, D. (2018). Spectral radius of graphs. In A. M. Encinas, & M. Mitjana (Eds.), *Combinatorial matrix theory* (pp. 83–130). Cham, Switzerland: Springer.
- Szádóczki, Z., & Bozóki, S. (2025). Optimal sequences for pairwise comparisons: The graph of graphs approach. *Annals of Operations Research*, 353(3), 1099–1122.
- Szádóczki, Z., Bozóki, S., Juhász, P., Kadenko, S. V., & Tsyganok, V. (2023). Incomplete pairwise comparison matrices based on graphs with average degree approximately 3. *Annals of Operations Research*, 326(2), 783–807.
- Szádóczki, Z., Bozóki, S., Sipos, L., & Galambosi, Z. (2025). An experimental approach: Converting verbal expressions to numerical scales. Manuscript. DOI: 10.48550/arXiv.2507.04539.
- Szádóczki, Z., Bozóki, S., & Tekile, H. A. (2022). Filling in pattern designs for incomplete pairwise comparison matrices: (Quasi-)regular graphs with minimal diameter. *Omega*, 107, 102557.
- Szybowski, J., Kulakowski, K., & Prusak, A. (2020). New inconsistency indicators for incomplete pairwise comparisons matrices. *Mathematical Social Sciences*, 108, 138–145.
- Tekile, H. A., Brunelli, M., & Fedrizzi, M. (2023). A numerical comparative study of completion methods for pairwise comparison matrices. *Operations Research Perspectives*, 10, 100272.
- Temesi, J., Szádóczki, Z., & Bozóki, S. (2024). Incomplete pairwise comparison matrices: Ranking top women tennis players. *Journal of the Operational Research Society*, 75(1), 145–157.
- Tummala, V. M. R., & Ling, H. (1998). A note on the computation of the mean random consistency index of the analytic hierarchy process (AHP). *Theory and Decision*, 44, 221–230.
- van Ours, J. C. (2024). Nontransitive patterns in long-term football rivalries. *Journal of Sports Economics*, 25(7), 802–826.
- van Ours, J. C. (2025). On bogey teams and circular triads: Psychological factors in team performance. *Sports Economics Review*, 10, 100051.
- van Ours, J. C. (2026). Circular triads of overperformance in top European football leagues. *Sports Economics Review*, 14, 100070.