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Vibration characteristics of rolling elements in the presence of different friction and creep models

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Abstract. Dynamics of wheels and other rolling elements in machines are significantly affected by the friction phenomena between the surfaces. When modeling the transitions between rolling and slipping motion, the simplest choice is applying the Coulomb friction model or the similar but improved models like the Stribeck model. However, when the local deformations are not negligible, often the so-called creep force models are used, which contain a coupled nonsmooth effect of the kinematic variables. This study investigates the effects of the different contact models on the properties of the vibrations. The models are demonstrated on 2D test problems of rolling elements. Mathematical methods of nonsmooth dynamics and singular perturbation are utilized to explore the details of the phenomena.

1 Introduction

Contact problems between bodies are challenging in analysis of multibody systems both because of the uncertainty of the mechanical models, and the nonsmooth dynamical effects caused by these models. When modelling contact between two bodies, the effects of local deformations can be taken into account. If these effects are transmitted to rigid body systems, then resultant contact force models can be formulated. Under *friction models*, we mean both the classical models of dry friction (see e.g. [1]) and the creep force models (see e.g. [2, 3]).

Compared to the contact of two flat surfaces, the contact of rolling bodies is significantly challenging. In engineering applications, we can find various rolling elements, such as wheels, rollers, and bearing balls.

In this paper, we try to understand the fundamental differences in the predicted behaviour of rolling elements caused by different friction models. For this purpose, a minimal test model is formulated. Then, three important friction models are applied, and the resulting behaviour of the test problem is analysed. Despite the simplicity of the test problem, the Coulomb friction model and creep force models lead to interesting dynamical behaviour. In each case, specific mathematical methods are utilised, such as the theory of piecewise smooth systems [4], the singular perturbation method [5], and the concept of extended Filippov systems [6]. Specifically, we focus on the decay of the vibrations and the tendencies to approach or reach the stationary state of the system. The results are checked by numerical simulations.



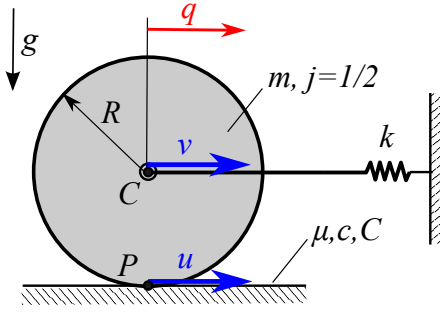


Figure 1: Sketch of the test problem analysed in the paper. The rolling disc is attached to a rough horizontal plane, where various contact models are considered. The system is a prototype problem of rolling elements in machines.

2 Test problem

Consider the test problem in Fig. 1. The rigid disc is supported horizontally by a spring at C and vertically at P by a rigid surface that induces friction forces. The equations of motion of the disc can be written in the form

$$\begin{aligned}\dot{q} &= v, \\ \dot{v} &= -\frac{k}{m}q + \frac{F_f}{m}, \\ \dot{u} &= -\frac{k}{m}q + \frac{1+j}{j} \cdot \frac{F_f}{m}\end{aligned}\tag{1}$$

where q is the position of C , and variables v, u are the horizontal velocities of C and P , respectively. In (1), k is the stiffness of the spring, m is the mass of the disc, and j is the dimensionless mass moment of inertia whose value is $j = 1/2$ for a homogeneous rigid disc. Moreover, F_f denotes the friction forces determined by the different models below.

The system (1) is a theoretical test problem, but is strongly related to the vibrations of the vehicle wheels and rolling elements in machines. This system has the possibility of stick-slip transitions and has minimal but sufficient degrees of freedom for oscillatory behaviour even in the stick phase.

Let us introduce the following new state variable

$$w := v - \frac{j}{1+j}u,\tag{2}$$

which has the dimension of velocity. This transformation eliminates the friction forces from the second component of (1). Then, by introducing the state vector $\mathbf{x} = (q, w, u)$, (1) becomes

$$\underbrace{\begin{bmatrix} \dot{q} \\ \dot{w} \\ \dot{u} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q + 3F_f/m \end{bmatrix}}_{\mathbf{F}(\mathbf{x})}\tag{3}$$

where $\omega = \sqrt{k/m}$, and we substituted the value $j = 1/2$ of a homogeneous disc. Our goal is to analyse the dynamics of (3) in the presence of different friction models of the force F_f .

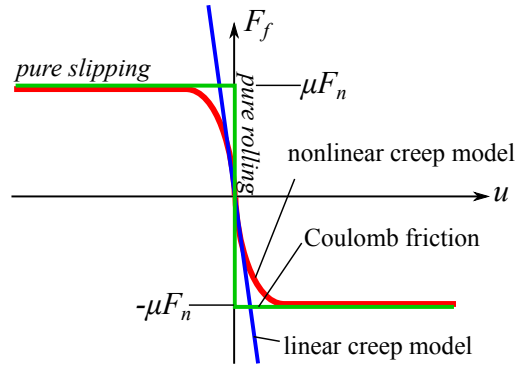


Figure 2: Sketch of the friction and creep force models analysed in the paper. Green line: Coulomb friction model. Blue line: linear creep force models. Red lines: nonlinear creep force models. For the *absolute* creep force models, the steepness of the curve at $u = 0$ is a fixed parameter c . For the *relative* creep force models, the steepness at $u = 0$ is given by $C/|w|$, where w is another variable.

