

Steering-Based Balancing Control of a Riderless Electric Scooter at Constant Speed with Emphasis on Pitch Dynamics

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Abstract: This study investigates the stabilization of a riderless electric scooter at constant forward speed using a steering-based control strategy, with emphasis on the handling of pitch dynamics. A spatial mechanical model, based on the Whipple bicycle model, is constructed, and the nonlinear equations of motion are derived using Kane's method. A hierarchical steering controller with feedback delay is applied to balance the upright position of the e-scooter. Stability is analyzed using a Nyquist-based and also an affine mapping method. The control gains are optimized by maximizing the spectral abscissa. The results show that the proposed controller significantly extends the stable speed range and ensures rapid decay of vibrations. Furthermore, alternative pitch handling approaches are compared, highlighting the trade-off between computational efficiency and accuracy in stability predictions.

Keywords: Micromobility, E-scooter, Pitch Dynamics, Delayed System, Steering Control, Stability Analysis

1. Introduction

The spread of micromobility vehicles (such as bicycles, electric scooters and electric unicycles) has become obvious over the past decade. On the one hand, e-scooters are environmentally-friendly and extremely cost-effective, offering a convenient solution to the “first and last mile problem”. On the other hand, they are inherently unstable and are involved in a rising number of road accidents.

The current understanding of micromobility dynamics is limited in several ways. The complex structure of the geometric and kinematic constraints leads to a set of nonlinear differential equations that cannot be handled analytically. Most studies rely on simplified in-plane models and are restricted to linear stability analysis and numerical methods. However, a fully spatial model is required to capture all relevant nonlinear phenomena.

In this study, we investigate the dynamics of a riderless e-scooter that is balanced at a constant forward speed using steering control. Although riderless operation may appear artificial, it allows us to investigate the dynamics of the e-scooter independently of the rider. Therefore, it can be useful for the development of autonomous micromobility functions (e.g., self-parking) and for establishing baseline stability characteristics. We use a spatial mechanical model that is based on the Whipple bicycle model⁽¹⁾. The nonlinear governing equations are derived using Kane's method⁽²⁾. A steering-based balancing control is introduced and the performance of the steering controller is evaluated using eigenvalue analysis and numerical simulations. The effect of pitch handling is examined.

2. Modelling and Governing Equations

The riderless e-scooter rolls on rigid, flat ground. We use a spatial mechanical model of the multibody system^(3,4), which consists of four rigid bodies: the front and the rear wheels, the body (also called the frame), and the handlebar and fork assembly, see Fig. 1. The spatial position of the e-scooter can be described with the X, Y , and Z coordinates of the center point of the rear wheel, the yaw angle ψ , the lean angle φ , the steering angle δ , the pitch angle ϑ , and the rotational angles ϕ_f and ϕ_r of the front and the rear wheels around their own axes.

We define six frames. The ground-fixed global reference frame

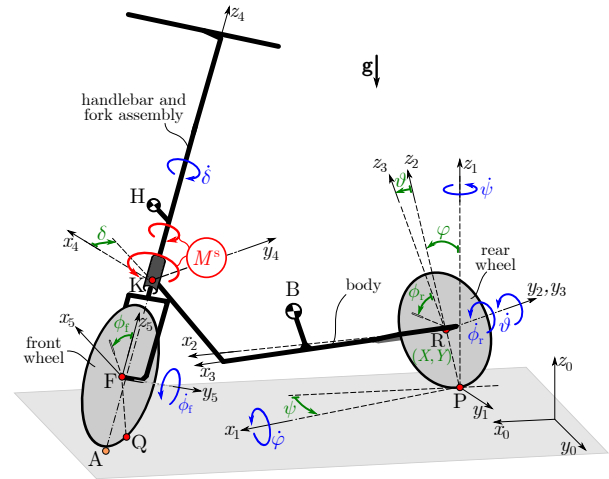


Fig. 1: The mechanical model of e-scooters^(3,4)

is referred to $\mathcal{F}_0$ with the coordinate system (x_0, y_0, z_0) . The other five reference frames $\mathcal{F}_k$ are provided by means of subsequent angular rotations: with the yaw angle ψ about the z_0 -axis, with the lean angle φ about the x_1 -axis, with the pitch angle ϑ about the y_2 -axis. Therefore, (x_3, y_3, z_3) , denoted by $\mathcal{F}_3$, is the body-fixed reference frame and (x_5, y_5, z_5) denoted by $\mathcal{F}_5$ is the reference frame attached to the front wheel. The geometric and inertial parameters of the e-scooter are shown in Fig. 2 and their reference values are summarized in Table 1.

