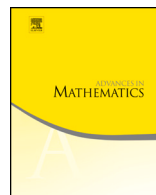




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Exponential valuations on lattice polygons valued at formal power series



Károly J. Böröczky^{a,*}, Mátyás Domokos^b, Ansgar Freyer^c,
Christoph Haberl^d, Jin Li^e

^a HUN-REN Alfréd Rényi Institute of Mathematics, and Loránd Eötvös University, Institute of Mathematics, Budapest, Hungary

^b HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary

^c FU Berlin, Fachbereich Mathematik und Informatik, Arnimallee 2, 14195 Berlin, Germany

^d Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10/104, 1040 Vienna, Austria

^e Department of Mathematics, and Newtown Center for Mathematics, Shanghai University, China

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ABSTRACT

We classify valuations on lattice polygons with values in the ring of formal power series that commute with the action of the affine unimodular group. A typical example of such valuations is induced by the Laplace transform, but as it turns out there are many more. The classification is done in terms of formal power series that satisfy certain functional equations. We align our classification with the decomposition into so-called dilative components.

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* Corresponding author.

E-mail addresses: boroczky.karoly.j@renyi.hu (K.J. Böröczky), domokos.matyas@renyi.hu (M. Domokos), a.freyer@fu-berlin.de (A. Freyer), christoph.haberl@tuwien.ac.at (C. Haberl), jinlishu@shu.edu.cn (J. Li).

1. Introduction

By a *polytope* P in $\mathbb{R}^n$, we mean the convex hull of finitely many points of $\mathbb{R}^n$. The polytope P is called a *segment* if $\dim P = 1$, and P is called a *polygon* if $\dim P = 2$. Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^n$ such that if $P \cup Q$ is convex for $P, Q \in \mathcal{F}$, then $P \cup Q \in \mathcal{F}$ and $P \cap Q \in \mathcal{F}$. Examples of such families are the family $\mathcal{K}^n$ of all convex compact sets in $\mathbb{R}^n$, the family $\mathcal{P}^n$ of all polytopes in $\mathbb{R}^n$ and the family $\mathcal{P}(\mathbb{Z}^n)$ of all lattice polytopes; namely, convex hulls of finitely many points of $\mathbb{Z}^n$ (see McMullen [22]). If $\mathcal{A}$ is a cancellative monoid (cancellative commutative semigroup with identity element $0_{\mathcal{A}}$), then a function $Z : \mathcal{F} \rightarrow \mathcal{A}$ is called a *valuation* if the following holds: if $P \cup Q$ is convex for $P, Q \in \mathcal{F}$, then

$$Z(P \cup Q) + Z(P \cap Q) = Z(P) + Z(Q). \quad (1)$$

We say that the valuation $Z : \mathcal{F} \rightarrow \mathcal{A}$ is *simple* if $Z(P) = 0_{\mathcal{A}}$ for any $P \in \mathcal{F}$ with $\dim P \leq n - 1$. Typically, one would consider valuations intertwining with some natural group actions, as we will shortly see.

While the idea of valuations on convex polytopes played a crucial role in Dehn's solution of Hilbert's Third Problem already around 1900, after sporadic results, the systematic study of valuations only started with Hadwiger's celebrated characterization of the intrinsic volumes as the basis of the space of continuous isometry invariant valuations from 1957. For the breathtaking developments of the last seven decades, see for example the monograph Alesker [2], and the survey papers Alesker [1], Ludwig [18] and Ludwig, Mussnig [19]. The theory of valuations on lattice polytopes has been flourishing since the classical paper by Betke, Kneser [6] in 1985 characterizing unimodular invariant valuations on lattice polytopes, see for example, Böröczky, Ludwig [8,9], Jochemko, Sanyal [12,13], Berg, Jochemko, Silverstein [5] and Ludwig, Silverstein [20].

