



Effect of elliptical profiles on the steady motions of loose rolling wheels

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Abstract

In this paper, the model of a rigid wheel is analysed with a pure rolling contact with a horizontal rigid plane. The profile of the wheel is assumed to be an ellipse, which shape can be used for approximating various profiles in vehicle wheels. The elliptic profile can still be a basis of analytical solution, and in the same time, includes qualitatively new types of behaviour compared to the special cases of thin wheels and toroidal wheels known in the literature. The full nonlinear differential equations are derived and compared to the related known cases of equations. The two-parametric set of steady motions are calculated. We show that because of the elliptical profile, the qualitative dynamics significantly changes compared to the classical thin wheel and even to the toroidal wheel. Through bifurcations, four scenarios are found with different dynamical behaviours and stability of the solutions.

Keywords Rolling wheel · Steady motion · Stability · Elliptic wheel tread

1 Introduction

The problem of rolling wheels on a horizontal plane is a standard classical model in analytical mechanics. However, this model serves as a basis for the modelling of wheels and simple vehicles even in recent publications [1, 30]. The core of the problem is the dynamics of the disc (*thin wheel*) on the plane. This problem appears in the literature from different theoretical aspects, the rolling disc serves as benchmark problems for analytical and numerical methods and for understanding more general systems. The present paper focusses on the effect of the *wheel profile*, which can lead to a significant qualitative change in dynamics compared to the classical *thin wheel* case. We analyse the wheel with *elliptical* profile. With its varying parameters, this model is already capable to approximate several wheel shapes in practice, and it is still keeps the possibility for analytical solutions and parameter studies.

The literature on rolling thin wheels/discs is very diverse. Here, only a few important works are mentioned; for a detailed list of the literature, we refer to [10, 27] and [1].

The differential equations of the rolling disc were derived in the late XIX. century. The integration of the equations was performed by Chaplygin [13], Appell [5], and Korteweg [21] independently at the same time. The existence of steady motions and their stability analysis was first presented by [12] and [29]. Later, the stability analysis was combined with the integrability results by Mindlin [25] and [15]. The papers of O'Reilly [27] and Cushman et. al. [14] contain a detailed qualitative analysis of the problem, including bifurcations. The very recent paper of Vizi et. al. [30] contains the concise formulation of the equations of motion using the Appell–Gibbs equations.

The problem of rolling disc can be considered as a special case in a larger class of dynamical systems. In [6–8], the authors analyse the general properties arising from the integrability of dynamical systems also focusing on nonholonomic systems including rolling discs [10]. The rolling of general bodies of revolution on a horizontal plane was analysed already in [13] and later in [18, 25, 26]. The authors in [9] present a thorough general analysis of bodies of revolution including general theorems of integrability, conserved quantities and nonlinear stability of special solutions.

It is well-known in the literature that the rolling disc and similar related classes or problems possess *two-parameteric families of steady motions*. That is, in the four-dimensional reduced phase space, there are two-dimensional surfaces filled with equilibria. The integrability of the system causes

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this two-parametric surface to consist of stable regions filled by degenerate (nonlinear) centre and unstable regions filled by degenerate saddles (see, e.g. [9]). Each of these equilibria of the reduced space corresponds to a *steady motion* of the full mechanical system; the disc is rolling while its kinematic components are fixed in an appropriate coordinate frame. This structure of the phase space is common in similar problems that contain a rigid rolling body; for example, the authors analysed the families of steady solution in the dynamics of basketball [2] and golf ball [16]. The motion of rolling ellipsoids in the plane can be found in recent papers by Kilin and Pivovarova [19, 28].

In this paper, we focus on the effect of the wheel profile on the dynamics. Despite the existing papers such as [9] contain general theorems and conditions for any wheel profiles, the detailed analysis and analytic parameter study of the simple but non-trivial cases is important to see the effects of the parameters. The behaviour of the rolling disc is strongly related to vehicle dynamics, especially in the dynamics and control of *micromobility* devices such as unicycles [11, 30], scooters [17, 20], and bicycles [24]. As these works typically keeps the assumption of a *thin wheel*, the present results demonstrate the consequences for considering *elliptical profiles*.

In the recent paper of Agundez et. al. [1], the authors already consider in detail how the rolling wheel changes when then thin wheel assumption is generalised to *toroidal wheel*. The long calculation of the paper shows the significant change in algebraic complexity of the toroidal equations compared to the thin wheels. But, finally, the qualitative behaviour of the system does not change compared to the thin wheel.

In the present paper, we show that by extending the wheel profile model to the elliptical shape, the situation significantly changes. We will see that through bifurcations, several scenarios appear with qualitatively different solutions and stability. The first results of the present analysis was published in the conference paper [22]. Now, a significantly extended analysis is presented that contains the investigation of stability, the exploration of different scenarios, and the outlook of more general wheel profiles.

The main novelty of this study lies in the calculation of the steady motions and their stability for elliptical profiles. We demonstrate that, unlike the transition from thin to toroidal wheels – which leaves qualitative dynamics unchanged – the extension to elliptical profiles introduces significantly different types of qualitative dynamics predicted by general theories of rolling bodies. The study bridges the gap between simple benchmark examples and more realistic tire geometries while preserving the possibility of analytical calculations.

The paper is structured as follows: In Sect. 2, we present the mechanical model with the elliptical profile and calcu-

late the conditions of geometric contact and pure rolling. In Sect. 3, the static equilibria of the wheel are calculated. This step is important because unlike in the thin and toroidal wheels, non-trivial equilibria can occur, and the number and stability of equilibria determine the typical scenarios of the steady motions, as well. In Sect. 4, we derive the equations of motion of the wheel to the elliptic wheel profile. In Sect. 5, the structure of the governing equations are analysed and compared to the results in the literature. The main results are found in Sect. 6: For the elliptical profile, we explore the steady motions in the known form two-parametric families in the phase space but having specific features generated by the elliptic profile. Compared to the more simple geometries, the shape of the elliptical profile is shown to create significant new cases of the structure and stability of this set of solutions.

2 Mechanical model and kinematics

The main structure in the configuration and kinematics of the elliptical wheel is similar to those of the more simple cases of thin [30] and toroidal [1] wheels. In addition, some works contain [9] the results in a general form valid for any profiles. However, we provide now a full detailed derivation to explore the consequence of the elliptical profile in each derivation step.

The rough structure of the system can be predicted from analytical mechanic studies: the normal contact and rolling condition reduces the system to 5 degrees of freedom (DoF) in the configurational space and 3 DoFs in the velocity space. The resulting 8-dimensional phase space is reduced because of the planar $\mathcal{SO}(2)$ symmetry of the plane (by three) and the rotational symmetry of the body (by one), all resulting in a four-dimensional phase space.

The interesting results originate from the *geometric* features of the *elliptical* profile. This profile makes the equations significantly more complicated but still analytically solvable, which shows algebraically the background of the new types of behaviour.

2.1 Configuration

Consider a fixed rigid horizontal plane with an attached coordinate system (x, y, z) , where the axis z points vertically upward and $z = 0$ at the plane (see Fig. 1). As a short-hand notation, (x, y, z) also denote the coordinates of the centre of gravity G of the wheel, and then the location of G is given by

$$\mathbf{r}_G = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (1)$$

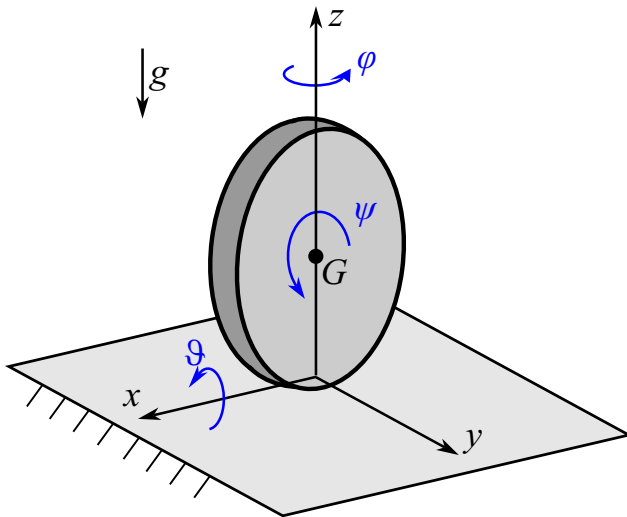


Fig. 1 Coordinate directions and Euler angles of the wheel. Starting from the reference configuration in the figure, the rotation of ϕ is applied about z (yaw); then, the rotation of ϑ is applied about the transformed x (tilt), and finally, the wheel is rotated by ψ about its transformed symmetry axis (pitch)

Assume that in a *reference configuration* the wheel is vertically positioned and its symmetry axis is parallel to the direction y (as in Fig. 1). For the orientation of the wheel, consider the Euler,¹ angles ϕ , ϑ , ψ using the intrinsic rotation sequence zxy . Then, we can express the rotation tensor

$$\mathbf{R} = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3, \quad (2)$$

where

$$\mathbf{R}_1 = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3)$$

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & -\sin \vartheta \\ 0 & \sin \vartheta & \cos \vartheta \end{bmatrix}, \quad (4)$$

$$\mathbf{R}_3 = \begin{bmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{bmatrix}. \quad (5)$$

The Euler angles have diverse naming conventions. In this paper, we use the following names:

- ϕ is the *yaw angle* of the wheel that describes the rotation about the vertical axis,
- ϑ is the *tilt angle* that measures the deviation of the vertical orientation,

¹ In the literature, this sequence of rotations (e.g. zyx) is often called the *Tait-Bryan angles* which is different from *classical Euler angles* (e.g. zyz). Here, we use the term *Euler angles* in a general sense.

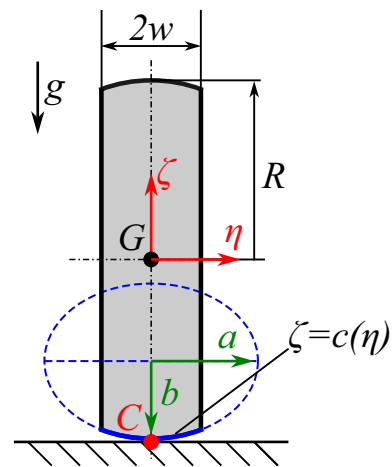


Fig. 2 Definition of the wheel profile by the local coordinates η , ζ in the vertical meridian section of the wheel. The ellipsoidal profile is defined by the axes a , b . By the function $\zeta = c(\eta)$, more general wheel profiles can be defined

- and ψ is the *pitch angle* that measures the rotation of the wheel about its symmetry axis.²

Finally, the configuration of the wheel is described by the location $\mathbf{r}_G$ of its centre and the rotation tensor $\mathbf{R}$.

2.2 Wheels with elliptical wheel profiles

The idea of elliptical profiles comes from the generalisation of toroidal profiles for the case where the meridian curve is not a circle but an ellipse. This assumption

- can cover a wider range of practical cases (for example, thick wheels of motorbikes),
- can still be solved analytically,
- and moreover, it introduces significant new qualitative behaviour compared to the thin or toroidal wheels.

2.2.1 Expression of elliptical profiles

Assume that the surface of the wheel is axisymmetric and has a *nominal wheel radius* R (see Fig. 2). The *profile* of the wheel is described by a local coordinate system in the *vertical meridian section*,³ where η is the axial direction and the ζ is the radial direction (see Fig. 2) measured from G . In the reference configuration, η , ζ are parallel to the global axes y , z , respectively.

² In the literature, the notation ϕ , ϑ , ψ of the rotation angles are quite common. But, for example the ϕ , ψ angles can be found in both combinations for the yaw and pitch angles..

³ The axes η , ζ are neither inertial nor body-fixed, but they are related to the intermediate frame defined by the rotation $\mathbf{R}_1 \mathbf{R}_2$.

