# Rainbow copies of $F$ in families of $H$

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**Abstract**

We study the following problem. How many distinct copies of $H$ can an $n$-vertex graph have without containing a rainbow $F$, that is, a copy of $F$ where each edge is contained in a different copy of $H$? The case $H = K_r$ is equivalent to the Turán problem for Berge hypergraphs, which has attracted several researchers recently. We also explore the connection of our problem to the so-called generalized Turán problems. We obtain several exact results. In the particularly interesting symmetric case where $H = F$, we completely solve the case $F$ is the 3-edge path, and asymptotically solve the case $F$ is a book graph.

## 1 Introduction

Given a graph $G$ and a family $\mathcal{H}$ of its subgraphs, we obtain an edge-coloring, where each subgraph in $\mathcal{H}$ has its unique color and each edge of $G$ has a list of colors of all the subgraphs containing it. We say that a subgraph $F$ is *rainbow* (with respect to $\mathcal{H}$) if we can pick a distinct color for each edge of $F$ from the corresponding list. Equivalently, $F$ is rainbow (with respect to $\mathcal{H}$) if there is an injection $\phi : E(F) \to \mathcal{H}$ such that each edge is contained in its image. In other words, we pick the edges of $F$ from different members of $\mathcal{H}$.

In this paper, $\mathcal{H}$ will consist of copies of the same graph $H$. Given an integer $n$ and simple graphs $H$ and $F$, we are interested in the largest cardinality of a family $\mathcal{H}$ of copies of $H$ in an $n$-vertex graph $G$ such that there is no rainbow $F$. We denote this number by $\mathrm{rb}(n, H, F)$. We remark that recently Imolay, Karl, Nagy and Váli [32] introduced a very similar notion, but they require that the graphs in $\mathcal{H}$ are edge-disjoint.

Let us mention some examples that almost fit our setting. Note that we do not allow the same copy of $H$ to be counted multiple times, while the results below do. Aharoni and Berger [1] showed that $2n - 1$ matchings of size $n$ in bipartite graphs contain a rainbow matching of size $n$. Barát, Gyárfás and Sárközy [5] studied how many matchings of size $m$ can be in a bipartite graph without a rainbow matching of size $m - k$. Aharoni, Briggs, Holzman and Jiang [2] showed that every family of $2\lceil n/2\rceil - 1$ odd cycles contain a rainbow odd cycle. Dong and Xu [7] showed that every family of $\lfloor 6(n-1)/5\rfloor + 1$ even cycles contain a rainbow even cycle. Cheng, Han, Wang and Wang [6] studied the existence of rainbow $K_t$-factors. Note that in most of these examples the order of the graphs we count or forbid rainbow copies of increases with $n$.

In contrast, Goorevitch and Holzman [27] considered the fixed graph $K_3$ and did not allow the same copy counted multiple times. They showed $\mathrm{rb}(n, K_3, K_3) = n^2/8$.

Observe that in most of the above examples, $H$ and $F$ were the same graph or the same family of graphs. Here we study the asymmetric version as well. Our main motivation for doing that is that in the special case $H = K_r$, an equivalent problem has attracted a lot of attention in hypergraph Turán theory.

Given a graph $F$, we say that a hypergraph $\mathcal{F}$ is a *Berge copy of* $F$ (in short: *Berge-*$F$) if there is a bijection $E(F) \to E(\mathcal{F})$ such that each edge is contained in its image. This is how Berge defined hypergraph cycles, and the definition was extended to arbitrary graphs by Gerbner and Palmer [21]. Several papers studied the maximum number $\mathrm{ex}_r(n, \text{Berge-}F)$ of hyperedges in an $r$-uniform $n$-vertex hypergraph that does not contain a Berge copy of $F$, see Subsection 5.2.2 of [24] for a slightly outdated survey.

Consider the $r$-uniform hypergraph that contains the vertex sets of copies of $K_r$ as hyperedges. Clearly, a collection of copies of $K_r$ contains a rainbow $F$ if and only if the corresponding $r$-uniform hypergraph contains a Berge-$F$, thus we have that

$$\mathrm{rb}(n, K_r, F) = \mathrm{ex}_r(n, \text{Berge-}F). \tag{1}$$

One can see our problem as a generalization of Berge hypergraph where hyperedges are replaced by graphs.

Győri [29] showed $\mathrm{ex}_3(n, \text{Berge-}K_3) \leq n^2/8$ if $n$ is large enough (implying most of the main result in [27]). Moreover, he studied multi-hypergraphs, thus his result implies the same bound if we allow a triangle to appear multiple times. This proves a conjecture from [27].

Another related topic is generalized Turán problems. We denote by $\mathcal{N}(H, G)$ the number of copies of $H$ in $G$, and let $\mathrm{ex}(n, H, F)$ denote the largest $\mathcal{N}(H, G)$ among $F$-free $n$-vertex graphs $G$. Note that $\mathrm{ex}(n, K_2, F) = \mathrm{ex}(n, F)$ is the ordinary Turán number. The systematic study of this generalized version was initiated by Alon and Shikhelman [3] after several sporadic results, and has also attracted several researchers recently, see e.g. [15, 16, 17, 22, 26, 31, 34, 35, 36] and [23] for a survey.

We can see our problem as a rainbow generalization of generalized Turán problems. Note that the oldest and generally most accepted notion of rainbow Turán number is in [33], and two different rainbow generalizations for counting subgraphs have already appeared in the literature [19, 30]. However, in those papers, rainbowness comes from a proper edge-coloring, not the copies of $H$.

Clearly we have $\mathrm{rb}(n, H, F) \geq \mathrm{ex}(n, H, F)$, since an $F$-free graph cannot contain a rainbow $F$. We will show a much stronger connection between the above two parameters.

We will often consider $F$ and $H$ fixed when $n$ grows. In particular, $o$ and $O$ are always used this way.

Our main results are the following.

**Lemma 1.1.** *For any graphs $H$ and $F$ and for any $n$, we have* $\mathrm{ex}(n,H,F) \leq \mathrm{rb}(n,H,F) \leq \mathrm{ex}(n,H,F) + \mathrm{ex}(n,F)$.

Since $\mathrm{ex}(n,H,F)$ often has larger order of magnitude than $\mathrm{ex}(n,F)$, this lemma determines the asymptotics of $\mathrm{rb}(n,H,F)$ for several pairs of graphs $H,F$. We also present a strengthening of the upper bound in Lemma 2.3.

In the case $\mathrm{ex}(n,H,F)$ is superquadratic, for example if $\chi(H) < \chi(F)$ and $|V(H)| \geq 3$ (in which case the Turán graph $T(n,\chi(F)-1)$ contains $\Theta(n^{|V(H)|})$ copies of $H$), our goal is to obtain exact results. We do this in the next theorem in the case a well-studied stability property holds, see Section 3 for the precise definition.