3 Friction models

We assume that there is permanent normal contact and that events of impact or loss of contact do not occur. Moreover, we assume that the normal force F_n at P is constant and can be calculated simply from the weight of the disc, $F_n = mg$. For an overview of dry friction models, see, e.g., [1]. For a thorough explanation creep force models, see [2] or [3].

3.1 Coulomb friction model

Pure *rolling* motion is defined by the kinematic condition

$$u \equiv 0, \quad (4)$$

and then, the friction force F_f can be calculated from (3) as a constraint force.

For the *slip* case $u \neq 0$, consider the Coulomb friction model in the form

$$F_f(u) = -\mu F_n \frac{u}{|u|}, \quad (5)$$

where μ is the friction coefficient (see Fig. 2). Note that (5) is not defined in $u = 0$. Still, it can be shown that from the theory of Filippov systems, dynamical systems including (5) naturally contain both slipping and rolling cases including consistent transitions between them. In this study, we neglect the difference of the values between the static and dynamic friction coefficients.

We can mention velocity-dependent static friction characteristics commonly called the Stribeck model. However, from the approach of the type of discontinuity, this model behaves very similarly to the Coulomb model and is not analysed separately.

3.2 Absolute creep force models

The finite contact stiffness of the rolling bodies causes local deformations and deviations from the pure rolling behaviour. From a multiple-scale modelling approach, we can consider deformations only in a small contact region around the nominal contact point, and the resulting contact force is added as a contact force law on the *rigid body model*. Then, the model can contain a non-zero deviation of u from the rolling state $u = 0$ even in the case where the real system has no macro-slip behaviour. This effect is called *creep*, and in this context u is called *creep velocity*.

Table 1: Numerical values of the parameters used for the numerical simulations.

quantity	notation	value	unit
mass of the disc	m	1.0	kg
spring stiffness	k	900	N/m
gravitational acceleration	g	10.0	m/s ²
creep coefficient (absolute)	c	100.0	Ns/m
creep coefficient (relative)	C	30.0	N

The simplest creep force model provides a linear relationship between the friction force and the creep velocity,

$$F_f(u) = -cu, \quad (6)$$

where c is the creep coefficient. The linear creep force model (6) is valid for small creep velocities. Otherwise, the saturation of the force leads to the nonlinear creep force models. These models can be constructed as a natural combination of the Coulomb friction model (5) and the linear creep force model (6),

$$F_f(u) = -\mu F_n \frac{u}{|u|} \text{sigm} \left(\frac{c|u|}{\mu F_n} \right). \quad (7)$$

In (7), the notation $\text{sigm}(\cdot)$ refers to a *sigmoid function*. For example, $\text{sigm}(x) = \tanh(x)$ or $\text{sigm}(x) = x/\sqrt{1+x^2}$.

In this paper, we call both (6) and (7) *absolute* creep force models because they depend on the value u independently from other state variables.

3.3 Relative creep force models

In the literature, it is common to use such creep force models, where the friction force depends on the dimensionless combination of two variables. Consider the model

$$F_f(u, w) = -C \frac{u}{|w|}, \quad (8)$$

where the friction force is proportional to the *creepage* $u/|w|$. For this ratio of velocities, different definitions are possible. In the literature, we can often find $u/|v|$; the similarity and difference between the two alternatives can be seen in (2). Between the creep coefficients of (6) and (8), there is a formal relationship $c = C/|w|$. That is, (8) can be considered as an extension of (6) with varying c .

Based on (7), the nonlinear saturated creep force model of (8) can be written in the form

$$F_f(u, w) = -\mu F_n \frac{u}{|u|} \text{sigm} \left(\frac{C|u|}{\mu F_n |w|} \right) \quad (9)$$

In this paper, we call (8) and (9) *relative* creep force models to distinguish them from (6)-(7).

4 Numerical simulation and data

In subsequent sections, the analytical calculations are illustrated by numerical simulations. Due to the different nonsmooth and stiff properties of the differential equations, a simple solver of the second-order Runge-Kutta method (midpoint method) is applied with a fixed very small time step. By this approach, explicit capture of nonsmooth events can be avoided while retaining

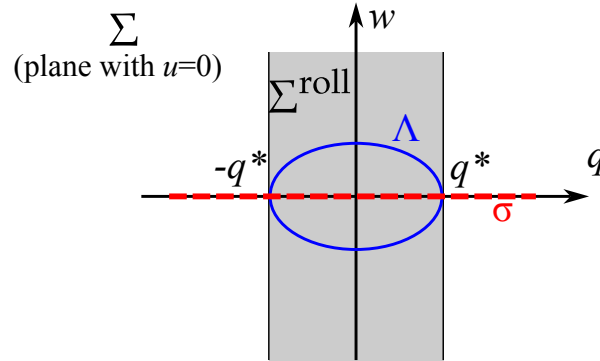


Figure 3: Schematic sketch of the the phase space of the system.

reasonable accuracy. (Higher-order methods, such as the fourth-order Runge-Kutta method, tend to behave oddly in the vicinity of discontinuities.)

The parameters chosen for the system in the simulations can be found in Table 1.