If we describe the lower portion of the profile, the wheel profile is an explicit function in the form

$$\zeta = c(\eta) \quad (6)$$

with the condition $c(0) = -R$ for obtaining the wheel radius. With the formulation (6), arbitrary wheel profiles can be described. For a systematic study of general profiles, see [9].

Now, we assume that the profile is an ellipse having a radial axis a and a vertical axis b (see Fig. 2). The profile curve can then be described by the implicit function $\eta^2/a^2 + (\zeta + R - b)^2/b^2 = 1$. From this, the explicit form of the profile curve (6) becomes

$$\zeta = c(\eta) = -R + b - b \cdot \sqrt{1 - \eta^2/a^2}. \quad (7)$$

2.2.2 Special cases and wheel thickness

The form (6) of the elliptical profiles contains several special cases from the literature:

- The limit case $a = b = 0$ leads to the classical problem of *thin wheel* (see, e.g., [27, 30]).⁴
- The case $a = b$ leads to a *toroidal wheel* (circular profile) recently analysed in [1].
- The case $a = b = R$ leads to a *spherical wheel*, an important degenerate case with a trivial behaviour.

Note that in (6), η is mathematically defined in the range $[-a, a]$, but the actual thickness $2w$ of the wheel is typically smaller in practice (see Figs. 2 and 5). With an appropriate thickness w , it is possible to have large values $a > R$ or $b > R$ on real wheels; however, the range of these parameters are limited. For $b < R$, the thickness is limited by $w \leq w_{\max} = a$, and in the limit case $w = a$, it is possible to have a wheel tilt ϑ up to a maximum value 90° . For $b > R$, the thickness is limited by the self-intersection of the elliptical torus. Then, it can be shown from (7) that $w \leq w_{\max} = a\sqrt{1 - (R - b)^2/b^2}$. Then, the maximum possible tilt appears at $w = w_{\max}$ with

$$|\vartheta|_{\max} = \arctan \left(\frac{b}{a} \cdot \frac{\sqrt{2b/R - 1}}{b/R - 1} \right). \quad (8)$$

In practice, the typically smaller actual wheel thickness is a limit for large tilt angles on the elliptical surface.

⁴ Although the expression (6) is not formally valid for $a = 0$, the final results of (6) evaluated at the limit $a, b \rightarrow 0$ lead to the result of the thin wheel.

2.2.3 Local polynomial expressions

Consider the local series expansion of the elliptical profile (7) in the form

$$c(\eta) = -R + \frac{A_2}{2}\eta^2 + A_4\eta^4 + \mathcal{O}(\eta^6), \quad (9)$$

where $\mathcal{O}(\eta^6)$ denotes the terms being sixth- or higher-degree in η . For the elliptical profile, the coefficients are

$$A_2 = \frac{b}{a^2}, \quad A_4 = \frac{b}{8a^4}. \quad (10)$$

In this way, the results of the elliptical profile can be compared to more general profiles with a similar form of expansion. In (10), we can see the effect of the ellipse axes a, b . The parameter A_2 is the curvature at $\eta = 0$ (the radius of the curvature is $\rho = 1/A_2$), and A_4 measures the change of the curvature along the profile curve.

2.3 Scalar expression of the normal contact condition

In several contact problems, it is straightforward to formulate the geometric contact conditions in an implicit vectorial form, but in the same time, it can be challenging to obtain the explicit scalar algebraic expressions needed for the subsequent calculations. In [9], the authors consider the location vector of the contact point and the normal vector, which, in a partial body-fixed description, become variables dependent on each other. In this paper, we show the approach applied previously to the normal contact of wheelsets with the rails (see [3, 4]). We first introduce further auxiliary variables of the local coordinates of contact points and apply the contact condition to the redundant set of variables.

2.3.1 Calculation method

Assume that in all wheel configurations, the contact point is located in the *vertical meridian section* of the wheel,⁵ where we can apply the variables η, ζ .

Let η^* denote the location of the contact point in the axial direction (see Fig. 3). Then, its radial coordinate is determined by $\zeta^* = c(\eta^*)$. In the global coordinate frame, the location of the contact point C is given by

$$\mathbf{r}_C = \mathbf{r}_G + \mathbf{r}_{GC} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \mathbf{R}_1 \mathbf{R}_2 \cdot \begin{bmatrix} 0 \\ \eta^* \\ c(\eta^*) \end{bmatrix}. \quad (11)$$

⁵ This assumption can be verified from formal calculations but can also be proven from symmetry properties.

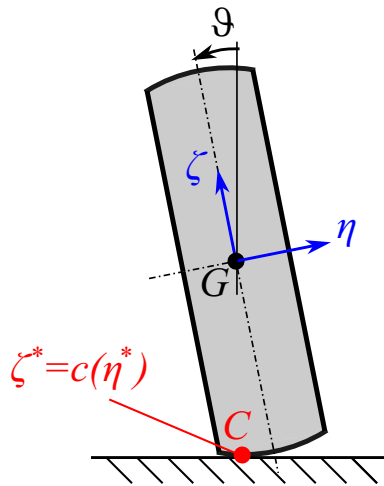


Fig. 3 Location of the contact point C in a tilted configuration of the wheel. The location of C is expressed by the local coordinates η^* , ζ^* in the vertical meridian plane

Now, we have an extended configuration containing the variables $(x, y, z, \phi, \psi, \vartheta, \eta^*) \in \mathbb{R}^7$; thus, the normal contact condition should eliminate two redundant variables to get a 5-dimensional space with minimum coordinates.

The normal contact condition at C is applied in two steps:

- First, we assume that the profile curve⁶ at C is tangent to the horizontal plane. That is, we can take

$$\frac{d\mathbf{r}_C}{d\eta^*} \cdot \mathbf{n} = 0, \quad (12)$$

where $\cdot$ is the scalar product and $\mathbf{n} = (0, 0, 1)^T$ is the unit vector in the normal (vertical) direction. It can be shown⁷ that (12) leads to the expression in the form

$$\eta^* = f_\eta(\vartheta), \quad (13)$$

and the radial coordinate of the contact point is then expressed by

$$\zeta^* = c(\eta^*) \equiv f_\zeta(\vartheta). \quad (14)$$

- Once the local coordinate η^* is found, we can prescribe that the contact point is located on the ground ($z = 0$). That is,

$$\mathbf{r}_C \cdot \mathbf{n} = 0, \quad (15)$$

⁶ The profile curve is assumed to be in the vertical meridian plane.

⁷ In addition to direct calculations, we can see that due to symmetry reasons, the tangency of the profile curve and the horizontal plane is not affected by the coordinates x, y, ϕ, ψ .

which after direct calculations, formally leads to the expression

$$z = f_z(\vartheta) \equiv -f_\eta(\vartheta) \sin \vartheta - f_\zeta(\vartheta) \cos \vartheta. \quad (16)$$

From (11), (12), and (15), the formulae of the functions f_η , f_ζ , f_z can be explicitly derived for both the elliptical and polynomial profiles. In each case, the wheel can be described as a set $\mathbf{q} = (q_1 \dots q_5) = (x, y, \phi, \vartheta, \psi)$ of five generalised coordinates.

2.3.2 Results for the elliptical profile

For the elliptical profile (7), the explicit solution of the normal contact condition in (13), (14) and (16) leads to

$$f_\eta(\vartheta) = -\frac{a^2 \sin \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}}, \quad (17)$$

$$f_\zeta(\vartheta) = -(R - b) - \frac{b^2 \cos \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}}, \quad (18)$$

$$f_z(\vartheta) = (R - b) \cos \vartheta + \sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}. \quad (19)$$

Note that in the special case of thin wheels ($a = b = 0$), the formulae (17)–(18) gives back the results $f_\eta(\vartheta) = 0$, $f_\zeta(\vartheta) = -R$, $f_z(\vartheta) = R \cos \vartheta$ known from the literature.

2.3.3 Results for the polynomial expansion

The calculation steps can be repeated to the local polynomial expansion form (9). Finally, for (13)–(14), we get

$$f_\eta(\vartheta) = -\frac{1}{A_2} \vartheta + \frac{24A_4 - 2A_2^3}{6A_2^4} \vartheta^3 + \mathcal{O}(\vartheta^5), \quad (20)$$

$$f_\zeta(\vartheta) = \frac{1}{2A_2} \vartheta^2 + \frac{2A_2^3 - 18A_4}{6A_2^4} \vartheta^4 + \mathcal{O}(\vartheta^6), \quad (21)$$

and the form of the expression (16) becomes

$$f_z(\vartheta) = C_2 \vartheta^2 + C_4 \vartheta^4 + \mathcal{O}(\vartheta^6), \quad (22)$$

where

$$C_2 = \frac{1 - A_2 R}{2A_2}, \quad C_4 = \frac{A_2^4 R + 2A_2^3 - 24A_4}{24A_2^4}. \quad (23)$$

By substituting the coefficients (10) of the elliptical profile into (23), we get

$$C_2 = \frac{a^2 - bR}{2b}, \quad C_4 = \frac{(2a^2 b^2 - 3a^4 + b^3 R)}{24b^3}. \quad (24)$$

for the elliptical profile.⁸

In (22), C_2 and C_4 are the quadratic and quartic coefficients in the expression of the height z of G as a function of the tilt angle ϑ , strongly related to the potential function. It is clear that the qualitative properties of the potential function has an essential effect on the dynamics (see also [9]). Thus, we can expect that the sign of the coefficients C_2 and C_4 will be important at determining the dynamical properties, as well.

2.4 Scalar expressions of the rolling constraint

Now, let us express the two scalar equations originating from the nonholonomic constraint from rolling.

The velocity state of the wheel can be described by the velocity $\mathbf{v}_G$ of the centre of gravity and the angular velocity $\boldsymbol{\omega}$ of the wheel given by

$$\mathbf{v}_G = \begin{bmatrix} \dot{x} \\ \dot{y} \\ f'_z(\vartheta)\dot{\vartheta} \end{bmatrix}, \quad \boldsymbol{\omega} = \mathbf{R}_1 \mathbf{R}_2 \cdot \begin{bmatrix} \dot{\vartheta} \\ \dot{\phi} \sin \vartheta + \dot{\psi} \\ \dot{\phi} \cos \vartheta \end{bmatrix}, \quad (25)$$

respectively.⁹ In (25), the dot denotes differentiation with respect to the time t , and $f'_z(\vartheta)$ denotes the derivative of $f_z(\vartheta)$ with respect to ϑ .

If we assume pure rolling then the velocity at the contact point C vanishes,

$$\mathbf{v}_C = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{GC} = \mathbf{0}, \quad (26)$$

where the vector

$$\mathbf{r}_{GC} = \mathbf{R}_1 \mathbf{R}_2 \cdot \begin{bmatrix} 0 \\ f_\eta(\vartheta) \\ f_\zeta(\vartheta) \end{bmatrix} \quad (27)$$

is expressed by the quantities defined in (13),(14).

Let us substitute (25) and (27) into (26) and multiply by $\mathbf{R}_1^{-1}$ from the left. Then we get

$$\begin{aligned} \dot{x} \cos \phi + \dot{y} \sin \phi - f_\zeta(\vartheta)(\dot{\phi} \sin \vartheta + \dot{\psi}) + \\ + f_\eta(\vartheta)\dot{\phi} \cos \vartheta \equiv a_{10} + \sum_{i=1}^5 a_{1i}\dot{q}_i = 0, \\ -\dot{x} \sin \phi + \dot{y} \cos \phi + f_\zeta(\vartheta)\dot{\vartheta} \cos \vartheta + \end{aligned} \quad (28)$$

⁸ It can be checked that by the direct series expansion of (19), we get the same result as (24).