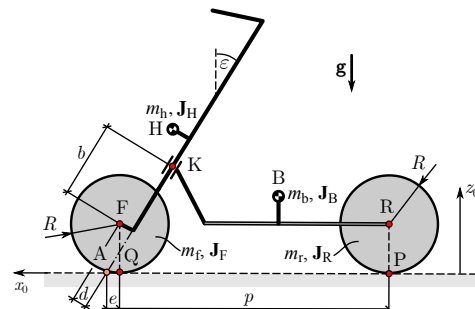


Fig. 2: The geometric and inertial parameters of the e-scooter

Table 1: Parameters of the e-scooter

Notation	Parameter name	Value
p	wheelbase	0.829 m
e	trail	0.0281 m
ε	rake angle	14.2°
g	gravitational acceleration	9.81 m/s ²
R	wheel radius	0.111 m
front wheel		
m_f	mass	2.894 kg
$\mathbf{J}_F$	mass moment of inertia in $\mathcal{F}_5$	$\begin{bmatrix} 62.8 & 0 & 0 \\ 0 & 113.4 & 0 \\ 0 & 0 & 62.8 \end{bmatrix}$ kg cm ²
rear wheel		
m_r	mass	1.136 kg
$\mathbf{J}_R$	mass moment of inertia in $\mathcal{F}_3$	$\begin{bmatrix} 29.0 & 0 & 0 \\ 0 & 53.6 & 0 \\ 0 & 0 & 29.0 \end{bmatrix}$ kg cm ²
body		
m_b	mass	9.5 kg
$\mathbf{J}_B$	mass moment of inertia in $\mathcal{F}_3$	$\begin{bmatrix} 1014 & 0 & -1074 \\ 0 & 4374 & 0 \\ -1074 & 0 & 3561 \end{bmatrix}$ kg cm ²
$\mathbf{r}_{RB}$	center of gravity in $\mathcal{F}_3$	$[0.4314, 0, 0.0782]^T$ m
handlebar		
m_h	mass	2.797 kg
$\mathbf{J}_H$	mass moment of inertia in $\mathcal{F}_5$	$\begin{bmatrix} 3412 & 0 & 1516 \\ 0 & 4018 & 0 \\ 1516 & 0 & 840.3 \end{bmatrix}$ kg cm ²
$\mathbf{r}_{KH}$	center of gravity in $\mathcal{F}_5$	$[0.01, 0, 0.2716]^T$ m
b	length	0.2418 m
d	offset	0 m

The front and rear wheels are attached to the ground, leading to two geometric constraints. The constraint equation ensuring that the rear wheel is on the ground can be used to eliminate coordinate Z as $Z = R \cos \varphi$. The constraint equation ensuring that the front wheel is on the ground allows us to express the pitch angle ϑ as a function of the lean φ and the steering δ angles. However, this equation for $\sin \vartheta$ is rather complicated, and its analytical solution is cumbersome. If one does not want to use this analytical solution, the geometric constraint equation for the pitch angle remains in the governing equations of the system as an algebraic relation. Thus, a system of differential-algebraic equations (DAEs) describes the dynamics. If one wants to keep on analytical handling of the equations of motion, the *dynamic inclusion method*⁽⁵⁾ provides a suitable alternative. In this approach, the pitch angle is kept as a generalized coordinate while a kinematic constraint is introduced by differentiating the geometric constraint with respect to time. This yields a constraint equation for $\dot{\vartheta}$. In this study, we use the dynamic inclusion method, accordingly, eight generalized coordinates are needed for the proper description⁽⁴⁾. Let us choose the coordinates X and Y , the yaw angle ψ , the lean angle φ , the steering angle δ , the rotational angles ϕ_f and ϕ_r of the front and the rear wheels around their own axes, and the pitch angle ϑ .

Four scalar kinematic constraining equations can be formulated for the two wheels rolling on the ground: one longitudinal and one lateral for each wheel. Since our goal is to balance the e-scooter at a constant forward speed ($v = \text{const}$), the rotational speed of the rear wheel is also considered constant. Together with the constraint equation for $\dot{\vartheta}$, this yields six kinematic constraints. The equations of motion of this nonholonomic system are derived using Kane's method⁽²⁾. For a unique description, $8 - 6 = 2$ pseudovelocities are required. In this study, we select the lean rate $\dot{\varphi}$ and the steering rate $\dot{\delta}$.