**Theorem 1.2.** *If $H$ is weakly $F$-Turán-stable, $F$ has a color-critical edge and $n$ is sufficiently large, then* $\mathrm{rb}(n,H,F) = \mathcal{N}(H,T)$ *for some complete $(\chi(F)-1)$-partite graph $T$ on $n$ vertices.*

Note that, among other results, this stability holds if $\chi(H) < \chi(F)$ and $H$ is a path [31] or a complete multipartite graph [17], or $H$ is arbitrary and the chromatic number of $F$ is sufficiently large [17].

We also deal with the symmetric case and prove the following theorems. Let $P_k$ denote the path on $k$ vertices and $B_t$ denote the *book graph*, which consists of $t$ triangles sharing an edge.

**Theorem 1.3.**

$$\mathrm{rb}(n,P_4,P_4) = \begin{cases} n-3 & \text{if } n \geq 5, \\ 2 & \text{if } n = 4, \\ 0 & \text{if } n \leq 3. \end{cases}$$

**Theorem 1.4.** *If $t,r \geq 2$, then we have* $\mathrm{rb}(n,B_t,B_r) = (1+o(1))n^2/8$.

The structure of the rest of the paper is the following. In Section 2, we generalize several known result on Berge hypergraphs to our setting. In particular, we present a connection to a variant of generalized Turán numbers. In Section 3, using the above mentioned connection, we show how some stability results on $\mathrm{ex}(n,H,F)$ can imply exact results on $\mathrm{rb}(n,H,F)$. In Section 4, we consider the symmetric case prove Theorems 1.3 and 1.4. We finish the paper in Section 5 with some concluding remarks.

# 2 Generalization of Berge results

Recall that by (1), the Turán number of Berge hypergraphs is a special case of our problem. In this section we extend some fundamental results on Berge hypergraphs to our setting. A simple technique often used in the theory of Berge hypergraphs is to pick the edges greedily. We illustrate this by presenting the simplest example in our setting.

**Proposition 2.1.** *Let $G$ be a graph, $\mathcal{H}$ a family of its subgraphs, and $F$ be a subgraph such that each of its edges is in at least $|E(F)|$ elements of $\mathcal{H}$. Then $F$ is rainbow.*

*Proof.* We go through the edges of $F$ in an arbitrary order. For each edge, we greedily pick a member of $\mathcal{H}$ containing it that we have not picked earlier. It is doable, as for the $i$th edge, we can pick one of at least $|E(F)|$ graphs containing it, and at most $i-1 \leq |E(F)|-1$ of those graphs were picked earlier. ■

The connection between Berge hypergraphs and generalized Turán problems was established by Gerbner and Palmer [22], who proved that for any graph $F$, any $r$, and any $n$, we have

$$\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, \text{Berge-}F) \leq \mathrm{ex}(n, K_r, F) + \mathrm{ex}(n, F). \tag{2}$$

A strengthening of the result was proved by Gerbner, Methuku and Palmer [18]. We say that a graph is *red-blue* if each of its edges is colored red or blue. Given a red-blue graph $G$, we denote by $G_{red}$ and (respectively, $G_{blue}$) the subgraph consisting of the red (resp. blue) edges. Let $\mathcal{N}^{col}(H_1, H_2; G) = \mathcal{N}(H_1, G_{red}) + \mathcal{N}(H_2, G_{blue})$, the number of red copies of $H_1$ plus the number of blue copies of $H_2$. Let $\mathrm{ex}^{col}(n, H_1, H_2; F)$ denote the largest $\mathcal{N}^{col}(H_1, H_2; G)$ in $F$-free $n$-vertex red-blue graphs $G$. Gerbner, Methuku and Palmer [18] proved that

$$\mathrm{ex}_r(n, \text{Berge-}F) \leq \mathrm{ex}^{col}(n, K_r, K_2; F). \tag{3}$$

An equivalent statement was proved by Füredi, Kostochka, and Luo [11].

We can generalize both above results to our setting. We start with Lemma 1.1, which generalizes the $K_r$'s in both inequalities of (2) to arbitrary $H$. Clearly Lemma 2.3 implies the second part of Lemma 1.1, but we present a proof of that part as well, since it is simple and makes the second proof easier to understand. Let us restate Lemma 1.1 now.

**Lemma.** *For any graphs $H$ and $F$ and for any $n$, we have* $\mathrm{ex}(n, H, F) \leq \mathrm{rb}(n, H, F) \leq \mathrm{ex}(n, H, F) + \mathrm{ex}(n, F)$.

*Proof.* To prove the first inequality, we take an $F$-free $n$-vertex graph with $\mathrm{ex}(n, H, F)$ copies of $H$. Let these copies of $H$ form our family $\mathcal{H}$, then clearly there is no rainbow copy of $F$ here. To prove the second inequality, we take an $n$-vertex graph $G$ and a family $\mathcal{H}$ of $\mathrm{rb}(n, H, F)$ distinct copies of $H$ in $G$ with no rainbow $F$. We go through the members of $\mathcal{H}$ in an arbitrary order, and pick a previously not picked edge from each member currently processed, if possible. If it is not possible (because all the edges of the copy of $H$ have been picked earlier), then we mark the copy of $H$. Let $G'$ be the graph formed by the edges we picked during this procedure. A copy of $F$ in $G'$ gets each of its edges from distinct copies of $H$, a contradiction, thus $G'$ is $F$-free. The copies of $H$ we could not pick an edge from are each subgraphs of $G'$. This shows that we could pick an edge at most $\mathrm{ex}(n, F)$ times and we could not pick an edge at most $\mathrm{ex}(n, H, F)$ times. ■

Observe that in the above proof, we picked a maximal matching between the copies of $H$ in $G$ and edges of $G$ greedily. To improve this, we will pick a largest matching. We follow the overall structure of the proof of (3) in [12]. In particular, we use the following lemma.

**Lemma 2.2** ([12]). *Let $\Gamma$ be a finite bipartite graph with parts $A$ and $B$ and let $M$ be a largest matching in $\Gamma$. Let $B'$ denote the set of vertices in $B$ that are incident to $M$. Then we can partition $A$ into $A_1$ and $A_2$ and partition $B'$ into $B_1$ and $B_2$ such that for every $a \in A_1$ we have that $ab \in M$ for some $b \in B_1$, and every neighbor of the vertices of $A_2$ is in $B_2$.*

We note that this lemma is equivalent to Kőnig's theorem. Now we state our generalization of (3).

**Lemma 2.3.** *For any graphs $H$ and $F$ and for any $n$, we have* $\mathrm{rb}(n, H, F) \leq \mathrm{ex}^{col}(n, H, K_2; F)$.