To state the result stimulating our research, let $L_c^1(\mathbb{R}^n)$ denote the family of Lebesgue integrable functions with compact support on $\mathbb{R}^n$. In addition, for measurable $\Omega \subset \mathbb{R}^n$, let $\mathcal{M}(\Omega)$ denote the family of Lebesgue measurable functions on Ω . For $f \in L_c^1(\mathbb{R}^n)$, its Laplace transform is

$$\mathcal{L}f(u) = \int_{\mathbb{R}^n} e^{-\langle u, v \rangle} f(v) dv.$$

Here, $\langle u, v \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Li, Ma [17] applied the definition of the Laplace transform to a convex body K by applying $\mathcal{L}$ to the characteristic function; namely,

$$\mathcal{L}K = \mathcal{L}\mathbf{1}_K = \int_K e^{-\langle u, v \rangle} dv.$$

Inspired by the properties of the Laplace transform applied to the characteristic functions of compact convex sets, Li, Ma [17] considered valuations $Z: \mathcal{K}^n \rightarrow C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ denotes the space of continuous functions on $\mathbb{R}^n$, which commute with affine transformations in the following sense: For any $K \in \mathcal{K}^n$ and any $\Phi \in \text{GL}(n, \mathbb{R})$, $w \in \mathbb{R}^n$ we have

$$Z(\Phi K + w)(u) = e^{-\langle u, w \rangle} |\det \Phi| \cdot Z(K)(\Phi^T u) \quad (2)$$

Theorem 1 (Li, Ma [17]). *Let $Z: \mathcal{K}^n \rightarrow C(\mathbb{R}^n)$ be a continuous (with respect to the Hausdorff metric) valuation satisfying (2). Then there exists a constant $c \in \mathbb{R}$ such that $Z = c\mathcal{L}$.*

In this paper, we consider valuations on lattice polygons (of $\mathbb{Z}^2$) valued at formal power series in two variables. We consider the action of the affine unimodular group (the group of affine transformations leaving $\mathbb{Z}^2$ invariant)

$$\mathcal{G}(\mathbb{Z}^2) = \mathbb{Z}^2 \rtimes \text{GL}(2, \mathbb{Z})$$

on $\mathbb{Q}[[x, y]]$ (the ring of formal power series in the variables x and y) where for $f \in \mathbb{Q}[[x, y]]$ and $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ with

$$\Xi(x, y) = (ax + by + \alpha, cx + dy + \beta),$$

$a, b, c, d, \alpha, \beta \in \mathbb{Z}$ and $ad - bc = \pm 1$, we have

$$(\Xi \cdot f)(x, y) = \exp(\alpha x + \beta y) \cdot f(ax + cy, bx + dy). \quad (3)$$

In this context the exponential function is considered as a formal power series $\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \in \mathbb{Q}[[t]]$, which still satisfies that $\exp(t + s) = \exp(t) \cdot \exp(s)$ for any formal power series s, t (cf. Sambale [23]).

On lattice polygons, $\mathcal{G}(\mathbb{Z}^2)$ acts in the natural way. Now a valuation $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is called $\mathcal{G}(\mathbb{Z}^2)$ equivariant, if $Z(\Xi P) = \Xi \cdot Z(P)$ holds for all $P \in \mathcal{P}(\mathbb{Z}^2)$ and $\Xi \in \mathcal{G}(\mathbb{Z}^2)$. This equivariance property is essentially analogue to the one in (2) except for a sign in the exponential function.

One example of a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation is the “positive Laplace transform” $\mathcal{L}_+$ defined for $P \in \mathcal{P}(\mathbb{Z}^2)$ as

$$\mathcal{L}_+(P)(x, y) = \int_{\mathbb{R}^2} \exp(\alpha x + \beta y) \cdot \mathbf{1}_P(\alpha, \beta) d\alpha d\beta.$$

This valuation has been studied (in general dimension) by Barvinok [3,4] and Lawrence [15,16], which has led to spectacular progress in computational geometry.

The main goal of this paper is to characterize $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ in a constructive way, providing a tool to produce such valuations on the one hand, and to be able to decide whether a valuation on $\mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is $\mathcal{G}(\mathbb{Z}^2)$ equivariant on the other hand. Our method is partially based on ideas in Böröczky, Domokos, Freyer, Haberl, Harcos, Li [7]. We note that Freyer, Ludwig, Rubey [10] characterized $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations valued at the formal power series in two variables, but their approach is less constructive (see below for a summary of their result).