Due to the complexity of the governing equations, we cannot

provide the explicit formulas of the nonlinear equations within the scope of this study. In contrast, the linearized equations of motion can be formulated simply as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{Q}, \quad (1)$$

where $\mathbf{x} = [\varphi \ \delta]^T$ and $\mathbf{Q} = [0 \ M^s]^T$, with internal steering torque M^s . Although the full spatial model includes yaw as a generalized coordinate, and the governing equations are derived for all generalized coordinates, for straight-line motion at constant forward speed the corresponding dynamics are decoupled, and the lateral (lean–steer) subsystem governs the stability. Therefore, the stability analysis can be restricted to the lean and steering dynamics without loss of generality. The linearized equations of motion (1) are in agreement with the results reported in the benchmark article of bicycles⁽⁹⁾. This agreement provides validation of the modelling approach in the linear regime. The nonlinear governing equations are derived using Kane's method, which is a well-established framework in multibody dynamics, and their detailed derivation is given in previous works^(3,4).

An uncontrolled, commercially available e-scooter with parameters summarized in Table 1 exhibits self-stability for forward speed $v \in (4.0, 5.7)$ m/s, see the real and imaginary parts of the eigenvalues in Fig. 3. Namely, asymptotically stable behavior can be observed, i.e., all eigenvalues have negative real parts for forward speeds between 4.0 and 5.7 m/s. Below 4.0 m/s, an unstable weave mode (a typical oscillatory mode of bicycles and motorcycles) can be observed, with frequencies up to 1 Hz. For forward speed from approximately 4.8 m/s, the weave mode diverges into two aperiodic branches. One of these aperiodic branches leads to an unstable motion from 5.7 m/s, which is similar to the capsize mode of bicycles. These results are in good agreement with the study on the stability of e-scooters with a fully rigid rider⁽¹⁰⁾.

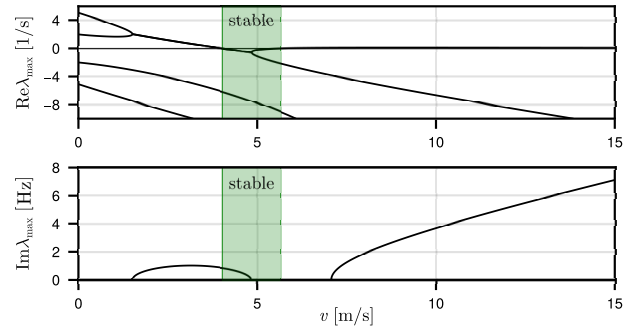


Fig. 3: Real and imaginary parts of the eigenvalues of the uncontrolled e-scooter

At $v = 4.8$ m/s, numerical simulations confirm that the vibrations decay over time, see the time graphs of Fig. 4. While these results demonstrate the presence of inherent self-stability, it is confined to a relatively narrow speed range. By applying control, this stable speed range can be significantly extended.

3. Steering-Based Balancing Control

Extending previous works^(3,4) on stabilizing e-scooters at zero forward speed, a hierarchical steering controller is employed to stabilize straight-line motion. Path following is not considered; instead, our aim is to maintain the e-scooter in an upright position via steering the handlebar.

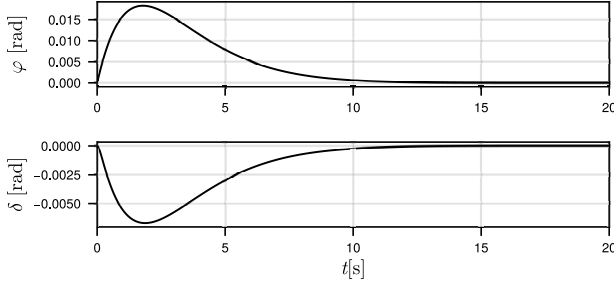


Fig. 4: Time graphs for the lean φ and the steering δ angles for the uncontrolled e-scooter for forward speed $v = 4.8$ m/s

3.1. Control Design

The higher-level controller computes a desired steering angle using the control gains $K_{p\varphi}^s$ and $K_{d\varphi}^s$ as

$$\delta_{\text{des}} = -K_{p\varphi}^s \varphi(t - \tau) - K_{d\varphi}^s \dot{\varphi}(t - \tau), \quad (2)$$

where τ is the feedback delay of the controller. The internal steering torque M^s is generated by a lower-level control law using the control gains $K_{p\delta}^s$ and $K_{d\delta}^s$ as

$$M^s = -K_{p\delta}^s (\delta(t) - \delta_{\text{des}}) - K_{d\delta}^s \dot{\delta}(t). \quad (3)$$

It is shown that the balancing of e-scooters can be achieved with this simple state feedback controller at zero forward speed, even in the presence of feedback delay^(3,4). This study investigates whether the proposed controller is suitable for stabilizing the e-scooter at non-zero forward speeds. Thanks to the feedback delay τ , delayed states appear in the equations of motion, and the dynamics are governed by delay differential equations (DDEs). Although the system is described by two second-order equations (1), its state space is infinite-dimensional. Therefore, special techniques are required to analyze this delayed dynamical system.