5 Dynamics with Coulomb friction model

5.1 The system

Static dry friction models, including the Coulomb model, commonly possess the property of having two distinct behaviours of *slipping* and *sticking/rolling*. By substituting (5) into (3), the slipping dynamics of the test problem becomes

$$\mathbf{F}^{\text{slip}}(\mathbf{x}) = \begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q - 3\mu g u/|u| \end{bmatrix}. \quad (10)$$

The vector field (10) is not defined in the *discontinuity set* $\Sigma = \{(q, w, u) : u = 0\}$ related to rolling (see Fig. 3). The rolling dynamics is given by

$$\mathbf{F}^{\text{roll}}(\mathbf{x}) = \begin{bmatrix} w \\ -2/3 \cdot \omega^2 q \\ 0 \end{bmatrix}. \quad (11)$$

This formula can be derived from the Newton-Euler equations of the test problem in the presence of (4). Alternatively, (11) can be generated from (10) using the theory of Filippov systems. For an overview of the Filippov systems, see [4] or [7]. A detailed overview of piecewise smooth systems can also be found in [8].

5.2 Dynamics as a Filippov system

Let $\mathbf{n}^{\pm} = (0, 0, \pm 1)$ be the normal vector of Σ on each side. Then, at a point $\hat{\mathbf{x}} = (q, w, 0) \in \Sigma$ of the discontinuity set, let us define the limits of the vector field (10),

$$\mathbf{F}^{\pm}(\hat{\mathbf{x}}) = \lim_{\rho \rightarrow 0^+} \mathbf{F}^{\text{slip}}(\hat{\mathbf{x}} + \rho \mathbf{n}^{\pm}) = \begin{bmatrix} w \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q \mp 3\mu g \end{bmatrix}. \quad (12)$$

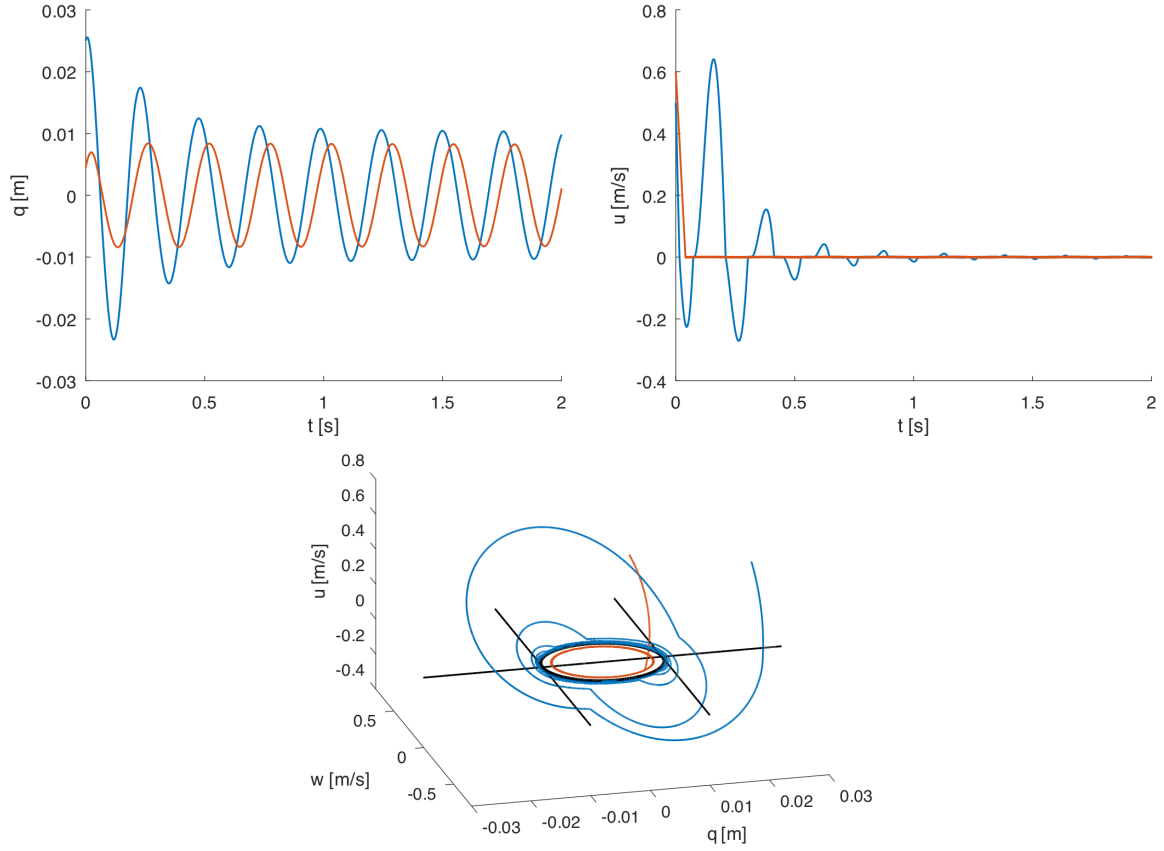


Figure 4: Numerical simulation of the test problem with the Coulomb model. Top panels: time evolution of the position q and the slip velocity u . Bottom panel: phase space of the system. Red lines: „Type A” trajectory. Blue lines: „Type B” trajectory.