*Proof.* Let $G$ be an $n$-vertex graph, $\mathcal{H}$ be a collection of copies of $H$ in $G$ such that $G$ is rainbow $F$-free. Let part $A$ of $\Gamma$ be $\mathcal{H}$ and part $B$ be $E(G)$, and we join $a \in A$ to $b \in B$ if $a$ contains $b$. Now we apply Lemma 2.2 to $\Gamma$ and an arbitrary largest matching $M$. The elements of $B'$ form an $F$-free graph $G'$. We color the edges in $B_1$ blue and edges of $B_2$ red. We have $|\mathcal{H}| = |A_1| + |A_2| \leq |B_1| + |A_2| = |E(G_{blue})| + |A_2| = \mathcal{N}(K_2, G_{blue}) + |A_2|$. As the copies of $H$ in $A_2$ have all their neighbors in $B_2$, they are all monored, showing $|A_2| \leq \mathcal{N}(H, G_{red})$. ■

Let us remark that $\mathrm{ex}^{col}(n, H_1, H_2; F)$ (and more generally an $r$-colored version) was studied in [13].

The Ramsey number $R(H, G)$ is the smallest integer $r$ such that in any red-blue coloring of $K_r$ there is either a red $H$ or a blue $G$. We say that a graph $H$ is *Ramsey* for another graph $F$ if in any red-blue coloring of $H$, there is either a monored $F$ or a monoblue $F$. Let $F \setminus e$ denote a graph obtained by deleting an arbitrary edge $e$ from $F$. Grósz, Methuku and Tompkins [28] proved that if $r \geq R(F, F \setminus e)$, then $\mathrm{ex}_r(n, \text{Berge-}F) = o(n^2)$. We can prove an analogous result in our setting as follows.

**Proposition 2.4.** ***(i)*** *Let us assume that $H$ contains $F$. Then* $\mathrm{rb}(n, H, F) = O(n^2)$.
***(ii)*** *Let us assume that $H$ is Ramsey for $F$. Then* $\mathrm{rb}(n, H, F) = o(n^2)$.

Recall that by (1), the Berge result corresponds to the case $H = K_r$, thus in that case the above proposition gives $\mathrm{ex}_r(n, \text{Berge-}F) = o(n^2)$ if $r \geq R(F, F)$. The result of Grósz, Methuku and Tompkins [28] is slightly stronger. We follow the proof of Grósz, Methuku and Tompkins [28]. In particular, we use the removal lemma [10], which states that if an $n$-vertex graph $G$ contains $o(n^{|V(H)|})$ copies of $H$, then there is a set $E_0$ of $o(n^2)$ edges in $G$ such that each copy of $H$ in $G$ contains an edge from $E_0$.

*Proof.* Observe that **(i)** is a simple corollary of Lemma 1.1. To prove **(ii)**, let us bound the number of copies of $F$ in $G$. Each copy of $F$ contains at least two edges, thus at least three vertices from an element of $\mathcal{H}$. This means we can pick every $F$ the following way. We pick an element of $\mathcal{H}$ ($O(n^2)$ ways because of **(i)**), three vertices of that element ($O(1)$ ways), $|V(F)|-3$ arbitrary other elements ($O(n^{|V(F)|-3})$ ways), and then a copy of $F$ on the $|V(F)|$ vertices ($O(1)$ ways). Therefore, there are $O(n^{|V(F)|-1})$ copies of $F$ in $G$.

By the removal lemma, there is a set $E_0$ of $o(n^2)$ edges such that each copy of $F$ contains an element of $E_0$. Let us pick an element of $\mathcal{H}$ and denote it by $H_0$. Let us color the edges of $H_0$ that are in $E_0$ red, and the other edges of $H_0$ blue. By the definition of $E_0$, there is no blue $F$, thus there must be a red $F$ inside $H_0$. By Proposition 2.1, at least one of the edges of this copy of $F$ is contained in at most $|E(F)|-1$ elements of $\mathcal{H}$.

We obtained that each element of $\mathcal{H}$ contains an edge of $E_0$ that is contained in at most $|E(F)|-1$ elements of $\mathcal{H}$. Therefore, we have that $|\mathcal{H}| \leq (|E(F)|-1)|E_0| = o(n^2)$. ■

# 3 Proofs using stability

In this section, we prove exact results on $\mathrm{rb}(n, H, F)$ using stability results on $\mathrm{ex}(n, H, F)$. Most of our statements will actually show that $\mathrm{ex}^{col}(n, H, K_2; F) = \mathrm{ex}(n, H, F)$.

Most stability results on $\mathrm{ex}(n, H, F)$ belong to one of two types. In the first type we happen to know an extremal graph $G$ witnessing $\mathcal{N}(H, G) = \mathrm{ex}(n, H, F)$ (or a family of extremal graphs), and we know that if an $F$-free graph $G'$ is not a subgraph of $G$, then $G'$ contains $\mathcal{N}(H, G) - \Omega(n^{|V(H)|-1})$ copies of $H$. Such a stability result immediately implies that $\mathrm{rb}(n, H, F) = \mathrm{ex}^{col}(n, H, K_2; F) = \mathrm{ex}(n, H, F)$ in conjunction with Lemma 2.3, if $|V(H)| \geq 4$, because for any red-blue coloring of a graph $G'$ that is not a subgraph of $G$, we have $\mathcal{N}^{col}(H, K_2, G') \leq \mathcal{N}(H, G') + \mathcal{N}(K_2, G') = \mathcal{N}(H, G') + O(n^2) < \mathcal{N}(H, G)$. Therefore, the extremal graph maximizing $\mathcal{N}^{col}(H, K_2, *)$ must be a subgraph of $G$, thus it is uniquely $G$. Such stability result can be found e.g. in [25] concerning $\mathrm{ex}(n, K_{a,b}, K_{s,t})$ for some values of $a, b, s, t$.

The other type of stability results is a generalization of the Erdős-Simonovits stability [8, 9, 39] for ordinary Turán problems. The *edit distance* of two graphs $G$ and $G'$ on the same vertex set is the smallest number of edges that we can add and delete from $G$ to obtain $G'$. Given graphs $H$ and $F$ with $\chi(H) < \chi(F)$, we say that $H$ is *$F$-Turán-stable* if for any $n$-vertex $F$-free graph $G$ with $\mathcal{N}(H, G) \geq \mathrm{ex}(n, H, F) - o(n^{|V(H)|})$, $G$ has edit distance $o(n^2)$ from the Turán graph $T(n, \chi(F) - 1)$. We say that $H$ is *weakly $F$-Turán-stable* if for any $n$-vertex $F$-free graph $G$ with $\mathcal{N}(H, G) \geq \mathrm{ex}(n, H, F) - o(n^{|V(H)|})$, $G$ has edit distance $o(n^2)$ from a complete $(\chi(F) - 1)$-partite graph. Note that it is not hard to see that there are weakly $F$-Turán-stable graphs that are not $F$-Turán-stable. For example, it was shown in [17] that stars are weakly $K_k$-Turán-stable for every $k > 2$. A simple calculation shows that for large enough $r$, the complete bipartite graph containing the most copies of the star $K_{1,r}$ is very unbalanced, thus has edit distance $\Omega(n^2)$ from $T(n, 2)$, showing that $K_{1,r}$ is not $K_3$-Turán-stable.