3.2. Stability Analysis Methods

The stability boundaries in these delayed systems can be formulated based on the characteristic equation $D(\lambda) = 0$. At the critical stability limits, the roots cross the imaginary axis ($\lambda = i\omega$), yielding two real-valued equations: $\text{Re}(D(i\omega)) = 0$ and $\text{Im}(D(i\omega)) = 0$.

It has been shown that this codimension-two problem can be solved highly efficiently to generate stability charts by means of the *Multi-Dimensional Bisection Method (MDBM)*^(7,8,12). Although this method constructs the chart with exceptional computational speed, it primarily identifies the general bifurcation lines where at least one characteristic root crosses the imaginary axis ($D(\lambda) = 0$). It does not inherently distinguish the true stability boundary, where the dominant (rightmost) characteristic root λ_{max} is located at the stability margin. Therefore, to evaluate the actual stability of the domains circumscribed by these curves, we use a Julia package called *InterpolatedNyquist*⁽¹¹⁾.

The stability of the delayed system is determined via the argument principle (Cauchy's integral theorem), which states that the number of unstable characteristic roots z (i.e., roots with $\text{Re } \lambda > \sigma$) can be expressed as

$$z = -\frac{1}{\pi} \int_0^\infty \frac{d}{d\omega} \arg D(i\omega) d\omega + \frac{n}{2}, \quad (4)$$

where $D(i\omega)$ is the characteristic function of the delayed system and n is the polynomial degree of D . The integrand is the instantaneous rate of change of the phase of D along the imaginary axis, evaluated

via automatic differentiation (ForwardDiff) in Julia language, which yields both the value $D(i\omega)$ and its exact derivative $\partial D / \partial \omega$ in a single function call. This derivative is used to formulate a scalar ODE for the accumulated phase, which is solved by a stiff adaptive-step solver (Rosenbrock23). Thanks to automatic step-size control, large steps are taken in smooth regions and small steps near rapid phase changes; integration up to a sufficiently large finite ω_{max} (e.g., 10^5 rad/s) is sufficient in practice, since the solver naturally takes large steps near the upper limit. A key advantage of the ODE formulation is that the closest characteristic root to the contour can be identified simultaneously with the encirclement count. At each step, the quantity $|D(i\omega)|^2$ is monitored; its local minimum over ω indicates the frequency ω^* closest to a root. The real part of that root is then estimated from the local gradient information as

$$\begin{aligned} \hat{\sigma}^* &= \text{Im} \left(\frac{D(i\omega)}{\partial_\omega D(i\omega)} \right) \Big|_{\omega=\omega^*} \\ &= - \left[\frac{\text{Re}(D) \partial_\omega \text{Im}(D) - \text{Im}(D) \partial_\omega \text{Re}(D)}{(\partial_\omega \text{Re}(D))^2 + (\partial_\omega \text{Im}(D))^2} \right] \Big|_{\omega=\omega^*} \end{aligned} \quad (5)$$

which follows from a first-order Taylor expansion of D in the σ -direction. Thus, the method not only counts unstable roots but also provides a continuous approximation of the spectral abscissa $\text{Re } \lambda_{\text{max}}$ throughout the parameter space. The approximation is most accurate near the stability boundary, where it matters most; deeper in the stable region the estimate may be less precise, which is acceptable since the exact value is not critical there.

The aforementioned Nyquist-based approach is highly efficient; however, to obtain high-fidelity results without the linear approximation of Eq. (5), we additionally employ a Julia package called *AffineMapDiffEq*⁽¹³⁾ for stability analysis using affine mapping and spectral analysis.

In general, this mapping technique is formulated for time-periodic delayed systems with period T , based on Floquet theory. It constructs a mapping from an initial state space, represented by a continuous function $\mathbf{x}_0(t)$ defined over the maximum delay interval $[-\tau_{\text{max}}, 0]$, to the state space one period later, $\mathbf{x}_1(t)$ over $[T - \tau_{\text{max}}, T]$. Although applying Floquet theory to a Linear Time-Invariant (LTI) system might seem to be unnecessarily complex, where an infinitesimally small period T could theoretically be sufficient, computational experience shows that choosing $T = \tau_{\text{max}}$ provides an optimal balance of efficiency and stability in the numerical implementation.

Using this approach, the characteristic exponents (eigenvalues) λ of the LTI system can be approximated from the Floquet multipliers μ of the mapped system according to the relation

$$\lambda = \frac{\ln \mu}{T}. \quad (6)$$

Because the chosen integration time T is relatively small, we do not need to account for the multi-valued periodicity of the complex logarithm (where the true roots could be shifted as $\lambda = \lambda^* + i2\pi k/T$ for an integer k).