⁹ The formula of the angular velocity can be obtained either formally as an axial vector of the tensor $\mathbf{R}\mathbf{R}^{-1}$, or, by the elementary calculation of the subsequent rotations.

$$+ f_\eta(\vartheta)\dot{\vartheta} \sin \vartheta \equiv a_{20} + \sum_{i=1}^5 a_{2i}\dot{q}_i = 0. \quad (29)$$

In this form of, (28)–(29) take the form of standard nonholonomic constraints as linear combinations of the generalised velocities $\dot{q}_i$ and the coefficients $a_{ji}(\mathbf{q})$.

3 Static equilibria of the tilted wheel

In this section, we examine the static equilibria of the wheel based on the standard study of the potential function. Obtaining the static equilibria is not an independent goal of the study, but the number and type of equilibria is strongly connected to the steady motions analysed later.

In the case of thin wheels, the only equilibrium is the vertical position ($\vartheta = 0$), which is unstable [14, 27, 30]. The toroidal wheel (circular profile) with $a = b < R/2$ was shown to give the same result [1]. However, in the present study of elliptical and polynomial profiles, several different scenarios occur.

The gravity having a coefficient g and pointing to the negative z direction (see Fig. 1). Thus, the form of potential energy is simply

$$U = mgz = mgf_z(\vartheta). \quad (30)$$

As usual, the static equilibria are given by solving $dU/d\vartheta = 0$ and the (static) stability are determined by the sign of the second derivative $d^2U/d\vartheta^2$.

3.1 Results for the elliptical profile

For the elliptical profile, (19) and (30) lead to

$$\begin{aligned} \frac{\partial U}{\partial \vartheta} = -mg(R-b) \sin \vartheta + \\ + mg \frac{(a^2 - b^2) \sin \vartheta \cos \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}}. \end{aligned} \quad (31)$$

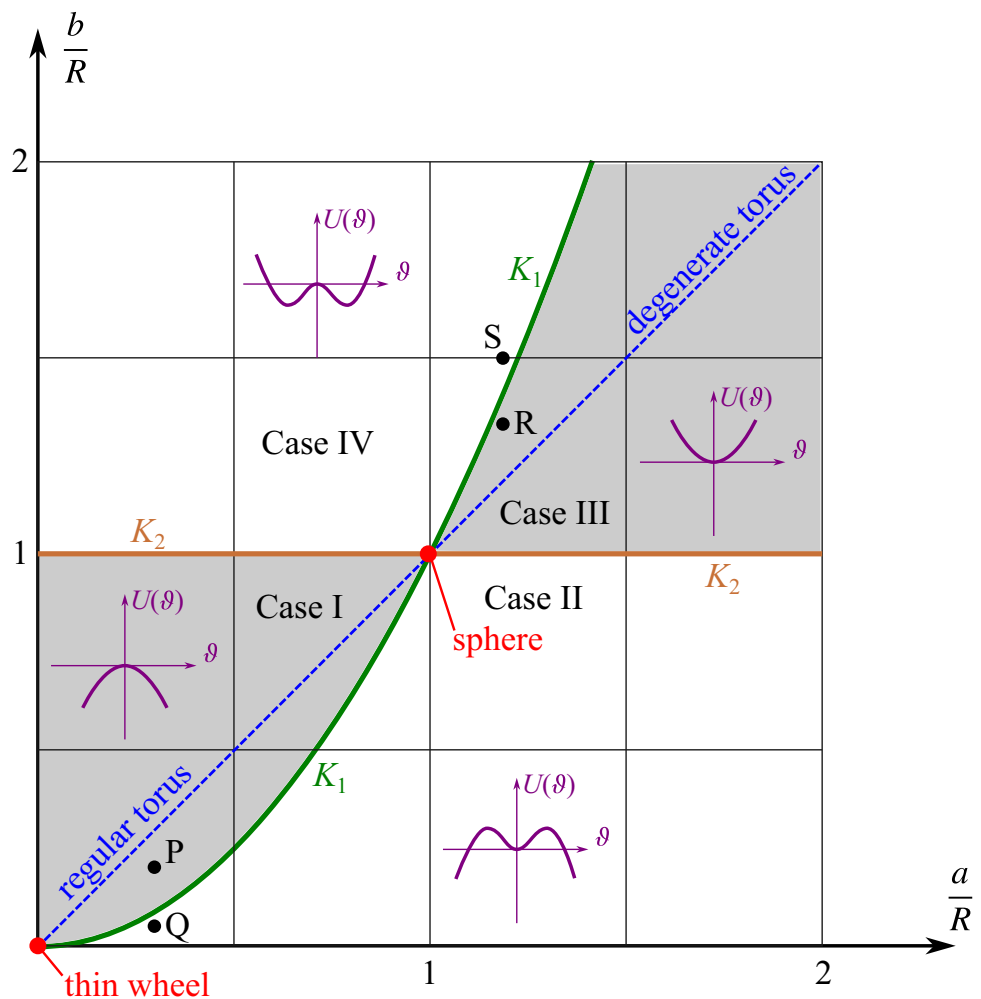
The equilibria are the roots of (31). The trivial equilibrium is the vertical orientation

$$\vartheta_1 = 0, \quad (32)$$

The non-trivial equilibria can be found by lengthy but straightforward calculations. In the physically relevant¹⁰

¹⁰ Mathematically, there are other non-trivial equilibria but they exist for the physically unrealistic range $|\vartheta| > \pi/2$. In addition (8) determines a more strict region of tilt angles for certain range of parameters.

Fig. 4 Regions in the parameter plane of a/R and b/R determined by the curves K_1 and K_2 . Cases I–IV are distinguished by the shape of the potential function $U(\vartheta)$ and the number of static equilibria. In the shaded regions (Cases I and III), there is only a trivial static equilibrium. In the unshaded regions (Cases II and IV), there are a pair of non-trivial equilibria, as well. Points denoted by P , Q , R , S correspond to the parameter values in Figs. 6 and 8. These parameter values can be seen in Table 1



region $-\pi/2 < \vartheta < \pi/2$, the two non-trivial equilibria are

$$\vartheta_{2,3} = \pm \arcsin \sqrt{\frac{(bR - a^2)(a^2 - 2b^2 + bR)}{(b^2 - a^2)(a^2 - 2bR + R^2)}}, \quad (33)$$

and they exist for the parameter range

$$(bR - a^2)(b - R) > 0. \quad (34)$$

Otherwise, the only static equilibrium is the trivial one $\vartheta_1 = 0$ in (32). In the plane of the dimensionless parameters a/R and b/R , (34) contains two boundary curves:

- The curve K_1 is defined by the implicit form

$$K_1\left(\frac{a}{R}, \frac{b}{R}\right) = \frac{b}{R} - \left(\frac{a}{R}\right)^2 = 0. \quad (35)$$

At this boundary, the two non-trivial equilibria merge with the trivial one at $\vartheta = 0$. In the meantime, the triv-

ial solution changes the stability (stable for $b < a^2/R$). Consequently, a *pitchfork bifurcation* occurs at K_1 .

- The curve K_2 is defined by

$$K_2\left(\frac{a}{R}, \frac{b}{R}\right) = \frac{b}{R} - 1 = 0. \quad (36)$$

This boundary corresponds to the case where non-trivial equilibria reach the degenerate limit $\vartheta = \pm\pi/2$ of the tilt by 90 degrees.

The results show that, compared to the thin wheel or regular toroidal wheel, the elliptical profile introduces two significant new phenomena:

- With a thin wheel, the trivial equilibrium is always unstable. With an elliptical profile, a stable trivial equilibrium is possible.
- With a thin wheel, the only equilibrium is the trivial one. With an elliptical profile, a pair of non-trivial equilibria can appear in the regions determined by (34).

The results can be depicted in the plane of $a/R \in \mathbb{R}^+$ and $b/R \in \mathbb{R}^+$ (see Fig. 4). The curves K_1 and K_2 divide the relevant parameter space into four regions, which are now denoted by Cases I–IV:

- In **Case I**, there is a single trivial unstable equilibrium. In this case, a and b are comparable and not larger than R . This case contains the regular *toroidal* wheel (circular profile, [1]) on the line $a/R = b/R < 1$. Moreover, we get the thin wheel in the limit case $a/R \rightarrow 0$ and $b/R \rightarrow 0$.
- In **Case II**, there is a trivial stable equilibrium and a pair of unstable equilibria. This case occurs when a is significantly larger than b , which can be called a flat wheel tread.
- In **Case III**, there is a single trivial stable equilibrium. This case occurs where both a/R and b/R are significantly large (surely larger than 1). On the line $a/R = b/R > 1$, we can find the degenerate barrel-shaped torus shapes.
- In **Case IV**, there is an unstable trivial equilibrium and two stable equilibria. This case occurs when b/R is greater than 1 and significantly larger than a/R .

The intersection of the bifurcation curves K_1 and K_2 at $a/R = b/R = 1$ is the secondary bifurcation which corresponds to the shape of the spherical wheel. The typical geometries of Cases I–IV can be found in Fig. 5. In Fig. 4, the points P , Q , E , S correspond to typical values from each case, which values can be found in Table 1 and used later for the diagrams.

3.2 Results for the polynomial approximation

In the case of the fourth degree polynomial profile (9), the derivative of (30) becomes

$$\frac{\partial U}{\partial \vartheta} = \vartheta \left(2C_2 + 4C_4\vartheta^2 + \mathcal{O}(\vartheta^4) \right). \quad (37)$$

In addition to the trivial equilibrium $\vartheta_1 = 0$, the *approximation* of the non-trivial equilibria are

$$\vartheta_{2,3} \approx \pm \sqrt{\frac{-C_2}{2C_4}} = \pm \sqrt{\frac{6A_2^3(A_2R - 1)}{A_2^4R + 2A_2^3 - 24A_4}}. \quad (38)$$

The approximation (38) is valid only if ϑ is sufficiently small. According to (38), the non-trivial equilibria exist if the signs of C_2 and C_4 are different. The two bifurcations are the following:

- A pitchfork bifurcation occurs at $C_2 = 0$. Having $\vartheta = 0$, this condition accurately reproduces the curve

K_1 obtained at (35). From (24)–(22), we can see that C_2 is proportional to $bR - a^2$ of (34).

- At $C_4 = 0$, the non-trivial equilibria reach the degenerate limit $\vartheta = \pm\pi/2$. As the polynomial expression is valid within a limited range of ϑ , this is only a theoretical boundary and it is not useful to approximate the elliptical case.

Apply the approximate formula (38) to the elliptical profile using (10). Then, we get

$$\vartheta_{2,3} \approx \sqrt{\frac{6b^2(a^2 - bR)}{3a^4 - 2a^2b^2 - b^3R}}. \quad (39)$$

It can be numerically verified that in the vicinity of the curve K_1 with $bR - a^2 = 0$, the results provided by the approximate formula (39) and the explicit formula (33) are close to each other.

3.3 Physical validity of the non-trivial equilibria

In case of the elliptical profiles, we have to consider the finite thickness $2w$ of the wheel (see Fig. 2). That is, the real wheel tread are determined a finite section of the profile curve $c(\eta)$ within the interval $-w \leq \eta \leq w$. From the inverse of the function (13), we can find the maximum tilt angle

$$\vartheta^{\max} = f_{\eta}^{-1}(w) \quad (40)$$

where the contact point reaches the edge of the wheel. The non-trivial equilibria are physically relevant only if their absolute value do not exceed the maximum tilt, $|\vartheta_{2,3}| < \vartheta^{\max}$. We also determined in (8), that the thickness w is also limited by the parameters a , b of the ellipse. A full analysis would include the combined geometric description including both cases where the contact point is located on the elliptical profile or at the edge. Then, the continuation of the non-trivial equilibria would be possible in the region of edge contact between the wheel and the road.