The first such stability result in generalized Turán problems is due to Ma and Qiu [35], who showed that $K_r$ is $F$-Turán-stable for every $F$ with chromatic number more than $r$. Other results appear in [15, 16, 17, 31, 34]. These results were used to prove sharp bounds on $\mathrm{ex}(n, H, F)$. In particular the author showed in [15] that if $H$ is weakly $F$-Turán-stable and $F$ has a color-critical edge (an edge whose deletion decreases the chromatic number), then $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for a complete $(\chi(F) - 1)$-partite graph $T$ (and some exact results were obtained in [16] in the case $F$ does not have a color-critical edge). We generalize this as follows. Note that this implies Theorem 1.2.

**Theorem 3.1.** *If $H$ is weakly $F$-Turán-stable, $F$ has a color-critical edge and $n$ is sufficiently* *large, then* $\mathrm{rb}(n,H,F) = \mathrm{ex}(n,H,F) = \mathcal{N}(H,T) = \mathrm{ex}^{col}(n,H,K_2;F)$ *for some complete* $(\chi(F)-1)$*-partite graph* $T$ *on* $n$ *vertices.*

*Proof.* Note that $\mathrm{rb}(n,H,F) \geq \mathrm{ex}(n,H,F)$ is implied by Lemma 1.1, $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T)$ follows from the result in [15] mentioned above, and $\mathrm{rb}(n,H,F) \leq \mathrm{ex}^{col}(n,H,K_2;F)$ follows from Lemma 2.3. Therefore, it is enough to prove that $\mathrm{ex}^{col}(n,H,K_2;F) \leq \mathcal{N}(H,T)$.

If $|V(H)|=2$, the statement is trivial, thus we assume that $|V(H)|\geq 3$. Let $G$ be an $n$-vertex red-blue colored graph such that $G$ is $F$-free and $\mathcal{N}^{col}(H,K_2;G) = \mathrm{ex}^{col}(n,H,K_2;F)$.

If $\mathcal{N}(H,G) = \mathrm{ex}(n,H,F) - \Omega(n^{|V(H)|})$, then $\mathcal{N}^{col}(H,K_2;G) < \mathrm{ex}(n,H,F) \leq \mathrm{ex}^{col}(n,H,K_2;F)$, a contradiction. Therefore, by the weak Turán stability, we have that $G$ has edit distance $o(n^2)$ from a complete $(\chi(F)-1)$-partite graph $T$. We pick $T$ among such graphs that has the smallest edit distance from $G_{red}$. Moreover, $G_{red}$ also must have $\mathcal{N}(H,G) = \mathrm{ex}(n,H,F) - o(n^{|V(H)|})$ by the same argument, thus we have $o(n^2)$ blue edges. Let $A_i$ denote the $i$-th part of $T$. Lemma 2.2 in [15] ensures that each part $A_i$ has order $\Theta(n)$.

Let $X_i$ be the set of vertices in $A_i$ that have $\Omega(n)$ non-neighbors in some $A_j$ with $j \neq i$, and $X := \cup_{i=1}^{\chi(F)-1} X_i$. Observe that $|X| = o(n)$. Let us pick an arbitrary set $B_i$ of $|V(F)|$ elements of $A_i \setminus X_i$ and observe that the common neighborhood of the vertices in $B_i$ contains all but $o(n)$ vertices from each $A_j$ with $j \neq i$.

Assume that $v \in A_i$ has $o(n)$ neighbors in some $A_j$ with $j \neq i$. Then $v$ must have $\Omega(n)$ non-neighbors in $A_i$, otherwise we could move $v$ to $A_j$ in $T$ to obtain a complete $(\chi(F)-1)$-partite graph with smaller edit distance from $G$, a contradiction. Having these properties of $G$, we delete all the edges incident to $v$ and join $v$ to the common neighborhood of $B_j$.

We claim that the resulting graph $G'$ is $F$-free. Indeed, if there is a copy of $F$, it must contain $v$, thus avoids a vertex $v' \in B_j$. We can replace $v$ by $v'$ to obtain another copy of $F$, since the neighborhood of $v$ is a subset of the neighborhood of $v'$. But this second copy of $F$ is in $G$, a contradiction.

Observe that there are $o(n)$ edges in $G$ not in $G'$ (since there are $o(n)$ neighbors of $v$ in $A_j$ in $G$, and $o(n)$ vertices in the other parts are not in the common neighborhood of $B_j$), and there are $\Omega(n)$ edges in $G'$ not in $G$ (inside $A_i$). Clearly, there are $o(n^{|V(H)|-1})$ copies of $H$ in $G$ that got deleted this way. We claim that we have added $\Theta(n^{|V(H)|-1}$ new copies of $H$. Indeed, we consider a proper coloring of $H$ with at most $\chi(F)-1$ colors, and embed a color class into the set of new neighbors of $v$ inside $A_i \setminus X_i$. We embed a vertex of another color class to $v$, and the rest of the vertices into the common neighborhood inside $A_j \setminus X_j$ of the vertices embedded into $A_i \setminus X_i$. Then we continue this way, we always embed a color class into a new part $A_\ell \setminus X_\ell$, choosing among the common neighbors of the already embedded vertices. Since the already embedded vertices are joined to all but $o(n)$ vertices in the new part, we always have $\Theta(n)$ choices. This shows that the number of embedding is $\Theta(n^{|V(H)|-1})$. As each copy is counted $O(1)$ times, $G'$ contains $\Theta(n^{|V(H)|-1})$ new copies of $H$, thus more than $G$ by $\Theta(n^{|V(H)|-1})$. If any of those contain blue edges, we recolor those blue edges to red to obtain $G''$. Since the number of blue edges is $o(n^2) = o(n^{|V(H)|-1})$, we have that $\mathcal{N}^{col}(H,K_2;G'') > \mathcal{N}^{col}(H,K_2;G)$, a contradiction.

We claim that there is no edge $uv$ inside $A_i$ with $u \notin X$. Indeed, we consider a proper coloring with $\chi(F)-1$ colors of the graph we obtain from $F$ by deleting a color-critical edge.

We embed the color-critical edge into $uv$, and then the rest of the color class into $A_i \setminus X_i$. Then we embed the rest of $F$ as we embedded $H$ in the previous paragraph: we embed the new color classes to new parts $A_j$. We always pick vertices from the common neighborhood of the already embedded vertices, and not from $X_j$. Each of these restrictions forbids $o(n)$ vertices of $A_j$, except that $v$ may be in $X_i$ and in that case $v$ may have $\Omega(n)$ non-neighbors in $A_j$. However, $v$ has $\Omega(n)$ neighbors in $A_j$, thus altogether we still have $\Omega(n)$ choices.