The affine mapping method utilizes iterative eigenvalue solvers to find the leading Floquet multipliers ($\mu_1, \mu_2, \dots$) with the largest magnitudes. By employing state-of-the-art, matrix-free Krylov subspace methods (such as those provided by *KrylovKit.jl*), the most time-consuming explicit construction of the large, dense monodromy matrix is entirely avoided. Instead, the eigenvalue solver only requires the input and the output of the direct evaluation of the mapping operator itself:

$$\mathbf{x}_1 = \mathcal{M}(\mathbf{x}_0). \quad (7)$$

The core idea of the package is that the mapping operator $\mathcal{M}$ can be evaluated seamlessly by simply simulating the system with a

standard DDE integrator, which permits the use of highly efficient, adaptive-step-size solvers. In practice, the continuous state functions $\mathbf{x}_0$ and $\mathbf{x}_1$ are sampled over a dense temporal grid, allowing for the rapid computation of inner products required during the Krylov iterations. This strategy guarantees a high-fidelity solution as long as the sampling resolution is sufficiently fine and the tolerance of the DDE solver is strictly constrained.

Consequently, a dual-method workflow is established in this study: the *InterpolatedNyquist* package is used to rapidly obtain the approximate dominant characteristic root in the exact stability chart over the parameter space, while the *AffineMapDiffEq* package is utilised to validate the results at specific operational points with high fidelity.

3.3. Choice of the Control Gains

Due to the nature of the hierarchical controller, the physical interpretation of certain control gains is not obvious, and even their signs may be difficult to determine. The gains can be selected heuristically (based on engineering intuition) or optimized.

In the following analysis, the forward speed is maximized at $v = 15$ m/s. Even with heuristically chosen control gains $K_{p\delta}^s = -1$ Nm, $K_{d\delta}^s = -1$ Nms, $K_{p\varphi}^s = -5$, $K_{d\varphi}^s = -2.5$ s, the stable speed range can be extended drastically. Namely, the motion of the controlled e-scooter is stable for $v \in (4.1, 15)$ m/s. In addition, the vibrations decay more rapidly at the same forward speed of $v = 4.8$ m/s, even in the presence of non-zero feedback delay, see Fig. 5.

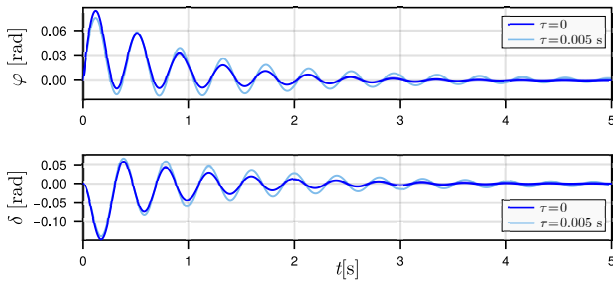


Fig. 5: Time graphs for the lean φ and the steering δ angles for the controlled e-scooter for forward speed $v = 4.8$ m/s, with $K_{p\delta}^s = -1$ Nm, $K_{d\delta}^s = -1$ Nms, $K_{p\varphi}^s = -5$, $K_{d\varphi}^s = -2.5$ s

As previously mentioned, the control gains can also be optimized. In this study, they are chosen to achieve the fastest decay of vibrations, i.e., by minimizing the real part of the rightmost characteristic root (equivalently, by maximizing the spectral abscissa). Our primary objective is to achieve stable balancing at low forward speeds; therefore, the optimization is initially performed at $v = 1$ m/s. In the long term, the theoretical results will be validated experimentally. Given the 100 Hz sampling frequency of the inertial measurement unit of our experimental e-scooter, the feedback delay is set to $\tau = 0.01$ s during the optimization.

For the optimization, the control gains were selected within the intervals $K_{p\delta}^s \in (-10, 10)$ Nm, $K_{d\delta}^s \in (-10, 10)$ Nms, $K_{p\varphi}^s \in (-150, 100)$ and $K_{d\varphi}^s \in (-100, 100)$ s. The spectral abscissa was determined with brute force using the Nyquist-based method, on a $6 \times 6 \times 300 \times 300$ mesh of the control gains. The optimization carried out on this coarse mesh took 1088 s. The maximum spectral abscissa and the corresponding control gains were determined. However, as the control gains cannot be identified or implemented with arbitrarily high numerical precision, they are rounded to physically meaningful and practically realizable values. The resulting control parameters are therefore referred to as "optimal" and are

summarized in Table 2. The subsequent analysis is performed using these four control gains.