Then, it can be shown that the rolling dynamics (10) is given by the linear combination

$$\mathbf{F}^{\text{roll}}(\hat{\mathbf{x}}) = (1 - \alpha(\hat{\mathbf{x}}))\mathbf{F}^+(\hat{\mathbf{x}}) + \alpha(\hat{\mathbf{x}})\mathbf{F}^-(\hat{\mathbf{x}}) \quad (13)$$

with

$$\alpha(\hat{\mathbf{x}}) = \frac{\mathbf{n}^+ \cdot \mathbf{F}^+}{\mathbf{n}^+(\hat{\mathbf{x}}) \cdot \mathbf{F}^+(\hat{\mathbf{x}}) + \mathbf{n}^-(\hat{\mathbf{x}}) \cdot \mathbf{F}^-(\hat{\mathbf{x}})}. \quad (14)$$

This calculation process seems obvious between (10) and (11), but it is already nontrivial for the original formulation (1), where the linear combination (13) finds the correct formula for $\dot{v}$.

Moreover, the Filippov theory easily defines the domain of the rolling behaviour (11). More specifically, we require (13) to be a *convex* combination with $\alpha \in [0, 1]$, which condition leads to $|q| \leq 3\mu g/\omega^2$. (This condition can be shown to be equivalent to the maximum available static friction force, see [9].)

5.3 Structure of the phase space

Finally, even for this simple system, we find an interesting structure of the three-dimensional phase space $(q, w, u) \in \mathbb{R}^3$: The rolling dynamics (11) is valid in the finite strip $\Sigma^{\text{roll}} \subset \Sigma$ with

$$\Sigma^{\text{roll}} = \{(q, w, 0) : |q| \leq 3\mu g/\omega^2 =: q^*\} \quad (15)$$

The slipping dynamics (10) is valid above (slip to right) and below (slip to left) of the plane Σ with $u = 0$. The following transitions can occur between the different types of behaviour:

- transition from left slip to right slip (or vice versa) where a trajectory of $\mathbf{F}^{\text{slip}}$ reaches Σ outside Σ^{roll} ;
- transition from slip to roll, where a trajectory of $\mathbf{F}^{\text{slip}}$ reaches Σ inside Σ^{roll} ;
- and transition from roll to slip, where a trajectory of $\mathbf{F}^{\text{roll}}$ reaches the edge of Σ^{roll} .

The trajectories of the rolling motion (11) are trivial; they can be written in the form of a harmonic oscillation, $q(t) = q_0 \cos(\omega_r t)$, where $\omega_r = \sqrt{2k/(3m)} = \sqrt{2/3} \cdot \omega$ is the angular frequency of the vibrations of the rolling disc. Consider the maximum amplitude trajectory Λ : $q(t) = q^* \cos(\omega_r t)$, which is just touches the edge of Σ^{roll} .

The solutions of the slipping motion (10) can also be expressed analytically. From the trajectories and the possible transitions, it is possible to show that there are two typical trajectories in the system:

- Type A trajectory: in this case, the trajectory of $\mathbf{F}^{\text{slip}}$ intersects Σ inside the ellipse Λ . Then, rolling motion is initiated with a constant amplitude.
- Types B trajectory: in this case, the trajectory of $\mathbf{F}^{\text{slip}}$ intersects Σ outside Λ . Then, it can be shown that a repeating pattern initiates with slip-left, roll, slip-right, roll. During the slip phases, the system loses energy and the trajectory is getting closer to Λ . It can be shown that Λ is a limit cycle of the system *from outside* and all trajectories of Type B tend to Λ . In the subsequent cycles, less and less time is spent in the slip phases, and through this pattern, the dry friction causes exponential decay.

From the approach of energy loss and decay, we can state that in both cases only a part of the kinetic energy is lost and the Coulomb friction model tends to a periodic oscillation of the system with pure rolling motion. In the cases of Type A trajectories, these oscillations have amplitude less than q^* and are reached in finite time. In contrast, Type B trajectories tend to a constant amplitude q^* with an exponential decay, while an infinite number of transitions occur between the slip and roll phases. In either case, the dissipation effect disappears and the system never reaches the trivial solution $(q, w, u) = (0, 0, 0)$.

6 Dynamics with absolute creep force model

6.1 The system

Introduce the notation $\gamma = c/m$ of the normalised creep coefficient. Then, by substituting the model (7) into (3) and by choosing the sigmoid function $\text{sigm}(x) = x/\sqrt{x^2 + 1}$, we get the nonlinear system

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q - 3 \frac{\mu g}{\gamma} \frac{u}{\sqrt{u^2 + \mu^2 g^2 / \gamma^2}} \end{bmatrix}, \quad (16)$$

For large creep velocities in the sense $|u| \gg \mu g / \gamma$, the nonlinear system (16) tends to the slip dynamics (10) of Coulomb friction model. In contrast, for $|u| \ll \mu g / \gamma$, (16) tends to the linear system

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q - 3\gamma u \end{bmatrix}. \quad (17)$$