We write e_1, e_2 to denote the orthonormal basis of $\mathbb{R}^2$ also generating $\mathbb{Z}^2$, and set $T = [e_1, e_2, o]$ where $o = (0, 0)$ stands for the origin, and $[x_1, \dots, x_k]$ stands for the convex hull of $x_1, \dots, x_k \in \mathbb{R}^2$. In the formulas below,

$$\frac{e^t - 1}{t} \text{ is identified with the formal power series } \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}.$$

In addition, we write D_4 to denote the subgroup of $\text{GL}(2, \mathbb{Z})$ generated by the matrices $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$. The reason for the chosen notation is that this group D_4 is isomorphic to the dihedral group of 8 elements. The subalgebra of $\mathbb{Q}[[x, y]]$ of D_4 invariant elements is (see Remark 21)

$$\begin{aligned} \mathbb{Q}[[x, y]]^{D_4} = & \{h \in \mathbb{Q}[[x, y]] : \Phi \cdot h = h \text{ for } \Phi \in D_4\} \\ & = \{g(2x^2 + 2xy + y^2, 4x^2y^2 + 4xy^3 + y^4) \in \mathbb{Q}[[x, y]] : \\ & \quad g(a, b) \in \mathbb{Q}[[a, b]]\} \end{aligned} \quad (4)$$

where the polynomials $2x^2 + 2xy + y^2$ and $4x^2y^2 + 4xy^3 + y^4$ are algebraically independent.

According to Proposition 6, any $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is determined by $Z(\{o\})$, $Z([o, e_1])$ and $Z(T)$.

Theorem 2. *For any $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, $Z(\{o\})$ is a constant power series $c \in \mathbb{Q}$, and*

$$f_1(x, y) := Z([o, e_1])(x, y) = g(x^2) \cdot \exp\left(\frac{1}{2}x\right) \quad (5)$$

holds for some $g \in \mathbb{Q}[[x]]$; moreover, Z satisfies

$$Z(T)(x, y) = f_2(x, y) + \frac{1}{2} f_1(x, y) + \frac{1}{2} f_1(y, -x) + \frac{e^x}{2} f_1(-x + y, -x) \quad (6)$$

where f_2 is defined by

$$f_2(x, y) = \frac{e^x}{y} \cdot \frac{e^{y-x} - 1}{y - x} \cdot \varrho(y - x, x) - \frac{1}{y} \cdot \frac{e^x - 1}{x} \cdot \varrho(x, y - x) \quad (7)$$

for a $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying

$$(2x + y)\varrho(x, y) = (x + y)\varrho(x, x + y) + x\varrho(x + y, x). \quad (8)$$

On the other hand, if $c \in \mathbb{Q}$, $f_1(x, y) = g(x^2) \cdot \exp(\frac{1}{2}x)$ for some $g \in \mathbb{Q}[[x]]$, and f_2 is defined by (7) for a $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying (8), then there exists a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that $Z(\{o\}) = c$, $Z([o, e_1]) = f_1$, and $Z(T)$ is defined by (6).

Remark. In particular, any $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is of the form $Z = Z_1 + Z_2$ where Z_1 is constructed as in Proposition 14 for $c = Z(\{o\})$ and $f_1(x, y) := Z([o, e_1])(x, y)$, and Z_2 is a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation with $Z_2(T) = f_2$ and f_2 is defined via (7) and (8).

Consequently, simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ are in bijective correspondence with formal power series $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying (8) via the formula (cf. (7))

$$Z(T)(x, y) = \frac{e^x}{y} \cdot \frac{e^{y-x} - 1}{y - x} \cdot \varrho(y - x, x) - \frac{1}{y} \cdot \frac{e^x - 1}{x} \cdot \varrho(x, y - x) \quad (9)$$

where the right hand side is a formal power series under these conditions. Here ϱ is even (it contains only terms of even degree) by (4), and actually, ϱ does not contain terms of degree 2 (cf. Corollary 25).

In order to characterize a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, Freyer, Ludwig, Rubey [10] say that Z is δ -dilative for $\delta \in \mathbb{Z}$ if for any (two-dimensional) lattice polygon P , we have

$$Z(mP)(x, y) = m^{-\delta} \cdot Z(P)(mx, my) \quad (10)$$

for any integer $m \geq 1$. As an example, the positive Laplace transform $\mathcal{L}_+$ is (-2) -dilative. It turned out that the dilative valuations give a natural decomposition to the space of $\mathcal{G}(\mathbb{Z}^2)$ valuations $\mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$. As in [10], we write $\overline{\text{Val}}$ for the vector space of $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $\mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, and moreover, $\overline{\text{Val}}_\delta$ for the subspace of δ -dilative valuations.