Table 2: The optimal control gains for forward speed $v = 1$ m/s and feedback delay $\tau = 0.01$ s with $\text{Re}\lambda_{\max} = -1.7022$ 1/s

$K_{p\delta}^s$	$K_{d\delta}^s$	$K_{p\varphi}^s$	$K_{d\varphi}^s$
-2 Nm	2 Nms	-33	-8 s

3.4. Performance of the Proposed Controller

With the optimal control gains of Table 2, high-fidelity spectral analysis was performed for forward speeds up to 15 m/s. The motion of the e-scooter is stable for $v > v_w = 0.6$ m/s, see the real and imaginary parts of the eigenvalues of the controlled e-scooter in Fig. 6. Above the weave speed v_w , the real part of the rightmost characteristic root decreases abruptly and then remains approximately at -5 1/s, indicating rapid decay of the vibrations. However, the vibration frequencies are relatively high, especially at higher forward speeds.

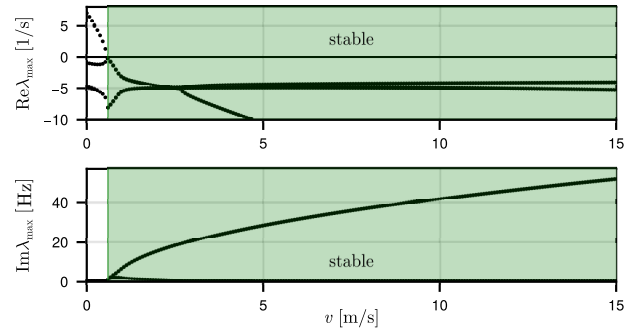


Fig. 6: Real and imaginary parts of the eigenvalues of the controlled e-scooter with a feedback delay $\tau = 0.01$ s

Since the control gains were optimized at a forward speed of $v = 1$ m/s, a more detailed linear stability analysis is performed at this speed. The feedback delay and the lower-level control gains are also fixed: $\tau = 0.01$ s, $K_{p\delta}^s = -2$ Nm and $K_{d\delta}^s = 2$ Nms. The bifurcation boundaries and the stable region are shown in the plane of the higher-level control gains $K_{p\varphi}^s$ and $K_{d\varphi}^s$ in Fig. 7. The bifurcation boundaries, plotted with the black curves, are obtained with the help of the MDBM⁽¹²⁾. One can observe a static stability boundary ($\omega = 0$) at approximately $K_{p\varphi}^s = 27$, see the vertical line. The other black curve corresponding to dynamic loss of stability ($\omega > 0$) is also plotted together with the direction of increasing angular frequency ω . The unstable region is shown in white, with the number of unstable roots indicated in gray boxes. The stable region is colored according to the real part of the rightmost characteristic root $\text{Re}\lambda_{\max}$, which is computed using the affine mapping method, with a 100×100 mesh of the higher-level control gains within the intervals $K_{p\varphi}^s \in (-150, 50)$ and $K_{d\varphi}^s \in (-40, 0)$ s. Dark purple regions correspond to control gain configurations that yield the best performance in terms of vibration decay. In this sense, the optimal higher-level control gain pair is $K_{p\varphi}^s = -32.93$ and $K_{d\varphi}^s = -7.59$ s, with $\text{Re}\lambda_{\max} = -4.25$ 1/s. Despite the fact that the Nyquist-based method only provides an approximation, the control gains reported in Table 2 lie within the dark purple region of Fig. 7, and can therefore be considered adequate to ensure stability.

Numerical simulations are performed to validate the theoretical results. For the optimal control gains of Table 2, the vibrations decay in 2 s with a frequency of 7 Hz, see the dark blue curves in the time graphs of Fig. 8.

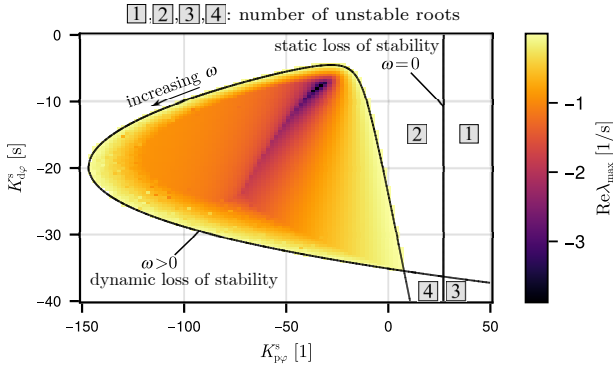


Fig. 7: Bifurcation boundaries and the stable region for the controlled e-scooter for forward speed $v = 1$ m/s and feedback delay $\tau = 0.01$ s