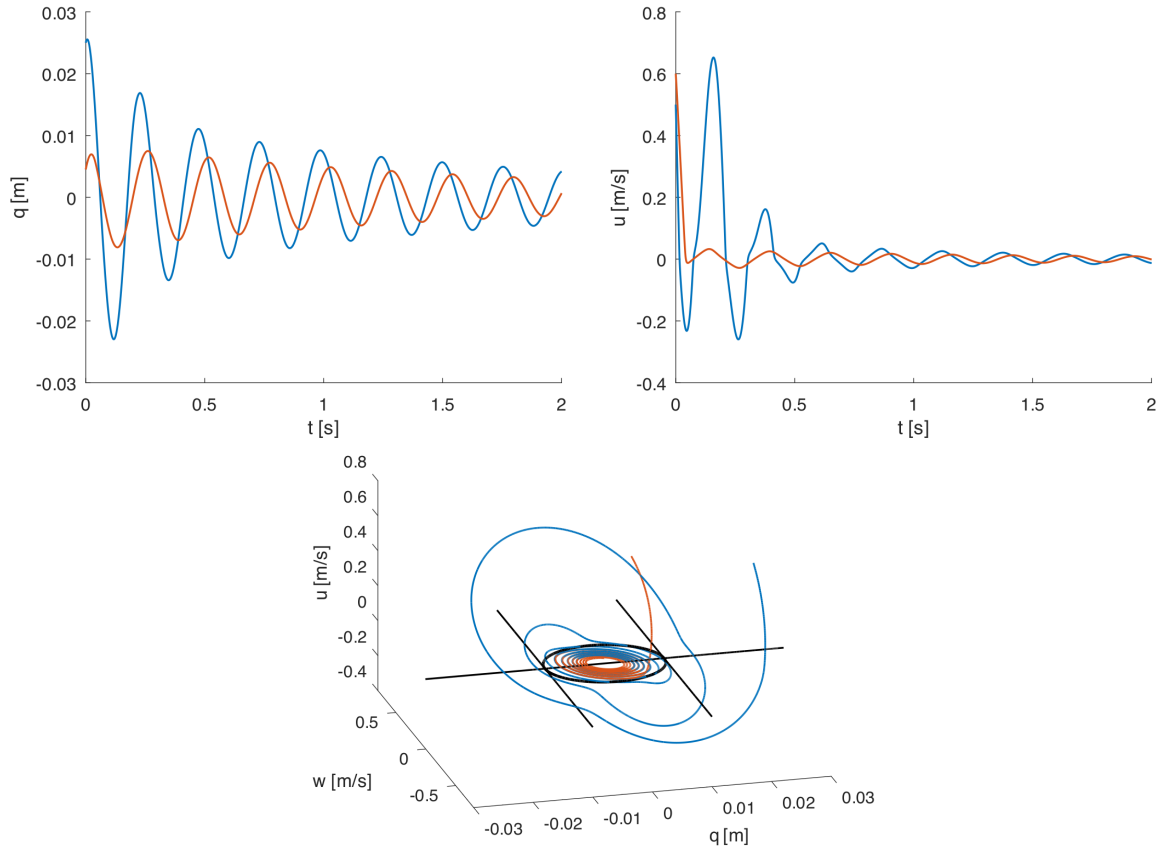


Figure 5: Numerical simulation of the test problem with the absolute creep force model. The notations of the diagrams are the same as those of Fig. 4.

Near the trivial solution $(q, w, u) = (0, 0, 0)$, the behaviour of the systems (16)-(17) is significantly different from that of the rolling dynamics (11) of the Coulomb model. Let us calculate the characteristic polynomial of (17),

$$\lambda^3 + 3\gamma\lambda^2 + \omega^2\lambda^2 + 2\gamma\omega^2 = 0. \quad (18)$$

There are no easy expressions for the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. However, it can be proven from the form of (18), that all three roots are stable (having negative real parts); moreover, there are one real and two complex conjugate eigenvalues. That is, the solutions converge to the trivial solution and thus the absolute creep force models make the disc stop at rest through an exponential decay of vibrations. If we assume that γ is typically a very large parameter, then, we can estimate the rate of exponential decay by using the approach of multiple time scales and singular perturbation theory (see [5]).

6.2 Dynamics with singular perturbation method

Let us introduce the parameter $\varepsilon = 1/\gamma$, which is a *small parameter* if γ is large. By multiplying the third component of (17) by ε , we get the following system,

$$\dot{q} = w + u/3, \quad (19)$$

$$\dot{w} = -2/3 \cdot \omega^2 q, \quad (20)$$

$$\varepsilon \dot{u} = g(q, w, u, \varepsilon) := -\varepsilon \omega^2 q - 3u. \quad (21)$$

If ε is small (i.e. γ is large), then (19)-(21) is a multiple time scale system (or slow-fast system) where q and w evolve on the *slow* time scale and u evolves on the *fast* time scale.

In these systems, there is often an *invariant manifold* in the form

$$u = h(q, w, \varepsilon), \quad (22)$$

within which the so-called slow dynamics can be defined. If this manifold is stable, then, the trajectories first approach the manifold on the fast time scale and then, the dynamics continues very close to the manifold on the slow time scale. In this way, the dynamics is approximately decomposed into two subsystems.

By using a singular perturbation theory, we consider the invariant manifold (22) as a polynomial function in the small parameter ε ,

$$h(q, w, \varepsilon) = h_0(q, w) + \varepsilon h_1(q, w) + \mathcal{O}^2(\varepsilon). \quad (23)$$

The first term in (23) is called the *critical manifold*, which can be obtained at the singular limit $\varepsilon \rightarrow 0$. From (21), we get the critical manifold $u = h_0(q, w) \equiv 0$. This result corresponds to the pure rolling case. That is, in the singular limit $\varepsilon \rightarrow 0$ ($\gamma \rightarrow \infty$), the system (17) gives back the behaviour of the rolling model (11).

By substituting (23) into the system (19)-(20) and performing implicit differentiation, the linear coefficient of (23) becomes $h_1(q, w) = -\omega^2 q/3$. That is, the first-order approximation of the invariant manifold (23) becomes

$$h(q, w, \varepsilon) = -\frac{1}{3} \omega^2 q \varepsilon + \mathcal{O}^2(\varepsilon). \quad (24)$$

This set is a plane with an inclination of ε above the plane Σ of $u = 0$. That is, as ε decreases, the invariant manifold approaches the plane Σ . (Determining the approximate invariant manifold is almost trivial in our example. For a general systematic process, see [5]; for a more complicated application, see [10].)