Theorem 3 (Freyer, Ludwig, Rubey [10]). *We have*

$$\overline{\text{Val}} = \prod_{\delta \geq -2} \overline{\text{Val}}_\delta,$$

as well as,

$$\{Z \in \overline{\text{Val}} : Z \text{ simple}\} = \prod_{\delta \geq -2, \delta \text{ even}, \delta \neq 0} \overline{\text{Val}}_\delta.$$

Moreover,

$$\dim \overline{\text{Val}}_\delta = \begin{cases} 0, & \delta < -2, \\ 1, & \delta > -2 \text{ odd}, \\ \lfloor \frac{\delta+2}{12} \rfloor + 1, & \delta \geq -2 \text{ even and } \delta \bmod 12 \neq 0, \\ \lfloor \frac{\delta+2}{12} \rfloor, & \delta \geq -2 \text{ even and } \delta \bmod 12 = 0. \end{cases}$$

It should be noted that in [10] power series with real coefficients are considered. However, the choice of coefficients does not play a crucial role in these results as the bases of the vector spaces involved may be chosen to be rational. Hence, Theorem 3 holds true for rational coefficients as well.

Given the above theorem, it is natural to ask for a characterization of δ -dilative valuations in terms of the data $c \in \mathbb{Q}$, $g \in \mathbb{Q}[[x]]$ and $\rho \in \mathcal{V}$ from Theorem 2, where $\mathcal{V}$ denotes the space of formal power series $\rho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying (8). We write $\mathcal{V}_d$ for the space of homogeneous polynomials of degree d in $\mathcal{V}$.

Theorem 4. *Let $\delta \geq -2$ be an integer and let $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ be a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation parametrized by $c \in \mathbb{Q}$, $g \in \mathbb{Q}[[x]]$ and $\rho \in \mathcal{V}$ as in Theorem 2.*

1. *Let δ be even and non-zero. Then, Z is δ -dilative if and only if $c = 0$, $g = 0$ and $\rho \in \mathcal{V}_{\delta+2}$.*
2. *Let δ be odd. Then Z is δ -dilative if and only if $c = 0$, $g(x) = \alpha x^{\delta/2} \sinh(\frac{\sqrt{x}}{2})$ for some $\alpha \in \mathbb{Q}$, and $\rho = 0$.*
3. *The valuation Z is 0-dilative if and only if $c = \alpha$, $g(x) = \alpha \cosh(\frac{\sqrt{x}}{2})$ for some $\alpha \in \mathbb{Q}$, and $\rho = 0$.*

Remark 5. Theorem 2 and Theorem 4 hold with real or complex coefficients, as well, instead of rational coefficients (actually, any field of characteristic zero can be chosen instead of $\mathbb{Q}$).

In view of Theorem 3 and Theorem 4, for even $\delta \geq 0$, the dimension of $\mathcal{V}_\delta$ (and thus of $\overline{\text{Val}}_{\delta-2}$) agrees with the dimension of the linear space of modular forms of weight δ with respect to the full $\text{SL}(2, \mathbb{Z})$. The latter space carries a natural product structure, while $\mathcal{V}$ is not closed under the usual multiplication in $\mathbb{Q}[[x, y]]$. It is not clear whether there exists a natural product on the space of simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations, or at least on the vector space $\mathcal{V}$.

Concerning the structure of the paper, results due to Böröczky, Domokos, Freyer, Haberl, Harcos, Li [7] needed for our paper are discussed in Section 2. Theorem 2 is proved in Section 3, and finally, Theorem 4 is verified in Section 4.

2. Preliminaries

In this section, we summarize some related results obtained by Böröczky, Domokos, Freyer, Haberl, Harcos, Li [7]. These results were obtained for $\mathcal{G}(\mathbb{Z}^2)$ equivariant valua-

tions that are valued at *measurable functions* in two variables, where from an algebraic perspective, the action of the affine unimodular group on modular function is equivalent to the definition here in the case of power series in two variables; it is also described by the formula (3). The results quoted here are not specific to measurable functions, which is why we can replace the target space by $\mathbb{Q}[[x, y]]$.