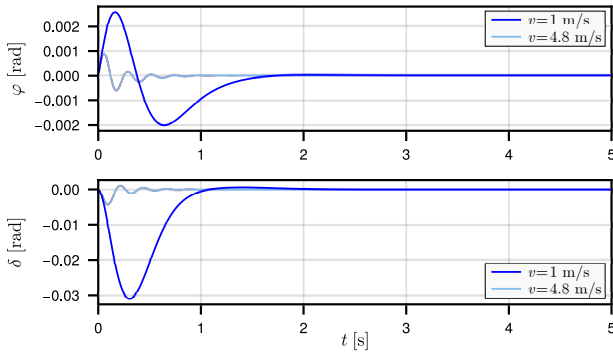


Fig. 8: Time graphs for the lean φ and the steering δ angles for the controlled e-scooter with the optimal control gains of Table 2 for forward speed $v = 1$ m/s and $v = 4.8$ m/s

As shown in Fig. 6, higher speeds are associated with more robust performance and higher vibration frequencies. Therefore, the bifurcation boundaries and the stable region are shown for $v = 4.8$ m/s in Fig. 9. Compared to the previous case, the stable region is significantly smaller, which implies that fewer lower-level control gain pairs ensure stability. In contrast, the dark purple region is much larger, indicating increased robustness against perturbations. For the optimal control gains of Table 2, the vibrations decay in 1 s with a frequency of 28 Hz, see the light blue curves in the time graphs of Fig. 8.

4. Handling the Pitch

In this study, the complex geometric constraint associated with the pitch motion was handled using the dynamic inclusion method. Besides this, several alternative approaches exist. The effect of the chosen pitch handling approach is analyzed in this section.

4.1. Pitch Handling Approaches

For the sake of clarity, the dynamic inclusion method is hereafter referred to as Approach A. Approach B is used, when the pitch is approximated as

$$\theta \approx \theta^{\text{approx}} = \frac{e}{4p} \delta^2 \sin(2\varepsilon) - \frac{e}{p} \varphi \delta \cos \varepsilon, \quad (8)$$

where e is the trail, p is the wheelbase and ε is the rake angle, see Fig. 2. It can be shown that Eq. (8) provides an approximate representation of the exact nonlinear governing equations. However, it

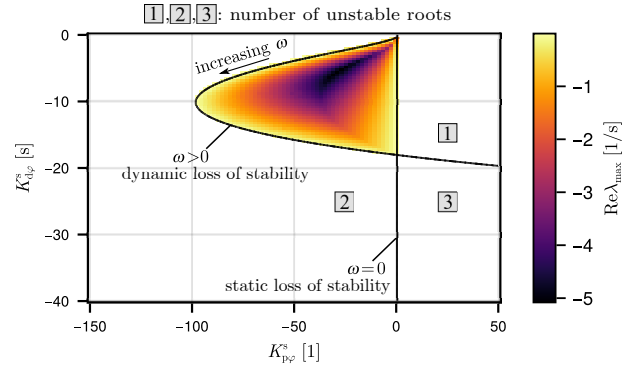


Fig. 9: Bifurcation boundaries and the stable region for the controlled e-scooter for forward speed $v = 4.8$ m/s and feedback delay $\tau = 0.01$ s

results in the *exact* linearized equations of motion of the e-scooter⁽⁴⁾. When using Approach C, one keeps the geometric constraint equation for the pitch within the governing equations of the system as a nonlinear algebraic relation and handles the resulting set of delay differential-algebraic equations (DDAEs) accordingly.

4.2. Effect of Pitch Handling and the Applied Stability Analysis Methods

The aforementioned three approaches are compared as follows. All geometric and inertial parameters of the e-scooter, as well as the forward speed, the feedback delay of the controller, and the control gains, are kept fixed. The real part of the rightmost characteristic root is then determined for all three approaches using both the Nyquist-based method and the affine mapping. The required CPU times and the resulting spectral abscissas are summarized in Table 3 and Table 4. The Nyquist-based method yields an approximate result, whereas affine mapping provides a high-fidelity one; these are denoted by the indices "approx" and "hf", respectively.

With the optimal control gains of Table 2, the investigated point lies relatively far from the stability boundary. Therefore, the Nyquist-based method is expected to be less accurate. The results confirm this: both methods predict stable motion, but the approximate spectral abscissas differ significantly from the high-fidelity ones, see the last two columns of Table 3. For a control gain pair, that is close to the stability boundary (i.e., $K_{\varphi}^s = -20$, $K_{\delta}^s = -6$ s), the Nyquist-based method is accurate, and the spectral abscissas differ only in the fourth digit, see the last two columns of Table 4. Notably, while the Nyquist-based method provides the same spectral abscissa for all three approaches, the results of the affine mapping method are affected by the chosen pitch handling approach.