We have obtained that each edge inside parts is inside $X$, in particular each vertex is adjacent to $o(n)$ edges inside parts. Assume that there is an edge $uv$ inside $X_i$. Then we delete all the edges incident to $v$, and join $v$ to the common neighborhood of the vertices of $B_i$. As earlier, the resulting graph is $F$-free since we could replace $v$ by a vertex in $B_i$ in any copy of $F$. We deleted $o(n)$ edges and added $\Omega(n)$ edges. As earlier, this comes with deleting $o(n^{|V(H)|-1})$ copies of $H$ and adding $\Theta(n^{|V(H)|-1})$ new copies of $H$, resulting in a contradiction.

Therefore, $G$ is a subgraph of $T$. Let $x$ denote the number of blue edges of $G$. Then, compared to the monored $T$, we lose $\Omega(n^{|V(H)|-2})$ red copies of $H$ for each such edge. As each copy of $H$ is counted at most $|E(H)| = O(1)$ times, we lose $\Omega(xn^{|V(H)|-2})$ red copies of $H$ and we gain $x$ blue edges between parts, a contradiction completing the proof. ■

Note that the above theorem determines $\mathrm{ex}_r(n, \text{Berge-}F)$ exactly for sufficiently large $n$ if $F$ has chromatic number more than $r$ and a color-critical edge. This is already known, but no simple proof exists. The $r$-uniform *expansion* $F^{+r}$ of a graph $F$ is the specific $r$-uniform Berge copy that contains the most vertices, i.e. the $r-2$ vertices added to each edge of $F$ are distinct for different edges, and distinct from the vertices of $F$. Pikhurko [38] determined the Turán number of $K_k^{+r}$ if $k > r$. According to the survey [37] on expansions, Alon and Pikhurko observed that Pikhurko's proof generalizes to the case $F$ is a $k$-chromatic graph with a color-critical edge. This gives the exact value of $\mathrm{ex}_r(n, \text{Berge-}F)$ for sufficiently large $n$, but the proof is more involved.

# 4 The symmetric case

Note that the results proved in the previous sections only imply the simple bound $\mathrm{rb}(n, F, F) \leq \mathrm{ex}(n, F)$ in the symmetric case (Lemma 1.1). In particular $\mathrm{rb}(n, F, F) = O(n^2)$, and for bipartite graphs $F$ we have $\mathrm{rb}(n, F, F) = o(n^2)$. We show that for non-bipartite graphs $\mathrm{rb}(n, F, F) = \Theta(n^2)$.

**Proposition 4.1.** *If $F$ is not bipartite, then* $\mathrm{rb}(n, F, F) = \Theta(n^2)$.

*Proof.* Let $g$ denote the odd girth of $F$, i.e., the length of the shortest odd cycle in $F$. For each vertex $v$, we define a vector $(r_0, r_1, \dots, r_{n-1})$, where $r_i$ denotes the number of cycles of length $g$ that are at distance $i$ from $v$. Let $u$ denote a vertex with the lexicographically first corresponding vector. By that we mean that we take the vectors with the largest $r_0$, among them the vectors with the largest $r_1$, and so on. If we obtain multiple vertices this way, we choose an arbitrary one.

Let $u_1, \dots, u_s$ be the neighbors of $u$. Let $F_0$ be the graph we obtain from $F$ by deleting $u$. Let $f = |V(F)|$, then we take $\lfloor n/f \rfloor$ vertex-disjoint copies $F_0^i$ of $F_0$ for $i \le \lfloor n/f \rfloor$. We also take vertices $w_\ell$ for $\ell \le n - \lfloor n/f \rfloor(f-1)$. We let $\mathcal{H}$ consist of copies of $F$ of the following form. We take a $w_\ell$ and an $F_0^i$ and connect $w_\ell$ to the vertices corresponding to the vertices $u_1, \dots, u_s$ in this copy of $F_0$. Clearly $\mathcal{H}$ consists of $\Theta(n^2)$ copies of $F$. We claim that there is no rainbow $F$ here. Let $G$ be the graph containing the edges in the elements of $\mathcal{H}$.

Consider a cycle $C$ of length $g$ in $G$ that contains a $w_i$. As $G$ has no edges between distinct $F_0^j$'s, $C$ decomposes into paths which start and end among the $w_i$'s, and whose interiors are entirely contained inside a single $F_0^j$ . Moreover, as $|C|= g$ is odd, at least one of these walks has an odd number of edges, say $w_i, x_1, x_2, \dots, x_q, w_\ell$ where $q$ is even. This corresponds to an odd cycle in F of length $q+1 \le g$, so by minimality of $g$, $q+1 = g$ and $w_i = w_\ell$. But only one element of $\mathcal{H}$ contains the edges $w_i x_1$ and $w_i x_q$, so $C$ cannot be rainbow. Therefore, each rainbow cycle of length $g$ must avoid each $w_i$. Let $d$ denote the distance from $u'$ to the vertices $w_\ell$.

Some of the cycles of length $g$ in our rainbow $F$ belong to $F_0^i$, let $\mathcal{F}$ denote the family of those cycles. Observe that cycles not in $\mathcal{F}$ are connected to $u'$ through some $w_\ell$, thus have distance at least $d+1$ from $u'$. We have $r_k$ cycles of length $g$ at distance $k$ from $u'$ in $F$, thus also in $\mathcal{F}$ for $k \le d$. By the choice of $u$, there are at most $r_0$ cycles of length $g$ in $F_0^i$ at distance 0 from $u'$, thus we are done unless there are exactly $r_0$ such cycles and they are in $\mathcal{F}$ each. The same holds for larger distances. More precisely, consider a cycle at distance $k \le d$ from $u$ in $F$. It is embedded to a cycle at distance $k$ from $u'$ in $F_0^i$. Distinct such cycles are embedded to distinct cycles, thus the $r_k$ such cycle in $F$ is embedded to $r_k$ such cycles in $F_0^i$. By the choice of $u$, we have that any vertex of $F_0^i$, in particular $u'$ has that there are at most $r_0$ cycles of length $g$ at distance 0 from $u'$. If we have equality here (which we do by the above), then there are at most $r_1$ cycles of length $g$ at distance 1 from $u'$. We again have equality here, thus there are at most $r_2$ cycles of length $g$ at distance 2 from $u'$. We again have equality, thus the same holds for distances $3, 4, \dots, d$.

Observe that in the above argument, we considered only cycles inside $F_0^i$, without using any $w_\ell$. Recall that $F_0^i$ together with $w_\ell$ forms a copy of $F$ where $w_\ell$ plays the role of $u$, thus by the choice of $u$, there is a cycle of length $g$ that goes through $w_\ell$ and is otherwise inside $F_0^i$. Let $d'$ denote the distance of that cycle and $u'$, then $d' \le d$. That means that there is a cycle of length $g$ in the graph we obtain by adding $w_\ell$ to $F_0^i$ that is at distance $d'$ from $u'$. In this graph there are at most $r_{d'}$ such cycle, thus in $\mathcal{F}$ there are at most $r_{d'} - 1$ such cycles, a contradiction completing the proof. ∎

We conjecture that $\mathrm{rb}(n, F, F)$ and $\mathrm{ex}(n, F)$ have the same order of magnitude even for bipartite graphs if they are connected.