According to Proposition 8 in [7], we have the following:

Proposition 6. *For any two $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $Z, Z' : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, the following statements hold:*

- (i) *If $Z(\{o\}) = Z'(\{o\})$ and $Z([o, e_1]) = Z'([o, e_1])$, then $Z - Z'$ is a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation;*
- (ii) *If $Z(\{o\}) = Z'(\{o\})$, $Z([o, e_1]) = Z'([o, e_1])$ and $Z(T) = Z'(T)$, then $Z = Z'$.*

It follows from Proposition 6 that in order to characterize a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, all we need to characterize are

$$Z(\{o\}) = f_0 \text{ and } Z([o, e_1]) = f_1 \text{ and } Z(T) = f_2. \quad (11)$$

The basic algebraic identities for f_0 and f_1 are described by Lemma 9 in [7].

Lemma 7. *Let $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ be a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation. Then $Z(\{o\}) = f_0$ and $Z([o, e_1]) = f_1$ satisfy the following properties:*

$$f_0(ax + cy, bx + dy) = f_0(x, y) \quad \text{for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z}); \quad (12)$$

$$f_1(-x, -y) = \exp(-x)f_1(x, y); \quad (13)$$

$$f_1(x, y) = f_1(x, x + y); \quad (14)$$

$$f_1(x, y) = f_1(x, -y). \quad (15)$$

In turn, Proposition 11 in [7] says that the conditions (12), (13), (14) and (15) are sufficient.

Proposition 8. *For $f_0, f_1 \in \mathbb{Q}[[x, y]]$ satisfying (12), (13), (14) and (15), there exists a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_1 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ satisfying $Z_1(\{o\}) = f_0$ and $Z_1([o, e_1]) = f_1$ and*

$$Z_1(T)(x, y) = \frac{1}{2} f_1(x, y) + \frac{1}{2} f_1(y, -x) + \frac{1}{2} \exp(x) \cdot f_1(-x + y, -x).$$

Proposition 6 (i) and Proposition 8 yield the following.

Corollary 9. *For any $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, let Z_1 be the $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation satisfying $Z_1(\{o\}) = Z(\{o\})$ and $Z_1([o, e_1]) = Z([o, e_1])$ constructed in Proposition 8. Then $Z_2 = Z - Z_1$ is a $\mathcal{G}(\mathbb{Z}^2)$ equivariant simple valuation.*

Our next goal is to understand the algebraic properties of simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$. The functional equations for $Z(T) = f_2$ are described by Lemma 13 in [7].

Lemma 10. *For any simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, $Z(T) = f_2$ satisfies the following properties:*

$$f_2(-x + y, -x) = \exp(-x) \cdot f_2(x, y); \quad (16)$$

$$f_2(x, y) + \exp(x + y) \cdot f_2(-x, -y) = f_2(x, x + y) + f_2(x + y, y); \quad (17)$$

$$f_2(x, y) = f_2(y, x). \quad (18)$$

Proposition 14 in [7] states the reverse statement.

Proposition 11. *For any $f_2 \in \mathbb{Q}[[x, y]]$ satisfying the properties (16), (17) and (18) in Lemma 10, there exists a unique simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that $Z(T) = f_2$.*

Finally, Lemma 21 in [7] observes that in Lemma 10 and Proposition 11, we can exchange (17) with (19).

Lemma 12. *Assuming that $f_2 \in \mathbb{Q}[[x, y]]$ satisfies (16) and (18), we have f_2 also satisfies (17) if and only if*

$$f_2(x, y) + \exp(x) \cdot f_2(y - x, y) = f_2(x, x + y) + f_2(y, x + y). \quad (19)$$

3. Parametrization of equivariant valuations

In this section, we combine the tools summarized in Section 2 with new results in order to prove Theorem 2. We note that according to Sambale [23], for any formal power series of the form $\varphi(x) = 1 + \sum_{n=1}^{\infty} b_n x^n \in \mathbb{Q}[[x]]$, there exists a formal power series $\psi \in \mathbb{Q}[[x]]$ such that

$$\exp(\psi) = \varphi. \quad (20)$$

The following statement is well-known in invariant theory.