4 Equations of motion

In this section, let us derive the equations of motion of the wheel. Obtaining the differential equations of the *thin wheel* is a standard benchmark example in the literature and it has been derived by using formulations and sets of variables. One of the interesting derivation methods is the usage of the Appell-Gibbs equations which results to an algebraically very simple form by using the components of the angular velocities (see e.g. [30]). For the elliptical profile, the essence of the model is not much different, but the algebraic derivations become significantly more complicated.

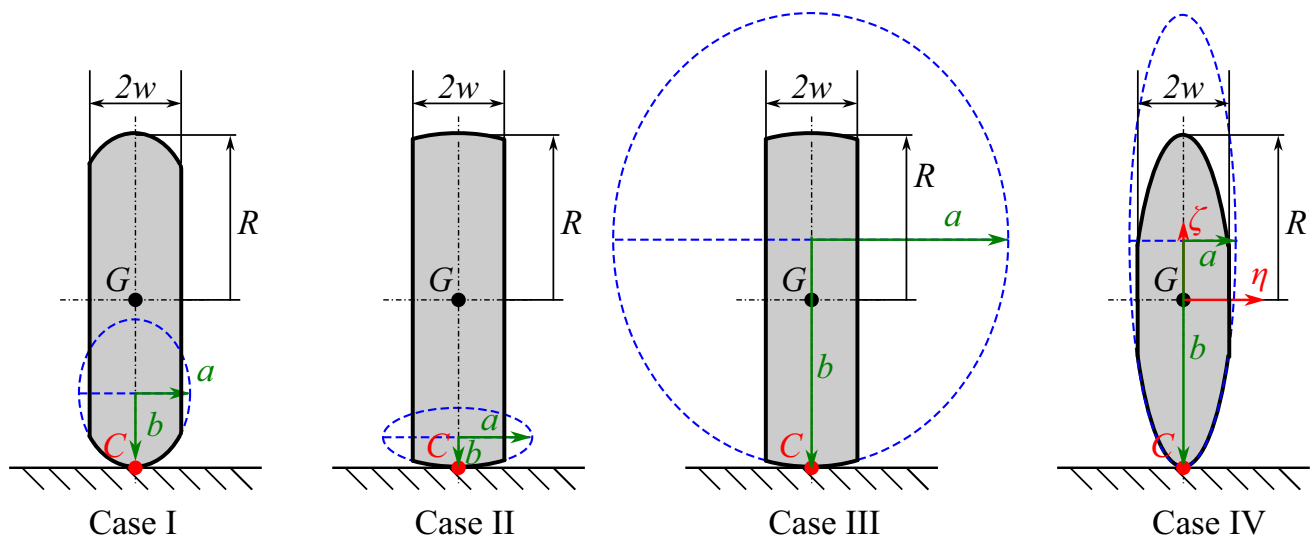


Fig. 5 Typical geometry of wheel profiles in Cases I–IV from the regions can be seen in Fig. 4

Table 1 Parameter values of the points P, Q, R, S

point	Case	a/R	b/R
P	Case I	0.3	0.2
Q	Case II	0.3	0.04
R	Case III	1.2	1.3
S	Case IV	1.2	1.5

These points are depicted in Fig. 4, and these values are used in Figs. 6 and 8. In all cases, further parameter values are $J_1/(mR^2) = 1/4$, $J_2/(mR^2) = 1/2$, $g = 10 \text{ m/s}^2$, $R = 0.3 \text{ m}$

In [9], the authors derive the equations of motions in a general form valid for any profiles of bodies of revolution. Those equations could be applied directly to the elliptical profile. However, [9] uses intuitive abstract state variables created by linear combination of the linear and angular momenta of the system. That formulation proved to be effective to show general properties of the system including integrability and conserved quantities. In the present paper, we focus mainly on the steady motions and want to keep the original variables for the sake of simple physical implementation. Hence, we present briefly the derivation of the equations of the motion by using standard methods.

Due to the presence of non-holonomic constraints (28)–(29), the derivation is performed by Lagrange's Equation of the second kind with constraint multipliers, which is also called the Routh-Voss equation. We consider the generalised coordinates $\mathbf{q} = (q_1 \dots q_5) = (x, y, \psi, \vartheta, \phi)$ with the two non-holonomic constraints (28)–(29).

Due to the rotational symmetry of the wheel, the inertia tensor in the body-fixed reference frame is given by

$$\hat{\mathbf{J}} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_1 \end{bmatrix}, \quad (41)$$

where the moments of inertia J_2 (about the symmetry axis) and J_1 (about the transversal directions) are computed with respect to the centre of gravity G . Then, the inertia tensor in the steady inertial frame becomes¹¹

$$\mathbf{J} = \mathbf{R}_1 \mathbf{R}_2 \hat{\mathbf{J}} \mathbf{R}_2^T \mathbf{R}_1^T. \quad (42)$$

The kinetic energy of the wheel is given by

$$T = \frac{1}{2} m \mathbf{v}_G^T \mathbf{v}_G + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega}, \quad (43)$$

which, by using (25), takes the form

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + f_z'(\vartheta)^2 \dot{\vartheta}^2) + \frac{1}{2} J_1 (\dot{\vartheta}^2 + \dot{\phi}^2 \cos^2 \vartheta) + \frac{1}{2} J_2 (\dot{\phi} \sin \vartheta + \dot{\psi})^2. \quad (44)$$

The potential energy U of the wheel was already calculated in (30). The Lagrangian of the system is given by

$$L = T - U = -mgf_z(\vartheta) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + f_z'(\vartheta)^2 \dot{\vartheta}^2) + \frac{1}{2} J_1 (\dot{\vartheta}^2 + \dot{\phi}^2 \cos^2 \vartheta) + \frac{1}{2} J_2 (\dot{\phi} \sin \vartheta + \dot{\psi})^2, \quad (45)$$

¹¹ The third rotation $\mathbf{R}_3$ can be omitted due to the symmetry of the wheel with respect to the rotation about its symmetry axis.

which contains the explicit expression $f_z(\vartheta)$ of the geometric constraint calculated in (19) and (23). Finally, as no further external forces are present, the Routh-Voss equations of the system are given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda_1 a_{1i} + \lambda_2 a_{2i}, \quad (46)$$

where $i = 1 \dots 5$, and λ_1, λ_2 are the Lagrangian multipliers related to the nonholonomic constraints (28)–(29).

The derivation of the equations of motion is lengthy but standard and straightforward. First, the variables $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ can be expressed from the constraints (28)–(29). Then, after evaluation of Eq. (46), a linear system is obtained for the variables $\ddot{\psi}, \ddot{\vartheta}, \ddot{\phi}, \lambda_1, \lambda_2$. After eliminating the multipliers λ_1, λ_2 , we finally arrive to three second-order differential equations (ODEs) in ψ, ϑ, ϕ .

This process could be performed on any profile while we know the explicit expressions of the geometric contact condition. Now, the calculations are done for the elliptical profiles we derived in (19).

4.1 Equations of motion of elliptical wheels

Let us perform the derivation steps above for the elliptical profile characterised by (7) and (17)–(19). Finally, we get a differential equation in the form

$$\mathbf{M} \begin{bmatrix} \ddot{\psi} \\ \ddot{\vartheta} \\ \ddot{\phi} \end{bmatrix} + \mathbf{h}\mathbf{\Omega} + \mathbf{g} = \mathbf{0}, \quad (47)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{h}$ is the coefficient matrix of inertial terms

$$\mathbf{\Omega} = [\dot{\vartheta}^2 \dot{\phi}^2 \dot{\psi}\dot{\vartheta} \dot{\vartheta}\dot{\phi} \dot{\psi}\dot{\phi}]^T, \quad (48)$$

and $\mathbf{g}$ contains the terms from gravity. The structure of these quantities becomes

$$\mathbf{M} = \begin{bmatrix} M_{11} & 0 & M_{13} \\ 0 & M_{22} & 0 \\ M_{13} & 0 & M_{33} \end{bmatrix}, \quad (49)$$

$$\mathbf{h} = \begin{bmatrix} 0 & 0 & h_{13} & h_{14} & 0 \\ h_{21} & h_{22} & 0 & 0 & h_{25} \\ 0 & 0 & h_{33} & h_{34} & 0 \end{bmatrix}, \quad (50)$$

$$\mathbf{g} = \begin{bmatrix} 0 \\ g_2 \\ 0 \end{bmatrix}. \quad (51)$$

The non-zero components of (49)–(51) are given by

$$M_{11} = J_2 + mC^2(\vartheta),$$

$$\begin{aligned} M_{22} &= J_1 + mA^2(\vartheta) + mB^2(\vartheta), \\ M_{33} &= J_1 \cos^2 \vartheta + J_2 \sin^2 \vartheta + mA^2(\vartheta), \\ M_{13} &= J_2 \sin \vartheta + mA(\vartheta)C(\vartheta), \end{aligned} \quad (52)$$

$$\begin{aligned} h_{13} &= -mC(\vartheta)D(\vartheta) \sin \vartheta, \\ h_{14} &= J_2 \cos \vartheta - mC(\vartheta)D(\vartheta) + 2mC(\vartheta)B(\vartheta), \\ h_{33} &= J_2 \cos \vartheta - mA(\vartheta)D(\vartheta) \sin \vartheta, \\ h_{34} &= 2(J_2 - J_1) \sin \vartheta \cos \vartheta - mA(\vartheta)D(\vartheta) \\ &\quad + 2mA(\vartheta)B(\vartheta), \end{aligned} \quad (53)$$

$$\begin{aligned} h_{21} &= -mA(\vartheta)D(\vartheta), \\ h_{22} &= -(J_2 - J_1) \sin \vartheta \cos \vartheta - mA(\vartheta)B(\vartheta), \\ h_{25} &= -J_2 \cos \vartheta - mB(\vartheta)C(\vartheta), \\ g_2 &= -mgA(\vartheta). \end{aligned} \quad (54)$$

In (52)–(54), the expressions of the geometric quantities $A(\vartheta)$ – $D(\vartheta)$ are given by

$$\begin{aligned} A(\vartheta) &= f_\eta(\vartheta) \cos \vartheta - f_\zeta(\vartheta) \sin \vartheta \\ &= \sin \vartheta \left(\frac{(b^2 - a^2) \cos \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}} + R - b \right), \end{aligned} \quad (55)$$

$$\begin{aligned} B(\vartheta) &= -f_\eta(\vartheta) \sin \vartheta - f_\zeta(\vartheta) \cos \vartheta \\ &= (R - b) \cos \vartheta + \sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}, \end{aligned} \quad (56)$$

$$C(\vartheta) = -f_\zeta(\vartheta) = \left(\frac{b^2 \cos \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}} + R - b \right), \quad (57)$$

$$\begin{aligned} D(\vartheta) &= \frac{f'_\zeta(\vartheta)}{2 \sin \vartheta} - \frac{f'_\eta(\vartheta)}{2 \cos \vartheta} \\ &= \left(\frac{a^2 b^2}{(a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta)^{3/2}} \right). \end{aligned} \quad (58)$$

The explicit expressions for the second derivatives $\ddot{\psi}, \ddot{\vartheta}, \ddot{\phi}$ could be obtained by multiplying (47) by the inverse matrix $\mathbf{M}^{-1}$. As the determinant

$$\begin{aligned} \det \mathbf{M} &= (J_1 + m(A^2(\vartheta) + B^2(\vartheta))) \cdot \\ &\quad \cdot [J_1(J_2 + mC^2(\vartheta)) \cos^2 \vartheta + J_2m(A(\vartheta) - C(\vartheta) \sin \vartheta)^2] \end{aligned} \quad (59)$$

is already a rather complicated expression, we prefer the form (47) for the subsequent analysis.

4.2 Special cases

From the ellipsoidal profile, we can get bank the special cases of the toroidal and thin wheels, which can be used as a comparison to the literature.