**Conjecture 4.2.** *For any connected graph $F$, we have $\mathrm{rb}(n, F, F) = \Theta(\mathrm{ex}(n, F))$.*

Note that the above conjecture does not hold for some disconnected graphs. Let $M_2$ denote the matching of size 2. It is easy to see that $\mathrm{ex}(n, M_2) = n-1$ while $\mathrm{rb}(n, M_2, M_2) = 3$ for $n \ge 4$.

We can prove the above conjecture for a class of bipartite graphs.

**Proposition 4.3.** *Let $F$ be a bipartite graph of diameter at most 3. Then* $\mathrm{rb}(n, F, F) \geq \mathrm{ex}(\lfloor n/|V(F)|\rfloor, F)/2$.

*Proof.* Consider an $F$-free graph $G$ on $\lfloor n/|V(F)|\rfloor$ vertices, with $\mathrm{ex}(\lfloor n/|V(F)|\rfloor, F)$ edges. It is well-known and easy to see that there is a bipartite subgraph $G'$ of $G$ with at least $\mathrm{ex}(\lfloor n/|V(F)|\rfloor, F)/2$ edges. Indeed, we consider a bipartite subgraph with the most edges, then every vertex $v$ has at least as many neighbors in the other part as in its part, thus at least half the edges incident to $v$ are in $G'$. This implies that $|E(G')| \geq |E(G)|/2$. Let $A$ and $B$ denote the two parts of $G'$.

Consider a bipartition of $F$ into two independent sets and let $s$ and $t$ be the order of those two sets. We replace each vertex $u$ of $A$ by $s$ vertices $u_1, \dots, u_s$, and replace each vertex $v$ of $B$ by $t$ vertices $v_1, \dots, v_t$. For each edge $uv$ of $G'$, we place a copy of $F$ on $u_1, \dots, u_s, v_1, \dots, v_t$ such that each edge is of the form $u_iv_j$. This way we created at least $\mathrm{ex}(\lfloor n/|V(F)|\rfloor, F)/2$ copies of $F$ on at most $n$ vertices. Let $G''$ denote the resulting graph, and observe that $G''$ is bipartite, with part $A'$ consisting of the vertices replacing the vertices in $A$ and part $B'$ consisting of the vertices replacing the vertices in $B$. Since $F$ is connected, each copy of $F$ in $G''$ has one part in $A'$ and the other part in $B'$.

We claim that there is no rainbow copy of $F$ here. Assume otherwise, and observe that since $G'$ is $F$-free, at least two vertices $u$ and $v$ of this copy of $F$ are replacements of the same vertex of $G'$. Then $u$ and $v$ are in the same part of $F$, thus they have a common neighbor $w$. Observe that the edges $uw$ and $vw$ appear only in the same copy of $F$, a contradiction completing the proof. ■

Let us continue with the proof of Theorem 1.3 that we restate here for convenience.

**Theorem.**

$$\mathrm{rb}(n, P_4, P_4) = \begin{cases} n-3 & \text{if } n \geq 5, \\ 2 & \text{if } n = 4, \\ 0 & \text{if } n \leq 3. \end{cases}$$

*Proof.* The lower bound is given by the following construction. We take vertices $a, b, c_1, \dots, c_{n-3}, d$ and paths $abc_id$. Assume that this graph contains a rainbow $P_4$. A vertex $c_i$ or $a$ can only be at one of the ends of this path as the edges incident to $c_i$ are of the same color and only one edge is incident to $a$. Therefore, $b$ and $d$ are the middle vertices, but they are not adjacent, a contradiction. In the case $n = 4$, obviously any two copies of $P_4$ avoid a rainbow $P_4$.

To prove the upper bound, consider a family $\mathcal{H}$ of copies of $P_4$ without a rainbow $P_4$, and let $G$ be the graph consisting of the edges appearing in elements of $\mathcal{H}$. We say that an edge of $G$ is *light* if it is contained in exactly one copy of $P_4$. Observe that for three different elements $H, H', H'' \in \mathcal{H}$, we cannot have that $H$ contains an edge of $H''$ and $H'$ contains another edge of $H''$. Indeed, those two edges have a different color and the third edge gets the color of $H''$, showing that $H''$ is rainbow. This implies that there is at least one light edge in every element of $\mathcal{H}$, as a different copy of $P_4$ cannot contain the same three edges.

Let us pick a light edge from each element of $\mathcal{H}$ and let $G'$ be the resulting graph, then $|E(G')| = |\mathcal{H}|$ and assume towards a contradiction that $|E(G')| \geq n-2$. $G'$ is rainbow, hence

$P_4$-free. We claim that $G'$ is also triangle-free. Indeed, if $uvw$ is a triangle in $G'$, then the $P_4$ corresponding to $uv$ has an additional vertex $w'$ adjacent to, say, $u$. This means that $uw'$ is an edge of $G$. Observe that $w \neq w'$ since $uw$ is a light edge. Then the edges $vw$, $wu$, $uw'$ form a rainbow $P_4$ in $G$, a contradiction. Therefore, each component of $G'$ is a star.

**Claim 4.4.** *If $|E(G')| \geq n-2$ and $n \geq 4$, then*

***(i)*** *Every star component of $G'$ with at least 3 leaves contains a vertex that is contained in exactly one element of $\mathcal{H}$.*

***(ii)*** *If $n > 4$, then there is a vertex contained in at most one element of $\mathcal{H}$, or there are two vertices contained only in the same two elements of $\mathcal{H}$.*

***(iii)*** *If $n = 6$, then there is a vertex contained in at most one element of $\mathcal{H}$.*

*Proof.* Let us assume that there is a star component of $G'$ with center $u$ and leaves $v_1, \dots, v_k$ with $k \geq 3$. Assume that $uv_i$ is contained in $H_i \in \mathcal{H}$ and there is $H' \in \mathcal{H}$ with $H' \neq H_1$ that contains $v_1$. Then we can go from $v_1$ to a vertex $w \neq u$ using $H'$. Without loss of generality, $w \neq v_2$. Then we can go from $w$ to $v_1$ using $H'$, then from $v_1$ to $u$ using $H_1$, and from $u$ to $v_2$ using $H_2$. This way we obtain a $P_4$, thus it is not rainbow, which implies that $H' = H_2$. Moreover, if $w \neq v_3$, then we could go from $u$ to $v_3$ instead, using $H_3$. Therefore, we have $w = v_3$.

Then $H'$ contains the edges $v_1v_3$ and $uv_2$, and does not contain $uv_1$ or $uv_3$ by the lightness of the edges of $G'$, thus either $H' = uv_2v_1v_3$ or $H' = uv_2v_3v_1$. In the first case, we can go from $v_2$ to $v_1$ using $H'$, then from $v_1$ to $u$ using $H_1$, and then from $u$ to $v_3$ using $H_3$. In the second case, we can go from $v_2$ to $v_3$ using $H'$, then from $v_3$ to $u$ using $H_3$, and then from $u$ to $v_1$ using $H_1$. We found a rainbow $P_4$, a contradiction completing the proof of **(i)**. In fact, we have shown the stronger statement that *each* leaf of a star component of $G'$ with at least 3 leaves is contained in exactly one element of $\mathcal{H}$.