Lemma 13. *If $f \in \mathbb{Q}[[x, y]]$ satisfies $f(x, y) = f(x, x + y)$, then there exists $\tilde{f} \in \mathbb{Q}[[x]]$ such that $f(x, y) = \tilde{f}(x)$.*

It follows that if both $f(x, y) = f(x, x + y)$ and $f(x, y) = f(x + y, y)$ hold for $f \in \mathbb{Q}[[x, y]]$, then f is a constant power series.

Next we show that all we need to understand are the simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations.

Proposition 14. *For any $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, $Z(\{o\})$ is a constant power series, and*

$$f_1(x, y) = Z([o, e_1])(x, y) = g(x^2) \cdot \exp\left(\frac{1}{2}x\right) \quad (21)$$

holds for some $g \in \mathbb{Q}[[x]]$; moreover, there exists a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_2 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that

$$Z(T)(x, y) = Z_2(T)(x, y) + \frac{1}{2}f_1(x, y) + \frac{1}{2}f_1(y, -x) + \frac{1}{2}\exp(x) \cdot f_1(-x + y, -x). \quad (22)$$

On the other hand, if $c \in \mathbb{Q}$ and $f_1(x, y) = g(x^2) \cdot \exp\left(\frac{1}{2}x\right)$ for some $g \in \mathbb{Q}[[x]]$, then there exists a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ satisfying $Z(\{o\}) = c$, $Z([o, e_1]) = f_1$ and (22) with $Z_2(T) = 0$.

Proof. For a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Z}[[x, y]]$, let $Z(\{o\}) = f_0$ and $Z([o, e_1]) = f_1$. According to Lemma 7, f_0 and f_1 satisfy the following properties:

$$f_0(x, y) = f_0(ax + cy, bx + dy) \quad \text{for any} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z}); \quad (23)$$

$$f_1(x, y) = \exp(x) \cdot f_1(-x, -y); \quad (24)$$

$$f_1(x, y) = f_1(x, x + y); \quad (25)$$

$$f_1(x, y) = f_1(x, -y). \quad (26)$$

It follows from (23) and Lemma 13 that f_0 is a constant power series.

For f_1 , (25) and Lemma 13 yield that $f_1(x, y) = \tilde{f}(x)$ for a $\tilde{f} \in \mathbb{Q}[[x]]$, and by (24) we have

$$\tilde{f}(x) = \exp(x) \cdot \tilde{f}(-x). \quad (27)$$

We may assume that $\tilde{f}$ is not the constant zero power series, and let $a_d x^d$, $d \in \mathbb{N}$, be the term with the smallest degree with non-zero coefficient a_d in $\tilde{f}(x)$. It follows that $\tilde{f}(x) = a_d x^d \cdot \varphi(x)$ where $\varphi(x) \in \mathbb{Q}[[x]]$ is of the form $\varphi(x) = 1 + \sum_{n=1}^{\infty} b_n x^n$. Substituting $\tilde{f}(x) = a_d x^d \cdot \varphi(x)$ into (27) and equating the coefficients of x^d on the two sides shows that $d = 2k$ for a $k \in \mathbb{N}$, and hence

$$\varphi(x) = \exp(x) \cdot \varphi(-x). \quad (28)$$

According to (20), there exists a formal power series $\psi \in \mathbb{Q}[[x]]$ such that $\exp(\psi) = \varphi$; therefore, (28) yields that $\psi(x) = x + \psi(-x)$. We deduce the existence of an $h \in \mathbb{Q}[[x]]$ such that $\psi(x) = \frac{1}{2}x + h(x^2)$, and hence $f_1(x, y) = g(x^2) \cdot \exp(\frac{1}{2}x)$ where $g(x) = a_k x^k \cdot \exp(h(x))$.

Now let us assume that $f_0, f_1 \in \mathbb{Q}[[x, y]]$ satisfy that $f_0 = c$ for a constant $c \in \mathbb{Q}$ and $f_1(x, y) = g(x^2) \cdot \exp(\frac{1}{2}x)$ for a $g \in \mathbb{Q}[[x]]$. In particular, f_0 and f_1 satisfy (23), (24), (25) and (26). It follows from Proposition 8 that there exists a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_1 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ satisfying $Z_1(\{o\}) = f_0$ and $Z_1([o, e_1]) = f_1$ and

$$Z_1(T)(x, y) = \frac{1}{2} f_1(x, y) + \frac{1}{2} f_1(y, -x) + \frac{1}{2} \exp(x) \cdot f_1(-x + y, -x). \quad \square$$

Given Proposition 14, our remaining task is to understand simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations; therefore, the rest of the section is dedicated to simple valuations.