4.2.1 Toroidal wheel

In the special case of the toroidal wheel (circular profile), we can take $b = a$. Then, the geometric expressions (55)–(58)

become much more simple,

$$\begin{aligned} A(\vartheta) &= (R - a) \sin \vartheta, \\ B(\vartheta) &= (R - a) \cos \vartheta + a, \\ C(\vartheta) &= R - a + a \cos \vartheta, \\ D(\vartheta) &= a. \end{aligned} \quad (60)$$

However, the overall structure of the equations does not become significantly easier.

In [1], the authors derive the equations of motions in redundant coordinates without eliminating the nonholonomic constraint and getting to the form (47) in the minimal coordinates. In [1], the analysis of steady motion becomes algebraically complicated. The authors of the present paper suggest that the derivation in the present formulation provide a more efficient path to the analysis of toroidal wheel itself, while it also contains the more general ellipsoidal profiles.

4.3 Special case of the thin wheel

For checking the calculations, we can consider the special case of $a = b = 0$, we get the classical thin wheel problem. Then, the coefficients (55)–(58) simplify to

$$\begin{aligned} A(\vartheta) &= R \sin \vartheta, \\ B(\vartheta) &= R \cos \vartheta, \\ C(\vartheta) &= R, \\ D(\vartheta) &= 0. \end{aligned} \quad (61)$$

By substituting (61) into (52)–(64), we get

$$\begin{aligned} M_{11} &= J_2 + mR^2, \\ M_{22} &= J_1 + mR^2, \\ M_{33} &= J_1 \cos^2 \vartheta + (J_2 + mR^2) \sin^2 \vartheta, \\ M_{13} &= (J_2 + mR^2) \sin \vartheta, \end{aligned} \quad (62)$$

$$\begin{aligned} h_{13} &= 0, \\ h_{14} &= (J_2 + 2mR^2) \cos \vartheta, \\ h_{33} &= J_2 \cos \vartheta, \\ h_{34} &= 2(J_2 - J_1 + mR^2) \sin \vartheta \cos \vartheta, \end{aligned} \quad (63)$$

$$\begin{aligned} h_{21} &= 0, \\ h_{22} &= -(J_2 - J_1 + mR^2) \sin \vartheta \cos \vartheta, \\ h_{25} &= -(J_2 + mR^2) \cos \vartheta, \\ g_2 &= -mgR \sin \vartheta. \end{aligned} \quad (64)$$

We can see that some components from (50) are vanishing. Moreover, the determinant (59) simplifies to

$$\det \mathbf{M} = J_1(J_1 + mR^2)(J_2 + mR^2) \cos^2 \vartheta. \quad (65)$$

The vanishing terms and the final simple form of (65) can be the reason why the equations of motion of the thin wheel can be found in a compact form in several works (see e.g. [14, 27, 30]). When the simplest non-trivial wheel profile appears, the final explicit form of the differential equation becomes complicated due to the form of the determinant (59) in the denominators.

5 Structure of the equations of motion

In this section, we analyse the formal structure of the governing equations without considering the exact algebraic formulae on geometric quantities. Then, we can compare the results to those in the literature [9] formulated in different variables.

5.1 Phase space of the wheel

The configuration of the system has five generalized coordinates $\mathbf{q} = (x, y, \psi, \vartheta, \phi)$, but due to the non-holonomic rolling constraint, the velocity state is determined by only the three derivatives $\mathbf{u} = (\dot{\psi}, \dot{\vartheta}, \dot{\phi})$. Then, we can say that the system has five degrees of freedom *geometrically* but only three degrees of freedom *kinematically*. Then, the dynamics could be described in the resulting phase space of $\bar{\mathbf{w}} = (\mathbf{q}, \mathbf{u}) \in \mathbb{R}^8$. However, the equations of motion (47) are independent from the x, y, ψ, ϕ , which can be explained through geometric symmetries. These so-called *cyclic coordinates* can be eliminated. Finally the dynamics of system of (47) in a reduced four dimensional phase space of

$$\mathbf{w} = (w_1 \dots w_4) = (\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi}) \in \mathbb{R}^4 \quad (66)$$

becomes

$$\dot{\mathbf{w}} = \mathbf{f}(\mathbf{w}) = \begin{bmatrix} \dot{\vartheta} \\ -\mathbf{K}\boldsymbol{\Omega} - \mathbf{L} \end{bmatrix}, \quad (67)$$

where

$$\mathbf{K} = \mathbf{M}^{-1} \mathbf{h} = \begin{bmatrix} 0 & 0 & K_{13} & K_{14} & 0 \\ K_{21} & K_{22} & 0 & 0 & K_{25} \\ 0 & 0 & K_{33} & K_{34} & 0 \end{bmatrix}, \quad (68)$$

$$\mathbf{L} = \mathbf{M}^{-1} \mathbf{g} = \begin{bmatrix} 0 \\ L_{12} \\ 0 \end{bmatrix}. \quad (69)$$

Note that due to the special form the mass matrix (49), the matrices (68)–(69) they has the same structure and non-zero components of (50) and (51), respectively.

By writing the scalar components of (67), we formally get

$$\begin{aligned}(\vartheta)' &= \dot{\vartheta}, \\(\dot{\psi})' &= \dot{\vartheta} (K_{13}(\vartheta)\dot{\psi} + K_{14}(\vartheta)\dot{\phi}), \\(\dot{\vartheta})' &= K_{21}(\vartheta)\dot{\vartheta}^2 + K_{22}(\vartheta)\dot{\phi}^2 + K_{25}(\vartheta)\dot{\psi}\dot{\phi} \\&\quad - L_{12}(\vartheta), \\(\dot{\phi})' &= \dot{\vartheta} (K_{33}(\vartheta)\dot{\psi} + K_{34}(\vartheta)\dot{\phi}).\end{aligned}\quad (70)$$

In (70), we can identify the overall quadratic dependence on the derivatives of the angles. Moreover, the derivative $\dot{\vartheta}$ of the tilt angle appears as a multiplier in three components, which property is the reason of the two-parametric families of steady motions calculated below.

5.2 Comparison to the literature

In [9], the authors derive the equations of motion of a general axisymmetric body rolling on the plane. Those equations are formulated for arbitrary profile curves. As that parametrisation is rather different from that of the present manuscript, its direct application of the ellipsoidal profile would be a further lengthy algebraic calculation beyond the extent of this paper. However, we can compare the general structure of the equations.

From [9], the equations of motion can be written in the form

$$\begin{aligned}\dot{\vartheta} &= \frac{1}{F_{\vartheta}(\vartheta)}\kappa_3, \\ \dot{\kappa}_1 &= F_1(\vartheta)\kappa_2\kappa_3, \\ \dot{\kappa}_2 &= F_2(\vartheta)\kappa_1\kappa_3, \\ \dot{\kappa}_3 &= F_{31}(\vartheta)\kappa_1^2 + F_{32}(\vartheta)\kappa_1\kappa_2 + F_{33}(\vartheta)\kappa_2^2 \\ &\quad + F_{34}(\vartheta),\end{aligned}\quad (71)$$

where the variables $\kappa_1, \kappa_2, \kappa_3$ are intuitively chosen linear combinations of linear and angular momenta, and, thus, they should be linear combinations of the derivatives $\dot{\psi}, \dot{\vartheta}, \dot{\phi}$. Note, that in the original form of (71), the authors of [9] use the tangent $\xi = \tan \vartheta$ instead of ϑ , but this transformation $\vartheta \in [-\pi, \pi] \rightarrow \xi \in [-1, 1]$ does not change the structure of the equations. Note, that in (71), the term F_{34} originates from the derivative of the potential energy function. As a further comparison, the geometric quantities f_{η} and f_{ξ} are related to f_1 and f_2 of [9] by

$$f_{\eta}(\vartheta) = -f_1(\tan \vartheta), \quad f_{\xi}(\vartheta) = -f_2(\tan \vartheta). \quad (72)$$

Let us check whether that the form of the present equations (70) are equivalent to (71) from [9]. First, trivially, $\kappa_3 = F_{\vartheta}(\vartheta)\dot{\vartheta}$. Then, the missing term κ_3^2 from the last component of (71) requires $K_{21}(\vartheta) = -F'_{\vartheta}(\vartheta)$, which could be

checked by long algebraic calculations. Then, κ_1 and κ_2 is a linear combination of $\dot{\psi}$ and $\dot{\phi}$. That is, the linear mapping between the variables of (70) and (71) can be written in the form

$$\begin{bmatrix} \vartheta \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & W_{11}(\vartheta) & 0 & W_{13}(\vartheta) \\ 0 & 0 & F_{\vartheta}(\vartheta) & 0 \\ 0 & W_{31}(\vartheta) & 0 & W_{33}(\vartheta) \end{bmatrix} \cdot \begin{bmatrix} \vartheta \\ \dot{\psi} \\ \dot{\vartheta} \\ \dot{\phi} \end{bmatrix}. \quad (73)$$

Then, it can be checked that the structure of Eqs. (70) and (71) are consistent to each other. By substituting the equations into each other through (73), we get algebraic equations for the ϑ -dependent coefficients of (70), (71) and (73). In this way, the detailed algebraic form of the present equations (70) could be checked from (71). This large amount of calculations is beyond the scope of the present paper.

In [9], Eq. (71) are the basis of a theoretical study focusing also on the conserved quantities, integrability and global dynamics. In the present study, we focus on the families of steady solutions. For this purpose, the form (70) is as appropriate as (71) due to the similar structure. In the meantime, we can preserve the variables $\dot{\psi}, \dot{\vartheta}, \dot{\phi}$ which have simple and clear kinematic interpretations as the derivatives of the Euler angles.

6 Steady motions of the wheel

6.1 Concept of steady motions

It is known from the literature (see e.g. [9, 27, 30]) that the four dimensional phase space of the rolling wheel possesses a two-parametric family of equilibria, which are physically the *steady motions* of the wheel. A similar two-parametric family of steady motions was found in the case of the basketball [2]. In [27] and [9], it was shown that this structure in the phase space is typical in a class of conservative mechanical systems containing a rigid body rolling on a rigid surface. We demonstrate that the known steady motions of the thin or toroidal wheels are significantly modified in the case of elliptical profiles.

Consider an equilibrium of (67) in the form

$$\mathbf{w}(t) = (\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi})(t) \equiv (\vartheta_0, \dot{\psi}_0, 0, \dot{\phi}_0) \quad (74)$$

In the original, full phase space, these equilibria correspond to *steady motions* in the form

$$\psi(t) = \dot{\psi}_0 t, \quad \vartheta(t) \equiv \vartheta_0, \quad \phi(t) = \dot{\phi}_0 t. \quad (75)$$

That is, the tilt angle ϑ_0 is constant while the yaw angle ϕ and the pitch angle ψ are changing uniformly in time. The wheel

is rotating about its symmetry axis with a constant angular velocity $\dot{\psi}_0$, while the travel direction of rolling is rotating about the vertical axis with a constant angular velocity $\dot{\phi}_0$.

Instead of the lengthy algebraic expressions in (67), the steady motions can also be calculated by substituting (74) into the original form (47) of the equations of motions. At the steady motion, the quadratic quantities become

$$\Omega(t) \equiv \Omega_0 = [0 \ \dot{\phi}_0^2 \ 0 \ 0 \ \dot{\psi}_0 \dot{\phi}_0]^T, \quad (76)$$

Then, the first and third components of (47) become identities and they are automatically satisfied for any steady motion (74). The only non-trivial condition is the second component of (47), which reduces to

$$G(\vartheta_0, \dot{\phi}_0, \dot{\psi}_0) = h_{22}(\vartheta_0)\dot{\phi}_0^2 + h_{25}(\vartheta_0)\dot{\phi}_0\dot{\psi}_0 + g_2(\vartheta_0) = 0. \quad (77)$$

Equation (77) provides a single scalar condition with respect to the three steady values $\vartheta_0, \dot{\phi}_0, \dot{\psi}_0$. This demonstrates the known fact that the equilibria are not distinct points in the phase space, but there is a two-parametric family of equilibria. Let us explore these equilibria through the important special cases:

- static equilibria of $\dot{\phi}_0 = \dot{\psi}_0 = 0$,
- special steady motions of straight line with $\dot{\phi}_0 = 0$,
- special steady motions of spin with $\dot{\psi}_0 = 0$,
- general steady motions with $\dot{\phi}_0, \dot{\psi}_0 \neq 0$.