By substituting the invariant manifold (24) into the slow subsystem (19)-(20) and using $\varepsilon = 1/\gamma$, we get the slow dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{w} \end{bmatrix} \approx \begin{bmatrix} w - 1/9 \cdot \omega^2 q / \gamma \\ -2/3 \cdot \omega^2 q \end{bmatrix}. \quad (25)$$

In the meantime, the fast dynamics (21) can be approximated by

$$\dot{u} \approx -3\gamma \cdot u. \quad (26)$$

Finally, utilising multiple time scales, the original linear system is approximately decomposed into subsystems (25)-(26). The characteristic polynomial of the decoupled system becomes

$$(\lambda + 3\gamma)(\lambda^2 + \omega^2/(9\gamma) \cdot \lambda + 2/3 \cdot \omega^2) = 0. \quad (27)$$

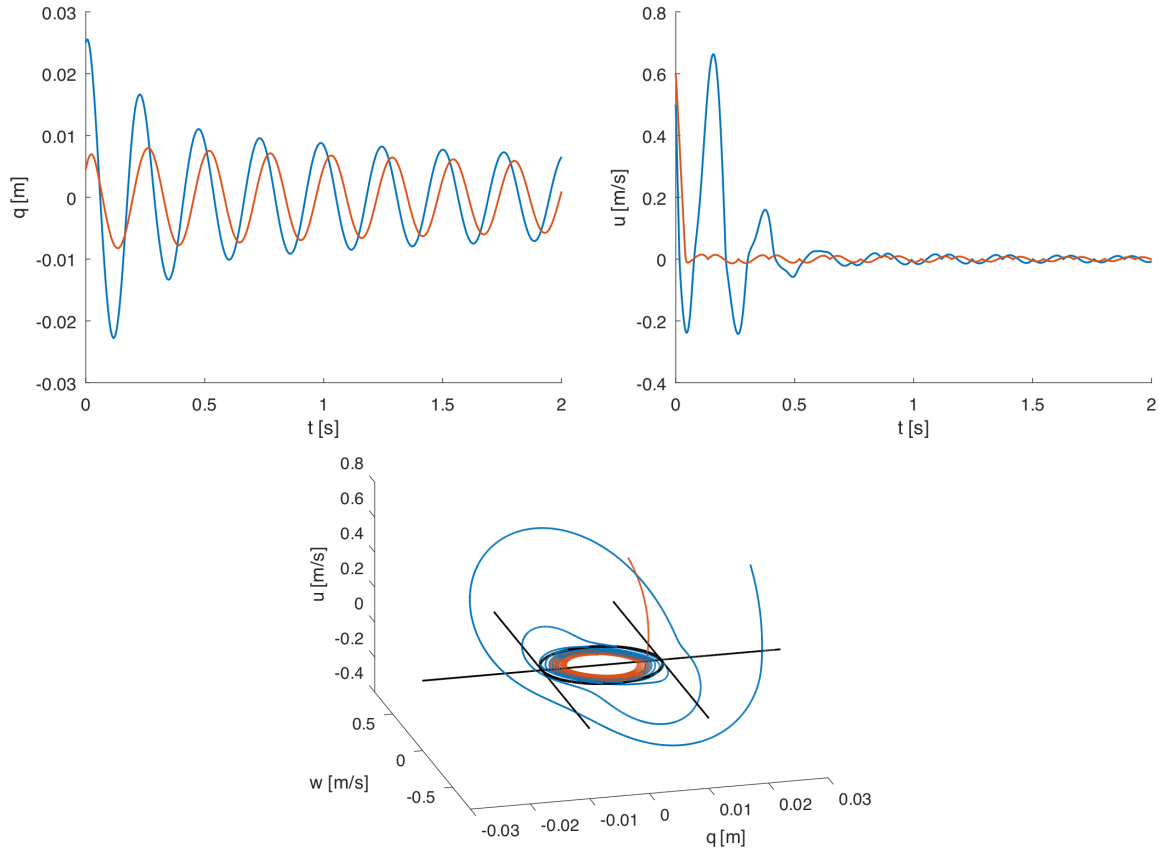


Figure 6: Numerical simulation of the test problem with the relative creep force model. The notations of the diagrams are the same as those of Fig. 4.

It is obvious that the polynomials (18) and (27) are not identical. However, it can be checked that their corresponding roots are very close to each other if γ is large. From (27), the real parts of the three roots are

$$\lambda_1 \approx -3\gamma, \quad \text{Re}(\lambda_2) = \text{Re}(\lambda_3) = -\frac{\omega^2}{18\gamma}. \quad (28)$$

That is, if γ increases, the real root λ_1 increases, and the slip velocity u decays faster. However, in the meantime, the real parts of the complex roots $\lambda_{2,3}$ decreases and the decay of the oscillations becomes slow. Consequently, as an interesting result, we can conclude that the exponent of the vibration decay is *inversely proportional* to the linear creep coefficient γ (and $c = m\gamma$). As the creep force model tends to the Coulomb model with infinite steepness, the dissipation dissolves once the system gets close to the periodic rolling solutions.