Assume now that there is no star component with at least three leaves, but there is a star component with center $u$ and exactly two leaves $v_1, v_2$, where $uv_i$ is contained in $H_i \in \mathcal{H}$. Assume that there is $H' \in \mathcal{H}$ with $H_1 \neq H' \neq H_2$ that contains $v_1$. Note that $u$ is not a neighbor of $v_1$ or $v_2$ in $H'$, since the edges $uv_i$ are light. If $v_1$ has a neighbor $w \neq v_2$ in $H'$, then we can go from $w$ to $v_1$ using $H'$, from $v_1$ to $u$ using $H_1$, and from $u$ to $v_2$ using $H_2$, a contradiction. If the only neighbor of $v_1$ in $H'$ is $v_2$, then $v_2$ has a neighbor $w \neq v_1$ in $H'$, and we can go from $w$ to $v_2$ using $H'$, from $v_2$ to $u$ using $H_2$, and from $u$ to $v_1$ using $H_1$, a contradiction. Therefore, in either case $v_1$ (and analogously $v_2$) are contained only in $H_1$ and $H_2$.

We are done with the proof of **(ii)** if a component has at least three vertices. If each component has at most 2 vertices, then there are at least three components, thus at most $n-3$ edges in $G'$, a contradiction that completes the proof of **(ii)**.

Assume now that $n = 6$. We are done unless we have exactly two components, a star with center $u$ and leaves $v_1, v_2$ and another star with center $u'$ and leaves $v_1', v_2'$. Indeed, otherwise, we either have a larger component and **(i)** completes the proof, or we have at least three components thus at most $n-3$ edges. Let $uv_i$ be in $H_i$ and $u'v_i'$ be in $H_i'$. The above description of star components with two leaves shows that for any $i$, $H_i'$ does not contain $v_1, v_2$ and $H_i$ does not contain $v_1', v_2'$. Therefore, $H_i$ contains $v_1, v_2, u, u'$ and $H_i'$

contains $v_1', v_2', u, u'$. Then we can go from $v_1'$ to $u'$ using $H_1'$, then from $u'$ to one of $u, v_1, v_2$ using $H_2'$, and then from this vertex to another vertex of $u, v_1, v_2$ using either the edge $uv_1$ and $H_1$, or the edge $uv_2$ and $H_2$. We obtained a rainbow $P_4$, a contradiction completing the proof of **(iii)**. ■

Let us return to the proof of the theorem. We apply induction on $n$. Let us start with the base cases $n \leq 5$. The cases $n \leq 3$ are trivial since we cannot take a $P_4$ on less than 4 vertices. If $n = 4$, we cannot have three edges in $G'$ without a triangle or $P_4$, unless $G'$ is a star with three leaves. However, in that case there is a vertex that is contained only in one element of $\mathcal{H}$ by **(i)** of Claim 4.4, and then the other elements of $\mathcal{H}$ cannot have four vertices. In the case $n = 5$, either $G'$ contains a star with at least three leaves, or $G'$ has to consist of a two-edge path and an isolated edge.

Assume that $G'$ contains a star with center $u$ and leaves $v_1, v_2, v_3$ with $H_i \in \mathcal{H}$ containing the edge $uv_i$. Without loss of generality, $H_1$ is the only element of $\mathcal{H}$ that contains $v_1$, using **(i)** of Claim 4.4. Then $v_2$ is a vertex of $H_3$, and its neighbor(s) in $H_3$ are neither $u$, nor $v_1$. Therefore, we can go from $v_1$ to $u$ using $H_1$, from $u$ to $v_2$ using $H_2$, and from $v_2$ to another vertex using $H_3$, a contradiction.

Assume now that $G'$ has to consist of a two-edge path $uvw$ and an isolated edge $u'v'$. Let $H$ be the copy of $P_4$ containing the edge $u'v'$. If an edge of $H$ from $u'$ or $v'$ goes to $u$ or $w$, then it forms a rainbow $P_4$ with $uv$ and $vw$. Therefore, the edge of $H$ adjacent to $u'v'$ goes to $v$. Without loss of generality, $v'v$ is an edge of $H$. Then the third edge of $H$ cannot be incident to $v$, as $uv$ and $vw$ are light edges, $vv'$ is already in $H$ and $vu'$ would create a triangle. Thus, the third edge of $H$ goes from $u'$ to $u$ or $w$, a possibility we have already ruled out. This contradiction proves the case $n = 5$.

Now consider an arbitrary $n \geq 6$. If $v$ is contained in at most one element of $\mathcal{H}$, then delete $v$. Then we obtain a rainbow $P_4$-free family $|\mathcal{H}'|$ of copies of $P_4$ on $n-1 > 4$ elements such that $|\mathcal{H}| \leq |\mathcal{H}'|+1$. By induction, $|\mathcal{H}'| \leq n-4$, completing the proof. If $n \geq 7$ and $v$ and $v'$ are contained in at most two elements of $\mathcal{H}$, then delete $v$ and $v'$. We obtain a family $|\mathcal{H}'|$ of copies of $P_4$ on $n-2 > 4$ elements such that $|\mathcal{H}| \leq |\mathcal{H}'|+2$. By induction, $|\mathcal{H}'| \leq n-5$, completing the proof. ■

Recall that $B_t$ denotes the *book graph*, which consists of $t$ triangles sharing an edge $uv$. We call $u$ and $v$ the *rootlet vertices* and the other $t$ vertices are the *page vertices*. Alon and Shikhelman [3] showed that $\mathrm{ex}(n, K_3, B_t) = o(n^2)$. The author [12] showed more generally that $\mathrm{ex}(n, K_r, B_t) = o(n^2)$. Here we give another simple bound.

**Proposition 4.5.** *For any $r$ and $t$, we have* $\mathrm{ex}(n, B_t, B_r) = o(n^2)$.

*Proof.* Let $G$ be a $B_r$-free graph on $n$ vertices. We count the copies of $B_t$ the following way. We pick a triangle $uvw$, $o(n^2)$ ways by the result of Alon and Shikhelman [3] mentioned above, then we pick $t-1$ other common neighbors of $u$ and $v$. As $u$ and $v$ have less than $r$ common neighbors, there are $O(1)$ ways to pick the additional vertices. Clearly every copy of $B_t$ is counted at least once, completing the proof. ■

The author [14] determined $\mathrm{ex}_3(n, \text{Berge-}B_r) = \mathrm{rb}(n, K_3, B_r)$ exactly for $n$ sufficiently large. Its value is $\lfloor n^2/8 \rfloor$ if $r \le 2$ and $\lfloor n^2/8 \rfloor + (r-1)^2$ for $r \ge 3$. We determine the asymptotics of $\mathrm{rb}(n, B_t, B_r)$ for every $r, t \ge 2$ in Theorem 1.4, which is restated below.