6.2 Static equilibria revisited

By substituting $\dot{\psi}_0 = \dot{\phi}_0 = 0$ into (75), the form of the steady motions become

$$(\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi})(t) \equiv (\vartheta_0, 0, 0, 0), \quad (78)$$

and from (77), we get $g_2(\vartheta_0) = 0$. From (30) and (51), this can be shown to be equivalent to the solutions of $U(\vartheta_0) = 0$. Thus, we find that static equilibria are special cases of steady motions, where the wheel stands with a tilt angle ϑ_0 . In addition to the trivial case $\vartheta_0 = \vartheta_1 = 0$, we showed that in Cases II and IV of an elliptical wheel tread, it is possible to have non-trivial tilted positions $\vartheta_0 = \vartheta_{2,3}$ (see Figs. 4 and 5). We will see that the existence or non-existence of the non-trivial static equilibria modifies the whole structure of steady motions, as well.

6.3 Steady motions of straight line rolling

With $\dot{\phi}_0 = 0$, the steady motions take the form

$$(\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi})(t) \equiv (\vartheta_0, \dot{\psi}_0, 0, 0), \quad (79)$$

and the condition (77) reduces to $g(\vartheta_0) = 0$, again. That is, for each static equilibrium with a tilt angle ϑ_0 , we can choose an arbitrary pitch rate $\dot{\psi}_0 \in \mathbb{R}$, and we get a steady motion where the wheel is tilted by ϑ_0 and rolling on a straight line by an uniform angular velocity $\dot{\psi}_0$. Compared to the case of the thin or toroidal wheel, the main difference that the steady motion on a straight line can be possible in a tilted position in the cases (II and IV, see Figs. 4 and 5) where there are non-trivial static equilibria.

6.4 Steady motions of spin

With $\dot{\psi}_0 = 0$, the form of the steady motions becomes

$$(\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi})(t) \equiv (\vartheta_0, 0, 0, \dot{\phi}_0), \quad (80)$$

and the condition (77) simplifies to

$$h_{22}(\vartheta_0)\dot{\phi}_0^2 + g_2(\vartheta_0) = 0. \quad (81)$$

It can be shown from (53)–(58) that (81) can be written in the product form

$$\sin \vartheta_0 \cdot \underbrace{(h_{22}^*(\vartheta_0)\dot{\phi}_0^2 + g_2^*(\vartheta_0))}_{\hat{f}(\vartheta_0, \dot{\phi}_0)} = 0, \quad (82)$$

where $h_{22}(\vartheta_0) = h_{22}^*(\vartheta_0) \sin \vartheta_0$, $g_2(\vartheta_0) = g_2^*(\vartheta_0) \sin \vartheta_0$, and $\hat{f}(\vartheta_0, \dot{\phi}_0)$ is smooth in $(\vartheta_0, \dot{\phi}_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$. It can be seen that the trivial vertical tilt angle $\vartheta_0 = \vartheta_1 = 0$ is always a solution of (81)–(82). Then, the wheel is spinning in vertical position with an arbitrary yaw rate $\dot{\phi}_0$.

It can also be possible that the second term $\hat{f}(\vartheta_0, \dot{\phi}_0)$ of (82) vanishes. Then, we get

$$\dot{\phi}_0 = \pm \sqrt{\frac{g_2(\vartheta_0)}{h_{22}(\vartheta_0)}} = \pm \sqrt{\frac{g_2^*(\vartheta_0)}{h_{22}^*(\vartheta_0)}}. \quad (83)$$

Consequently, if $g_2(\vartheta_0)$ and $h_{22}(\vartheta_0)$ have the same sign for some tilt angle $\vartheta_0 \in (0, \pi/2)$ then we can find a yaw rate $\dot{\phi}_0$ such that the steady spinning motion is realised.

Let us check the limit of (82) for small yaw rates $\dot{\phi}_0 \rightarrow 0$. Then, the second term $\hat{f}(\vartheta_0, \dot{\phi}_0)$ of (82) tends to $g_2^*(\vartheta_0) = 0$, which is equivalent to the condition of the non-trivial equilibria of the potential function. That is, the explicit form (83) is valid in Cases II and IV where non-trivial equilibria exist.

In the other limit case of (82), we can take $\dot{\phi}_0 \rightarrow \infty$ of very large yaw rates. Then (82) tends to $h_{22}^*(\vartheta_0) = 0$. The formula of $h_{22}^*(\vartheta_0) = 0$ is long and due to the dependence on several parameters (a, b, R, m, J_2, J_1) it is not easy to obtain general analytical results for the roots ϑ_0 at $\dot{\phi}_0 \rightarrow \infty$. By performing numerical experiments in Cases II and IV, we can get interesting results: For some combinations of the

parameters, $h_{22}^*(\vartheta_0) = 0$ has two distinct roots $\pm\vartheta_0^\infty$ close to but not equal to $\vartheta_{2,3}$. That is, there is the possibility for the wheel to spin in a tilted position even with very large yaw rates $\dot{\phi}_0$.¹² For another parameter combinations, $h_{22}^*(\vartheta_0) = 0$ has no roots, so only the trivial tilt angle ϑ_0 is possible for very large yaw rates $\dot{\phi}_0$.¹³

6.5 General steady motions

Consider the general case of the steady motions (74),

$$(\vartheta, \dot{\psi}, \dot{\vartheta}, \dot{\phi})(t) \equiv (\vartheta_0, \dot{\phi}_0, 0, \dot{\psi}_0). \quad (84)$$

Then, from (77), we can express the pitch rate

$$\dot{\psi}_0 = \frac{-g_2(\vartheta_0) - h_{22}(\vartheta_0)\dot{\phi}_0^2}{h_{25}(\vartheta_0)\dot{\phi}_0}. \quad (85)$$

That is, by choosing a tilt angle ϑ_0 and a uniform rotation speed $\dot{\phi}_0 \neq 0$, the formula (85) provides the necessary pitch rate $\dot{\psi}_0$ to maintain the steady motion. In this way, (85) is the formula of the two-parametric family of steady motions.

This set of steady motions can be depicted in the space of the variables $(\vartheta_0, \dot{\phi}_0, \dot{\psi}_0)$,¹⁴ which can be seen in Fig. 6. In the top panel, we can see a parameter combination from Case I in Fig. 4. In this case, there is a single (trivial) static equilibrium of the vertical orientation of the wheels. The static equilibrium is located at the intersection of the two one-parametric steady motions of straight line rolling (80) and spin (79). The general two-parametric family form a twisted-shape smooth surface passing through the two previous lines of steady motion.¹⁵ This shape of the graph is known from the literature for the thin wheel (see, e.g. [30]), and based on the present study, it is qualitatively valid for all geometric parameters chosen from the region of Case I (see Fig. 4). It can be checked that in Case III, we got a surface qualitatively similar to Case I.

¹² This effect seems to contradict physical expectations because if the centre of gravity is not located on the rotation axis, infinitely large centrifugal force is expected. However, in this parameter region, the tilted inertia tensor creates a gyroscopic torque in the opposite direction which compensates the moment of the centrifugal force. Still, an infinitely large friction force would be needed which makes a limitation in this result.

¹³ With a simplified analytical study of given values of J_1 and J_2 , we find that in these cases, there is a limited value of $\dot{\phi}_0$, where the two non-trivial steady spin branches meet in a pitchfork bifurcation. Above this yaw rate, no non-trivial spin steady state is realizable.

¹⁴ The fourth state variable $\dot{\vartheta}$ is trivially zero, see (74).

¹⁵ Although (85) formally does not defined at $\dot{\phi}_0 = 0$, the limit of the expression exist for $g_2(\phi_0) \rightarrow 0$ which are just the locations of the static equilibria. The surface in Fig. 6 can thus intersect the plane $\dot{\phi}_0 = 0$ exactly at the static equilibria.

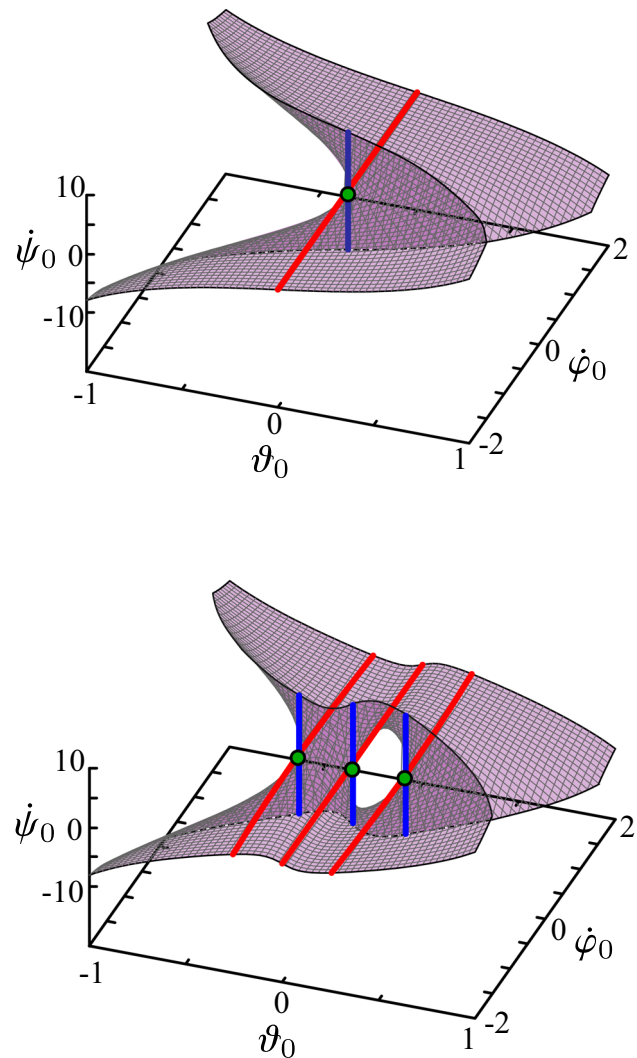


Fig. 6 Two-parametric family (85) steady motions of the wheel visualised in the space of the variables $(\vartheta_0, \dot{\phi}_0, \dot{\psi}_0)$. The numerical values of the parameters can be seen in Table 1. Top panel: Point P, which corresponds to Case I (see Fig. 4), and where only the single trivial equilibrium exists. Bottom panel: Point Q, which corresponds to Case II (see Fig. 4), and where the non-trivial equilibria appear. In both panels, the equilibria (78) are denoted by green dots. The special steady motions of straight rolling (80) are denoted by vertical blue lines. The special steady motions of spin (79) are denoted by red curves

When we choose a parameter combination from Case II in Fig. 4, the structure of the set of steady motions significantly changes (see the bottom panel of Fig. 6). We can observe the appearance of the two non-trivial equilibria corresponding to tilted angles. Through both new equilibria, there are vertical straight (blue) lines of steady motions of straight line motion, and slightly curved (red) lines of steady motions of spin. The surface of general steady motions passes through all these lines, forming a strange three-dimensional surface containing two holes! This structure of steady motions is valid in Case II where the profile curve is sufficiently flat ($b < a^2/R$). The bifurcation between Cases I and II happens at the line K_1 in

Fig. 4 at $b = a^2/R$, where the three static equilibria collapses in a pitchfork bifurcation and modify the whole set of steady motions. Note that in Case IV, we can find a qualitatively similar set of steady motion to Case II.