7 Dynamics with relative creep force model

7.1 The system

Let $\Gamma = C/m$ denote the normalised creep coefficient of the relative creep force models (8)-(9). By substituting (9) into (3) and taking $\text{sigm}(x) = x/\sqrt{x^2 + 1}$, we get

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q - 3\frac{\mu g}{\Gamma} \frac{u}{\sqrt{u^2 + w^2 \mu^2 g^2 / \Gamma^2}} \end{bmatrix}, \quad (29)$$

and its linearisation at $|u|/|w| \ll 1$ becomes

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} w + u/3 \\ -2/3 \cdot \omega^2 q \\ -\omega^2 q - 3\Gamma u/|w| \end{bmatrix}. \quad (30)$$

For the system (30), we cannot use the singular perturbation method as we did for (17), because instead of a parameter, a further variable w appears in the denominator. However, the form of (29) is a special nonsmooth vector field, and the theory of *extended Filippov system* [6, 9] can be used to explore the dynamics.

7.2 Dynamics as an extended Filippov system

The system (10) a Filippov system, which has a discontinuity in the two-dimensional surface Σ with $u = 0$. In the three-dimensional phase space, this is a *codimension-1* discontinuity. The system (29) has a discontinuity in the line σ where $u = w = 0$. In the phase space, this is a *codimension-2* discontinuity, because two variables have to vanish at the same time. For such a system called an *extended Filippov system* by the author [6, 11], specific properties of the discontinuity can be exploited.

First, let us transform the variables w and u into polar coordinates. Let

$$r = \sqrt{u^2 + w^2 \mu^2 g^2 / \Gamma^2} \quad (31)$$

be the generalised magnitude of relative motion of the disc by applying an appropriate two-norm. In addition, let

$$\varphi = \arctan_2(\Gamma u, \mu g w) \quad (32)$$

be the generalised magnitude of the relative motion by applying the two-parameter arcus tangent function $\arctan_2$. The inverse transformation of (31)-(32) gives

$$u = r \sin \varphi, \quad w = \frac{\Gamma}{\mu g} \cos \varphi. \quad (33)$$

Then, from (29), the time derivative of the new variables can be expressed as

$$\dot{r} = -\omega^2 q \left(\frac{2\Gamma}{3\mu g} \cos \varphi + \sin \varphi \right) - \mu g \sin^2 \varphi, \quad (34)$$

$$r\dot{\varphi} = \omega^2 q \left(\frac{2\Gamma}{3\mu g} \sin \varphi - \cos \varphi \right) - \mu g \sin \varphi \cos \varphi. \quad (35)$$

For small r , the form of (35) resembles the fast dynamics of (21), and (34)-(35) indeed forms a multiple time scale system with the fast variable φ and the slow variable r . However, in

(35), the small quantity r is not a parameter but a variable, which requires different methods of analysis.

For the qualitative description of these systems, we can introduce the concept of *limit directions* [6, 11]. Let us choose a point $(q, w, u) = (q, 0, 0) \in \sigma$. Then, the limit directions are the roots $\hat{\varphi}_1 \dots \hat{\varphi}_n$ of (35). These values are the fixed points of the fast dynamics (35), and in the meantime they characterise the directions where the trajectories of (29) reach the discontinuity σ . By substituting each value $\hat{\varphi}_i$ into (34), the sign of the resulting values $\dot{r}_1 \dots \dot{r}_n$ shows whether the limit direction is radially attracting ($-$) or repelling ($+$). If all limit directions of a point q are radially attracting, then the neighbouring trajectories are trapped into the discontinuity σ , resulting in sliding dynamics similar to the Filippov systems. If at least one limit direction is repelling, then the trajectories can leave σ . For more details, see [6] or [11].

In our system, the equation (35) typically has two or four roots in $\varphi \in [0, 2\pi)$. These limit directions can be expressed by lengthy analytical expressions. Instead, it is sufficient for us to explore the rough structure of the dynamics. Consider the two limit cases:

- For $\mu g \ll \omega^2 q$, (35) has *two* limit directions with $\tan \hat{\varphi}_1 = \tan \hat{\varphi}_2 = 2\Gamma/(3\mu g)$. Then, from (34), we can see that one of them is attracting ($\dot{r} < 0$) and the another is repelling ($\dot{r} > 0$).
- For $\mu g \gg \omega^2 q$, (35) has the *four* limit directions $\hat{\varphi}_1 = 0$, $\hat{\varphi}_2 = \pi/2$, $\hat{\varphi}_3 = \pi$, and $\hat{\varphi}_4 = 3\pi/2$. From (34), it follows that $\hat{\varphi}_2$, $\hat{\varphi}_4$ are attracting ($\dot{r} < 0$); and one of $\hat{\varphi}_1$, $\hat{\varphi}_3$ is attracting and the another is repelling. In the limit case $q \rightarrow 0$, both $\hat{\varphi}_1$, $\hat{\varphi}_3$ are at the boundary of attracting and repelling ($\dot{r} = 0$).

These results show two bifurcations of the limit directions [11]. First, there must be a value of q when the number of limit directions changes from 2 to 4, which happens in *fold of limit directions*. In addition, the value $q = 0$ is a special case of a *tangency bifurcation* (change of radial attracting or repelling behaviour).