Note that high-fidelity results are not available for Approach C (the DAE system with the complicated, nonlinear constraint equation for the pitch). This is because, when using the affine mapping method, the solution must be perturbed in order to determine the corresponding eigenvectors, while at the same time the initial conditions must also satisfy the constraint equation. Extending the affine mapping method to DAE systems in a way that resolves this issue is left for future work.

In terms of CPU time, the Nyquist-based method is approximately 50 times faster than the affine mapping method. For rapid stability analysis, the Nyquist-based method provides acceptable results with very low computational cost. For high-fidelity results, however, increased accuracy comes at the expense of higher computational effort. The computational cost of Approaches A and B is essentially the same. However, obtaining the results for Approach C

Table 3: The CPU times and the spectral abscissas for optimal control gains of Table 2, forward speed $v = 1$ m/s and feedback delay $\tau = 0.01$ s

Approach	$T_{\text{CPU}}^{\text{approx}}$	$T_{\text{CPU}}^{\text{hf}}$	$\text{Re}\lambda_{\text{max}}^{\text{approx}} [1/\text{s}]$	$\text{Re}\lambda_{\text{max}}^{\text{hf}} [1/\text{s}]$
Approach A	$183.097 \mu\text{s} \pm 159.140 \mu\text{s}$	$9.765 \text{ ms} \pm 2.255 \text{ ms}$	-1.7022	-3.0861
Approach B	$178.863 \mu\text{s} \pm 179.471 \mu\text{s}$	$13.543 \text{ ms} \pm 2.557 \text{ ms}$	-1.7022	-2.9380
Approach C	$601.720 \mu\text{s} \pm 298.950 \mu\text{s}$		-1.7022	

Table 4: The CPU times and the spectral abscissas for $K_{p\delta}^s = -2$ Nm, $K_{d\delta}^s = 2$ Nms, $K_{p\varphi}^s = -20$, $K_{d\varphi}^s = -6$ s, forward speed $v = 1$ m/s and feedback delay $\tau = 0.01$ s

Approach	$T_{\text{CPU}}^{\text{approx}}$	$T_{\text{CPU}}^{\text{hf}}$	$\text{Re}\lambda_{\text{max}}^{\text{approx}} [1/\text{s}]$	$\text{Re}\lambda_{\text{max}}^{\text{hf}} [1/\text{s}]$
Approach A	$216.146 \mu\text{s} \pm 189.824 \mu\text{s}$	$8.772 \text{ ms} \pm 2.072 \text{ ms}$	-0.3681	-0.3687
Approach B	$208.368 \mu\text{s} \pm 192.173 \mu\text{s}$	$8.572 \text{ ms} \pm 2.176 \text{ ms}$	-0.3681	-0.3831
Approach C	$721.560 \mu\text{s} \pm 223.949 \mu\text{s}$		-0.3681	

takes significantly more time, because handling the algebraic equation is much more involved.

Overall, the Nyquist-based method has both advantages and limitations. It is significantly faster than the affine mapping method. However, it provides less accurate estimates of the spectral abscissa when the investigated point lies far from the stability boundary, and it requires prior knowledge of the fixed point. The affine mapping method can be applied not only to linear systems but also to nonlinear ones. It does not require prior knowledge of the fixed point, and is therefore more general. In the long term, it can be used to obtain results not only for straight-line motion but also, for example, for cornering with a constant steering angle. However, it still requires further development for DAE systems, which is a future work.

The results show that the pitch handling approach affects the spectral abscissa when using high-fidelity analysis. A more precise linear stability analysis, as well as nonlinear analysis, is required for a deeper understanding of pitch handling.

5. Conclusions

The stabilization of a riderless e-scooter at constant forward speed was investigated using a steering-based control strategy. A spatial model was constructed, and the nonlinear governing equations were derived using Kane's method. It was shown that the proposed hierarchical controller with feedback delay significantly extends the stable speed range and ensures rapid decay of vibrations, even at low forward speeds. The results also show that the choice of pitch handling approach affects the spectral abscissa when high-fidelity analysis is applied. A clear trade-off was identified between computational efficiency and accuracy when comparing the Nyquist-based and affine mapping methods.

Future work includes the optimization of the control gains with respect to robustness at fixed forward speed and feedback delay. In addition, a more detailed investigation of pitch handling is planned, including both linear and nonlinear analyses, such as numerical bifurcation analysis and nonlinear simulations.

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