6.6 Stability of steady motions

At a steady motion $\mathbf{w}_0 = (\vartheta_0, \dot{\psi}_0, 0, \dot{\phi}_0)$ (see (84)), the linearized differential equations from (67) take the form

$$\dot{\mathbf{w}} = \mathbf{A}(\mathbf{w} - \mathbf{w}_0) + \mathcal{O}^2. \quad (86)$$

After lengthy calculations, the structure of the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & A_{23} & 0 \\ A_{31} & A_{32} & 0 & A_{34} \\ 0 & 0 & A_{43} & 0 \end{bmatrix}, \quad (87)$$

where the non-zero elements are expressed by

$$\begin{aligned} A_{23} &= K_{33}\dot{\psi}_0 + K_{34}\dot{\phi}_0, \\ A_{31} &= \frac{dK_{13}}{d\vartheta}\dot{\phi}_0^2 + \frac{dK_{25}}{d\vartheta}\dot{\phi}_0\dot{\psi}_0 + \frac{dL_{12}}{d\vartheta}, \\ A_{32} &= K_{25}\dot{\phi}_0, \\ A_{34} &= 2K_{22}\dot{\phi}_0 + K_{25}\dot{\psi}_0, \\ A_{43} &= K_{13}\dot{\psi}_0 + K_{14}\dot{\phi}_0. \end{aligned} \quad (88)$$

The characteristic polynomial of (87) takes the form

$$\lambda^2 (\lambda^2 + H) = 0, \quad (89)$$

where

$$H(\vartheta_0, \dot{\psi}_0, \dot{\phi}_0) = -A_{34}A_{43} - A_{23}A_{32} - A_{31}. \quad (90)$$

That is, out of the four eigenvalues, two are zeros, which fact was proven in [9] for general shape of profiles. It can be also shown that the eigenvectors related to the zero eigenvalues are tangent to the surface containing the steady motions. The other two eigenvalues are

- either pure complex (if $H > 0$) in the form $\lambda = \pm i\sqrt{-H}$, and the equilibrium is a neutrally stable centre;
- either real (if $H < 0$) in the form $\lambda = \pm\sqrt{H}$, and the equilibrium is an unstable saddle.

It is showed in [9] that for a class of systems including the present problem, the stability of the centre also leads to the nonlinear stability of the system. That is, the neighbouring trajectories around the steady solutions are all periodic solutions.

The full analytical expression of these stability boundaries is not possible, but some special cases can be easily analysed, and at the end, the general case is explored numerically.

6.6.1 Stability of the trivial straight line rolling

By considering the trivial steady straight line motion $(\vartheta, \dot{\phi}, \dot{\psi})(t) \equiv (0, 0, 0, \dot{\psi}_0)$, the components of the matrix $\mathbf{A}$ become

$$\begin{aligned} A_{23} &= A_{32} = 0, \\ A_{31} &= \frac{mg(R - a^2/b)}{J_1 + mR^2}, \\ A_{34} &= \dot{\psi}_0 \frac{J_2 + mR^2}{J_1 + mR^2}, \\ A_{43} &= -\dot{\psi}_0 \frac{J_2}{J_1}, \end{aligned} \quad (91)$$

and the expression of H reduces to

$$H = -mgJ_1(bR - a^2) + \dot{\psi}_0^2 J_2 b (J_2 + mR^2). \quad (92)$$

Then, the condition of the stable steady-state rolling is

$$\dot{\psi}_0 > \dot{\psi}_0^{\text{cr}} = \underbrace{\sqrt{\frac{g}{R}} \cdot \sqrt{\frac{mR^2 J_1}{J_2(J_2 + mR^2)}}}_{=\dot{\psi}_0^{\text{cr,thin}}} \cdot \sqrt{1 - \frac{a^2}{Rb}}. \quad (93)$$

In (93), $\dot{\psi}_0^{\text{cr,thin}}$ denotes the stability limit (critical pitch rate) for the thin wheel (see, e.g. [30]). Before drawing further conclusions about these results, let us calculate the stability of the trivial steady spin, which leads to a formula similar to (93).

6.6.2 Stability of the trivial spin motion

In the case of the trivial steady spin $(\vartheta, \dot{\phi}, \dot{\psi})(t) \equiv (0, \dot{\phi}, 0, 0)$, the components of the matrix $\mathbf{A}$ become

$$\begin{aligned} A_{34} &= A_{43} = 0, \\ A_{23} &= -\dot{\phi}_0 \left(1 + \frac{mR^2(1 - a^2/(bR))}{J_2 + mR^2} \right), \\ A_{31} &= \frac{mg(1 - a^2/(bR))}{J_1 + mR^2} \\ &\quad - \dot{\phi}_0^2 \frac{J_2 - J_1 + mR^2(1 - a^2/(bR))}{J_1 + mR^2}, \\ A_{32} &= \dot{\phi}_0 \frac{J_2 + mR^2}{J_1 + mR^2}, \end{aligned} \quad (94)$$

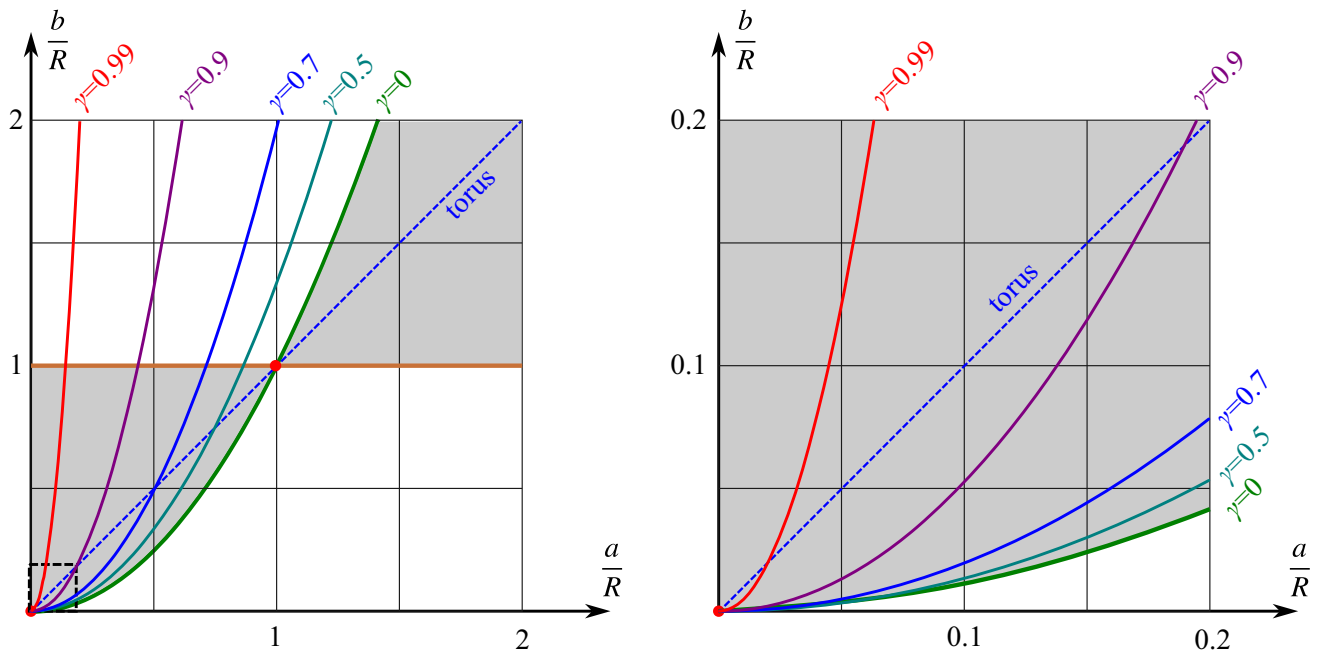


Fig. 7 Level sets of the expression γ from (97) depicted on the parameter plane $a/R - b/R$ of the profile parameters. This ratio expresses the change of the critical yaw and pitch rates compared to the thin wheel.

The notations in the figure are the same as those of Fig. 4. The right panel is a magnified view of the left panel in the vicinity of $a, b \ll 1$

and the expression of H simplifies to

$$H = -mg(bR - a^2) + \dot{\phi}_0^2 b(J_1 + mR^2). \quad (95)$$

Finally, the condition of stable steady-state spin is

$$\dot{\phi}_0 > \dot{\phi}_0^{\text{cr}} = \underbrace{\sqrt{\frac{g}{R}} \cdot \sqrt{\frac{mR^2}{J_1 + mR^2}}}_{=\dot{\phi}_0^{\text{cr,thin}}} \cdot \sqrt{1 - \frac{a^2}{Rb}}. \quad (96)$$

In (96), $\dot{\phi}_0^{\text{cr,thin}}$ is the stability limit (critical yaw rate) for the thin wheel (see, e.g. [30]). By comparing (93) and (96), we find that

$$\frac{\dot{\psi}_0^{\text{cr}}}{\dot{\phi}_0^{\text{cr,thin}}} = \frac{\dot{\phi}_0^{\text{cr}}}{\dot{\phi}_0^{\text{cr,thin}}} = \sqrt{1 - \frac{a^2}{Rb}} := \gamma. \quad (97)$$

This quantity γ shows the ratio of the minimal rotation speeds of the elliptical profile to the thin wheel. It can be seen that for $a, b > 0$ and $bR > a^2$, these critical values are smaller than those of the thin wheel. It is interesting that the critical values of the yaw rate $\dot{\phi}$ and the pitch rate $\dot{\psi}$ are scaled with the same factor γ . It can also be seen from (97) that the increase in both a and b typically decreases the critical values. Consequently, the "thick" wheel has a larger region of initial conditions of the stable steady motion. This fact was already identified in

[1] for the toroidal profile with $a = b \leq R/2$. For those values, the stability change was not very significant because even for $a = b = R/2$ (an aspect ratio of $\eta = 1$ called in [1]), the value of (97) becomes $\gamma = 1/\sqrt{2} \approx 0.71$. However, considering general elliptical profiles with $a \neq b$ and a, b being large, the expression (97) radically decreases when approaching the curve K_1 curve $bR = a^2$ in Fig. 4. When reaching this curve, the critical values decrease to zero, and the trivial solutions become stable even for zero velocities. This is equivalent to the appearance of stable static trivial equilibria at $b = a^2/R$. For $b < a^2/R$ (on the right side of the curve K_1 in Fig. 4), the expression (97) of γ becomes undefined. That is, there are no critical pitch and yaw rates, and the trivial roll and spin steady motions are stable for all velocities in a certain range about the trivial equilibrium.

An interesting further fact is that the limit of (97) is not defined for $a = b = 0$, although we expect to get back the value $\gamma = 1$ for the thin wheel. For a fixed ratio $a/b = \varepsilon$, the limit of (97) at $a = b\varepsilon \rightarrow 0, b \rightarrow 0$ already gives the expected value $\gamma = 1$. The limit is, however, undefined for $a = 0, b \rightarrow 0$ (approaching the origin horizontally in Fig. 4).

In Fig. 7, we can see the level sets of γ in the plane of the parameters a/R and b/R . Due to the square root expression in (97), the value of γ changes rapidly in the vicinity of the curve K_1 with $bR = a^2$ and $\gamma = 1$. That is, the region of stable trivial solutions and the critical yaw and pitch rates are sensitive to the shape of the profile. This sensitivity is

even more significant for small a, b , especially when b/a is small (see the right panel of Fig. 7). That is, although the ideal thin wheel has the well-known values of critical rotation speeds (see (93), (96)), slight finite curvatures of the profile can cause a significant drop in the critical speeds and thus a modified dynamic behaviour.