The results uncover the qualitative dynamics in the vicinity of the discontinuity σ . For all point $(q, 0, 0) \in \sigma$ except the case $q = 0$, the trajectories of the nonsmooth system (29) pass through the line σ . The only point where trajectories can permanently reach σ is the trivial solution $(q, w, u) = (0, 0, 0)$. But, due to the degenerate case between radially attracting and repelling behaviour, this is not happening directly, but through an infinite number of vibrations. That is, before the disc reaches the permanent rest of the trivial solution, its state goes through the static state $w = u = 0$ of σ infinitely many times.

8 Conclusion

We analysed the test problem of an elastically suspended rolling disc moving on a rough surface. We considered the Coulomb friction model and the creep force models with absolute and relative formulations. In each case, the test problem leads to dynamical systems with either nonsmooth or multi-scale behaviour. The careful application of mathematical tools provided insight into the predicted vibrations. We determined the speed of exponential convergence of the absolute creep force model and showed how the time scales are separating as the model approaches the Coulomb friction. The most interesting behaviour originates from the relative creep force model: the analysis of the limit directions showed that the exponential convergence slows down compared to the absolute creep force model, and the solutions pass through the static states infinitely many times before stopping. Summary of the results can be seen in Fig. 7.

Based on these results, several further research directions are possible. The current analysis requires a deeper understanding of the theoretical relations between different contact models and their extensions to further forms of motion. This analysis can be applied to complex practical engineering problems, e.g., in vehicle dynamics.

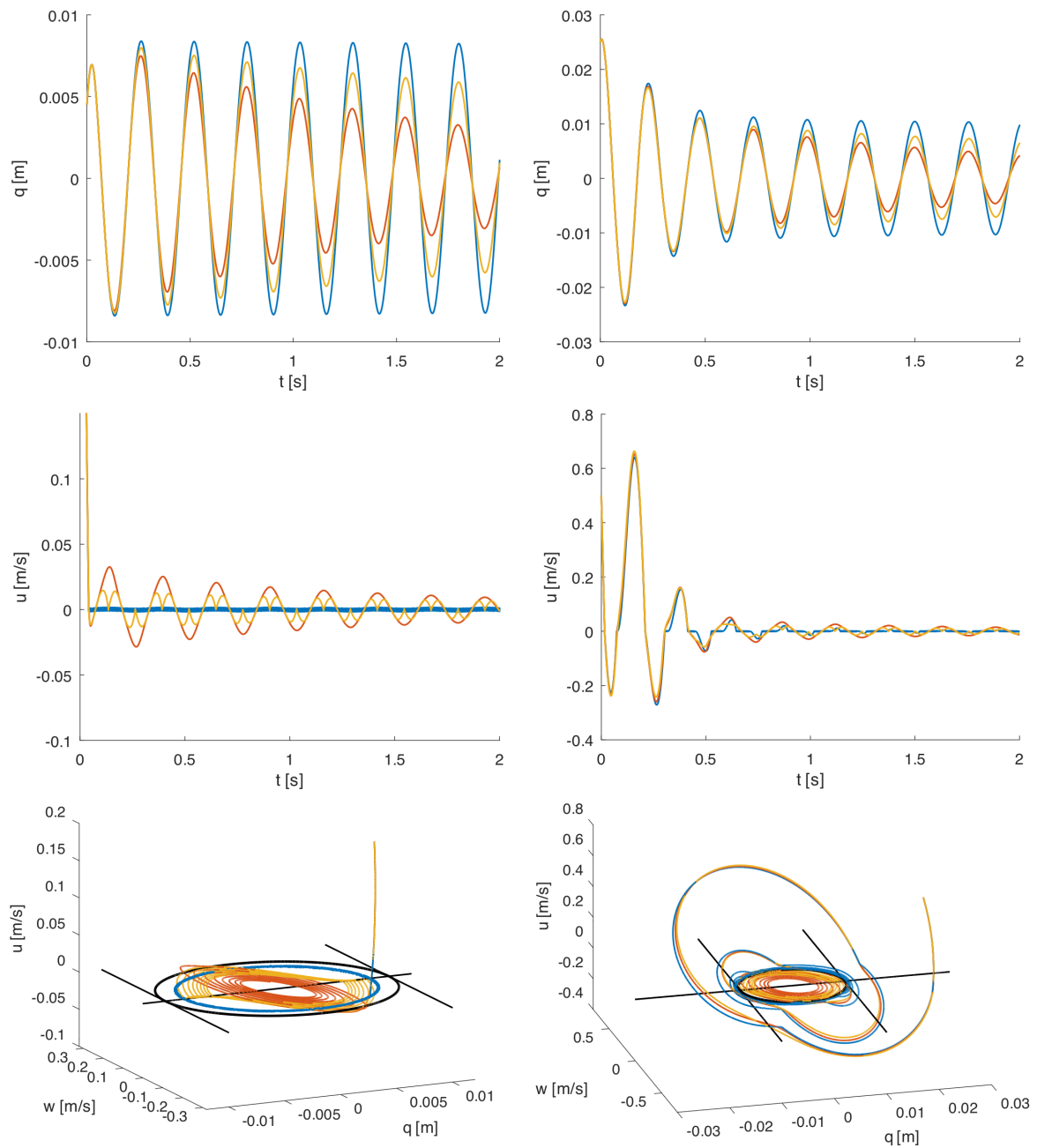


Figure 7: Comparison of trajectories of the three models. Blue lines: Coulomb model. Red lines: absolute creep force model. Yellow lines: relative creep force model. Left column sub-figures: „Type A”. Right column sub-figures: „Type B”

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