**Theorem.** *If $r, t \ge 2$, then we have $\mathrm{rb}(n, B_t, B_r) = (1 + o(1))n^2/8$.*

*Proof.* To prove the lower bound, let us take vertices $u_i$ and $v_i$ for $i \le n/4$, $w_1, \dots, w_{t-1}$ and $x_j$ for $j \le n - 2\lfloor n/4 \rfloor - t + 1$. For every $i$ and $j$, we take the copies of $B_t$ where the rootlet vertices are $u_i$ and $v_i$ and the page vertices are $w_1, \dots, w_{t-1}$ and $x_j$. We consider the graph formed by the edges of these copies of $B_t$.

The subgraph obtained by deleting $w_1, \dots, w_{t-1}$ is rainbow triangle-free (this is the construction from [29] and [27]). Therefore, in any rainbow $B_r$, one of the rootlet vertices is $w_\ell$ for some $\ell$. As the only neighbors of $w_\ell$ are the $u_i$'s and the $v_i$'s, the other rootlet vertex is, say, $u_1$. The only common neighbor of $w_\ell$ and $u_1$ is $v_1$, a contradiction since $r \ge 2$.

Let us continue with the upper bound. Let $\mathcal{H}$ be a family of copies of $B_t$ on $n$ vertices with no rainbow $B_r$. We say that a triangle or edge is *$p$-heavy* if it is contained in at least $p$ elements of $\mathcal{H}$ and *$p$-light* otherwise.

We partition $\mathcal{H}$ into three subfamilies. Let $\mathcal{H}_0$ denote the family of copies of $B_t$ that contain a 2-light triangle, $\mathcal{H}_1$ denote the family of copies of $B_t$ not in $\mathcal{H}_0$ that contain a 2-heavy, $3r$-light triangle, and $\mathcal{H}_2$ denote the rest of $\mathcal{H}$, i.e., the copies of $B_t$ where each triangle is $3r$-heavy.

**Claim 4.6.** $|\mathcal{H}_0| = (1 + o(1))n^2/8$.

*Proof of Claim.* We pick a 2-light triangle from each element of $\mathcal{H}_0$. Those triangles clearly cannot contain a rainbow $B_r$, thus there are at most $\mathrm{rb}(n, K_3, B_r) = n^2/8 + O(1)$ of them. We counted each triangle exactly once, completing the proof. ■

**Claim 4.7.** $|\mathcal{H}_1| = o(n^2)$.

*Proof of Claim.* Let us pick a 2-heavy $3r$-light triangle from each element of $\mathcal{H}_1$ and let $G_1$ denote the graph consisting of the edges of those triangles. Assume that $G_1$ contains a $B_{t(2r+1)}$ with rootlet vertices $u, v$ and page vertices $w_1, \dots, w_{t(2r+1)}$. As each edge is 2-heavy, we can take two elements of $\mathcal{H}$ to form a rainbow path with edges $uw_i$ and $w_iv$. Those two elements contain at most $2t$ other vertices.

We pick an element of $\mathcal{H}$ that contains the edge $uv$. It avoids without loss of generality $w_1, \dots, w_{2tr-t}$. Then we pick two elements of $\mathcal{H}$ to form a rainbow path $uw_1v_1$, they avoid without loss of generality $w_2, \dots, w_{2tr-3t}$. We continue this way, the elements of $\mathcal{H}$ we pick for the path $uw_iv$ avoid $w_{i+1}, \dots, w_{2tr-i2t-t}$, hence we can pick elements for the path $uw_rv$, obtaining a rainbow $B_r$, a contradiction.

We obtained that $G_1$ is $B_{t(2r+1)}$-free, thus contains $o(n^2)$ triangles. For each element of $\mathcal{H}_1$, we picked a triangle in $G_1$, and each triangle was picked at most $3r$ times, completing the proof. ■

**Claim 4.8.** $|\mathcal{H}_2| = o(n^2)$.

*Proof of Claim.* Observe first that the graph formed by the $3r$-heavy triangles must be $B_r$-free by Proposition 2.1, thus there are $o(n^2)$ such triangles by the result of Alon and Shikhelman [3] mentioned above. Also observe that if $t \geq r$, then $\mathcal{H}_2$ is empty by Proposition 2.1.

If each $3r$-heavy triangle is in at most $k = 3\binom{r-2}{t-1}$ elements of $\mathcal{H}_2$, then $|\mathcal{H}_2|$ is at most $k$ times the number of $3r$-heavy triangles and we are done. Assume that $uvw$ is in more than $k$ elements of $\mathcal{H}_2$. In particular, for one of the edges, say $uv$, $u$ and $v$ are the rootlet vertices of more than $k/3$ elements of $\mathcal{H}_2$. Each of those elements has $t-1$ vertices that are the page vertices different from $w$. This way, we assign distinct $(t-1)$-sets to these books $B_t$. We have assigned more than $\binom{r-2}{t-1}$ distinct $(t-1)$-sets, thus at least $r-1$ vertices of $G$ belong to some of these $(t-1)$-sets. Together with $w$, there are $r$ vertices such that the triangles they form with $u$ and $v$ are $k$-heavy. We found a $B_r$ consisting of $k$-heavy triangles, thus of $k$-heavy edges, hence Proposition 2.1 gives a rainbow $B_r$, a contradiction. ■

Combining the above claims completes the proof. ■

# 5 Concluding remarks

We have proved generalizations of some fundamental results concerning the Turán number of Berge hypergraphs. Probably several other results can be generalized the same way. We also mention some variants of Berge hypergraphs that can be studied in our setting.

The $t$-wise Berge hypergraphs [20] are defined similarly to Berge hypergraphs, but for each edge $uv$ of $F$, we take $t$ distinct hyperedges containing $uv$. The resulting $t|E(F)|$ hyperedges form a $t$-wise Berge copy of $F$. In our setting, we look for graphs $F$ such that we can pick $t$ distinct copies of $H$ for every edge of $F$ (altogether $t|E(F)|$ elements of $\mathcal{H}$).

A natural idea is to study the same problem for hypergraphs instead of graphs. This was already started in the Berge setting, see [4].

Another natural idea is to consider families $\mathcal{H}$ that consists of different graphs. This was also studied in the Berge setting, as taking cliques of different order corresponds to non-uniform hypergraphs.

We already mentioned in Section 2 the expansion, which is a specific Berge copy of a graph. In our setting, this corresponds to finding a rainbow copy of $F$ where vertices in the elements of $\mathcal{H}$ corresponding to the edges of $F$ are distinct outside the necessary intersections in $F$.

**Funding**: Research supported by the National Research, Development and Innovation Office - NKFIH under the grants KH 130371, SNN 129364, FK 132060, and KKP-133819, and by the János Bolyai scholarship.

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