6.6.3 Stability of the general steady motions

The stability boundary is given by the intersection of the surface $H(\vartheta_0, \dot{\phi}, \dot{\psi}) = 0$ (see (90)) of the stability condition with the surface $G(\vartheta_0, \dot{\phi}_0, \dot{\psi}_0) = 0$ (see (77)) of the set of steady motions. It can be shown that these two equations can be written in the form

$$\begin{aligned} H_{20}\dot{\psi}_0^2 + H_{11}\dot{\psi}_0\dot{\phi}_0 + H_{02}\dot{\phi}_0^2 + H_{00} &= 0, \\ G_{11}\dot{\psi}_0\dot{\phi}_0 + G_{02}\dot{\phi}_0^2 + G_{00} &= 0, \end{aligned} \quad (98)$$

which are second-degree expressions in $\dot{\phi}_0, \dot{\psi}_0$, and the coefficients H_{ij} and G_{ij} depend on the tilt angle ϑ_0 . This is consistent with Eq. (71) from [9], where the conditions of steady motions are also quadratic expressions of the variables κ_1, κ_2 , which are, in turn, linear expressions of ϕ, ψ . From (98), we can eliminate $\dot{\psi}_0$ (see (85)) and finally, we obtain a second-degree equation in $\dot{\phi}_0^2$:

$$X_2(\vartheta)\dot{\phi}_0^4 + X_1(\vartheta)\dot{\phi}_0^2 + X_0(\vartheta) = 0. \quad (99)$$

Then, from (99), it is possible to determine the expressions

$$\dot{\phi}_0 = f_\phi(\vartheta_0), \quad \dot{\psi}_0 = f_\psi(\vartheta_0), \quad (100)$$

which are parametric curves of the stability boundaries in the space $(\vartheta_0, \dot{\phi}_0, \dot{\psi}_0)$. The expressions in (100) can still be derived analytically, but the resulting formulae are extremely long, and thus the derivation was performed by the computer algebra system Maxima [23].

The plot of the stability boundaries can be seen in Fig. 8 for the points P, Q, R, S of Fig. 4 and Table 1. These curves could be plotted onto the surfaces in Fig. 6, but the resulting plots are not easy to understand. Instead, as in [30], projections of the stability boundaries are presented.

In the upper panels of Fig. 8, we can see the point P from Case I of a single unstable equilibrium. This case is structurally similar to the thin wheel [14, 27, 30] and the toroidal wheel [1]. Around the unstable equilibrium, we can find a closed unstable region, which looks like a bow tie from one direction (top view of the surface in Fig. 6), but clearly shows the closed shape projected onto the plane $(\dot{\psi}_0, \dot{\phi}_0)$. From the diagram, we can also read the values of the critical speeds $\dot{\phi}_0^{\text{cr}}$ and $\dot{\psi}_0^{\text{cr}}$. As the geometry of the unstable region varies for different parameters in Case I, the topological structure remains the same as presented for the thin wheel in [30].

Finally, we can conclude that for parameter values in Case I, the steady motion of the wheel is unstable within a region of small yaw and pitch rates.

In the middle panels of Fig. 8, an interesting structure of boundaries can be observed for Case II at point Q . We find that there are two closed unstable boundaries around each unstable (non-trivial) equilibrium. Between these regions, the steady solutions are stable around the stable trivial equilibrium. For sufficiently large yaw and pitch rates, the steady motions are stabilised again. This rich structure of dynamics is topologically different from both the thin wheel and the toroidal wheel.

In the bottom-left panel of Fig. 8, we can see the stability boundaries for Case III at the point R . For Case III, we may expect a fully stable set of steady motions since only a stable trivial equilibrium is present. However, unstable steady motions appear for very large values of $\dot{\phi}_0$ where the tilt angle ϑ_0 is close to $\pm 90^\circ$. In practice, these solutions are probably not realisable not just because of the limited friction force but because the numerical experiment shows that they lie beyond the limit (8) of maximum tilt (denoted by dashed red lines in the figure).

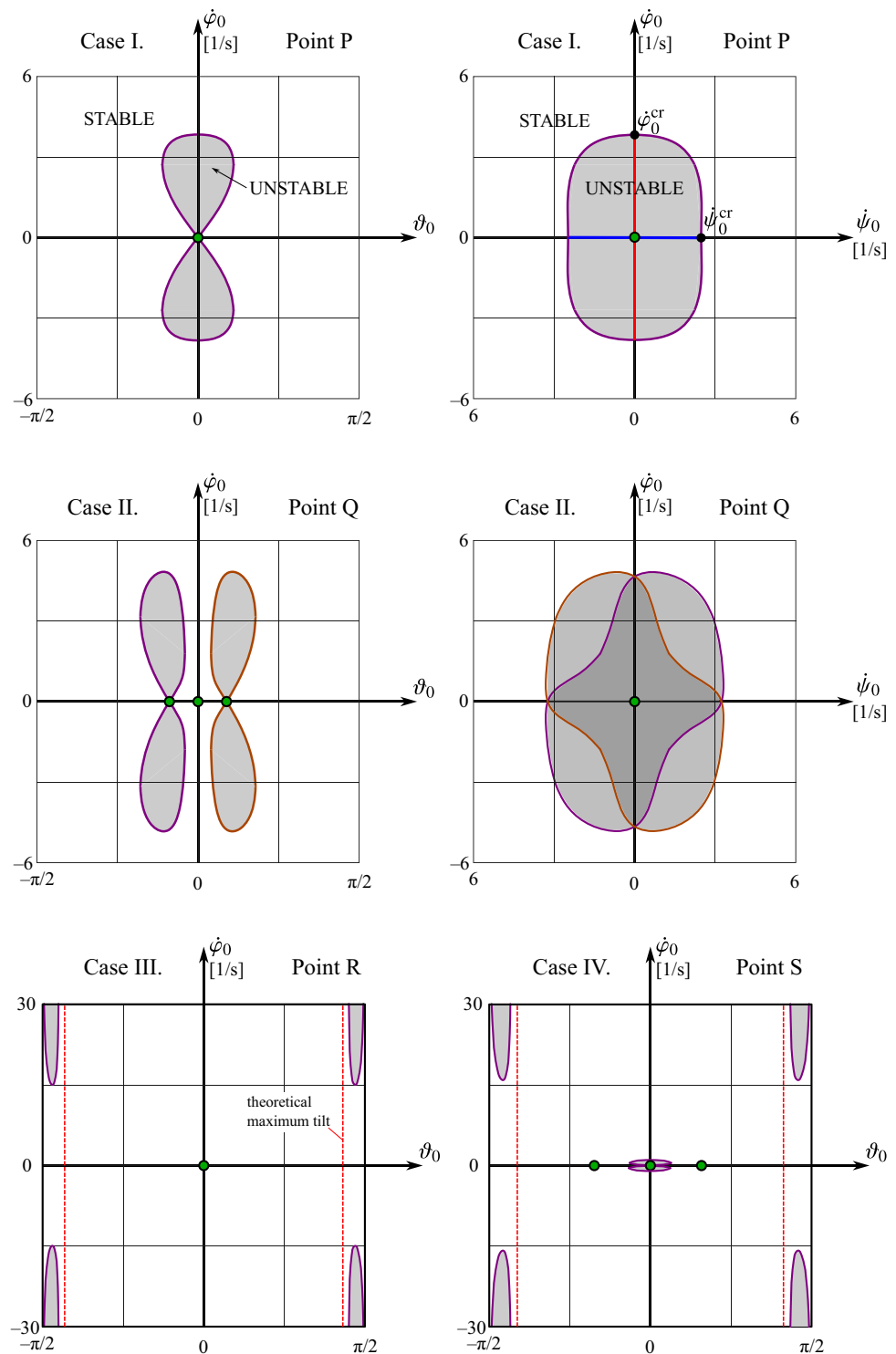
Finally, in the bottom-right panel of Fig. 8, we can see the point S from Case IV. This case shows the stability properties combined from Cases I and III. Similar to Case III, we can observe the unstable steady motions far from the equilibria, but these are, again in the region which is impossible due to the limited maximum tilt angle (8). Around the unstable trivial equilibrium, we can see an unstable region with a shape similar to that of Case I. The size of this unstable region is affected by the distance between the unstable trivial and the stable non-trivial equilibria. In practice, parameters from Case IV but far from the curve K_1 behave similarly to Case I: the stable non-trivial equilibria wander farther, and the unstable region of steady motions around the trivial equilibrium becomes more relevant.

6.6.4 Effect of moments of inertia

The model of this study are described by the seven parameters R, a, b, m, J_1, J_2, g (the wheel thickness w is not directly included). In most part of the paper, we focused on the effect of the profile geometry through the dimensionless parameters a/R and b/R (see Fig. 4). It was shown that these two parameters have an essential role for the qualitative changes in steady motions. For a complete parametric study, the number of mentioned seven parameters can be reduced to four by creating dimensionless quantities. Beyond a/R and b/R , we can consider the effect of moments of inertia in the dimensionless form $j_1 := J_1/mR^2$ and $j_2 := J_2/mR^2$.

In this paper, the numerically computed diagrams contains the assumption $j_1 = 1/4$ and $j_2 = 1/2$ of a solid thin

Fig. 8 Regions of stable and unstable general steady motions. The parameter values can be found in Table 1



disk. Due to the complexity of the formulae, it was not possible to obtain general formulae for the possible bifurcation points showed in similar problems e.g. in [19], [28]. However, numerical experiments showed no qualitative changes in the results for realistic parameters. The reason can be that unlike the papers [19], [28] or [9] of arbitrary shapes, the parameter

range of realistic wheels are more or less limited. Compared to a uniform solid body, the material distribution of vehicle wheels is slightly shifted towards the circumference, thus, we expect the dimensionless values between the solid thin disc and the ring, $1/4 < j_1 < 1/2$ and $1/2 < j_2 < 1$. Moreover, the real wheels are typically thin enough for approxim-

ing the moments of inertia by the theorem of perpendicular axes, $j_2 \approx 2j_1$. Even if this ratio is slightly deviated, the range of possible parameters for realistic wheels are limited. That can be the reason that numerical experiments cannot show qualitative changes due to realistic moments of inertia.

7 Conclusion

In the paper, we analysed the dynamics and stability of a rolling wheel in the case when the contacting geometry is not a single circle (thin wheel) or a toroidal surface but an elliptical tread. With the elliptical profile, several calculations of dynamics could still be performed analytically or semi-analytically. To highlight the results, we can draw the following conclusions.

- From the analysis of static equilibria of the wheel, it can be shown that unlike a thin wheel, the elliptical wheel tread can lead to non-trivial equilibria in tilted positions. In the parameter space of the ellipse parameters, four qualitatively different scenarios can be found regarding the number and stability of the equilibria (Cases I–IV). From these, Case I contains the cases of the thin wheel and the toroidal wheel.
- Using the Routh–Voss equations, the equations of motions were derived for the elliptical wheel tread. The resulting differential equations have a form similar to those of the thin wheel, but beyond the algebraic complexity, new terms appear. The structure of the governing equations found to be consistent to a known other formulation of the equations containing general profiles.
- We calculated the one-parametric families of steady motions in the phase space. These solutions correspond to the straight line rolling and spinning. In the case of three static equilibria, families of rolling and spinning steady motions exit for each equilibrium. We found that the unstable range around the unstable equilibria is always smaller compared to the thin wheel. In Cases II and III, the unstable region disappears.
- We determined the two-parametric family of general steady motions containing combined yaw-pitch motion. In Cases I and III, the resulting surface in the phase space is structurally similar to that of the thin wheel and toroidal wheel. In Cases II and IV, the surface of steady motions gets a strange, twisted with two holes, which structure cannot be found in the literature. Compared to the thin wheel, the elliptical profile shows a more complicated pattern of stable and unstable regions.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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