The following statement follows from Lemma 10, Proposition 11 and Lemma 12.

Proposition 15. *For any simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, $Z(T) = f_2$ satisfies the properties*

- (A) $f_2(x, y) + \exp(x) \cdot f_2(y - x, y) = f_2(x, x + y) + f_2(y, x + y)$;
- (B) $f_2(x, y) = f_2(y, x)$;
- (C) $f_2(y - x, -x) = \exp(-x) \cdot f_2(x, y)$.

On the other hand, for any $f_2 \in \mathbb{Q}[[x, y]]$ satisfying the properties (A), (B) and (C), there exists a unique simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that $Z(T) = f_2$.

In the formulas below, the expression $\frac{e^t - 1}{t}$ stands for $\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \in \mathbb{Q}[[t]]$, and its reciprocal $\frac{t}{e^t - 1}$ stands for $\sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot t^n \in \mathbb{Q}[[t]]$ where $B_0, B_1, \dots$ are the Bernoulli numbers (cf. Zagier [24]). In particular, we can also consider $\frac{e^y - e^x}{y - x} = \exp(x) \cdot \frac{e^{y-x} - 1}{y - x} \in \mathbb{Q}[[x, y]]$. Tacitly, we also use the property that $\mathbb{Q}[[x, y]]$ is an integral domain; namely, the product of non-zero elements is non-zero (cf. Sambale [23]).

The main idea to better understand simple power series valued valuations is to transform f_2 into a power series ϱ that satisfies some relations (see (A') and (E) in Theorem 16) that do not contain exponential components and are simpler than the conditions (A), (B) and (C) for f_2 in Proposition 15. In particular, for $f \in \mathbb{Q}[[x, y]]$, we consider the formal power series

$$f^\#(x, y) = \frac{x}{e^x - 1} \cdot \frac{x + y}{e^{x+y} - 1} \cdot \left[f(x, x + y) + \exp(x) \cdot f(y, x + y) \right]. \quad (29)$$

As a reverse notion, if $\varrho \in \mathbb{Q}[[x, y]]$ and $\frac{e^y - e^x}{y - x} \cdot \varrho(y - x, x) - \frac{e^x - 1}{x} \cdot \varrho(x, y - x)$ is of the form $y \cdot g(x, y)$ for a $g \in \mathbb{Q}[[x, y]]$, then we consider

$$\varrho^\dagger(x, y) = \frac{1}{y} \cdot \left[\frac{e^y - e^x}{y - x} \cdot \varrho(y - x, x) - \frac{e^x - 1}{x} \cdot \varrho(x, y - x) \right]. \quad (30)$$

We observe that $\varrho^\dagger(x, y)$ can't be defined, for example, for $\varrho(x, y) = x$; however, $\varrho^\dagger$ is a formal power series if either $\varrho = f^\sharp$ for an $f \in \mathbb{Q}[[x, y]]$ (cf. Lemma 17), or if ϱ satisfies (cf. Lemma 19)

$$(D) \quad \varrho(-x, -y) = \varrho(x, y)$$

where (D) is equivalent to saying that each term in ϱ with non-zero coefficient has even degree. We note that the operators $f \mapsto f^\sharp$ and $\varrho \mapsto \varrho^\dagger$ are essentially inverses of each other (cf. Lemma 17).

The first major step of our argument towards understanding simple valuations is the following statement:

Theorem 16. *If $f \in \mathbb{Q}[[x, y]]$ satisfies (A), (B) and (C) in Proposition 15, then $\varrho = f^\sharp$ satisfies $\varrho^\dagger = f$ and*

$$(A') \quad (x + y)\varrho(x, y - x) = y\varrho(x, y) + x\varrho(y, x) \text{ and}$$

$$(E) \quad \varrho(x, -2x - y) = \varrho(x, y).$$

On the other hand, if $\varrho \in \mathbb{Q}[[x, y]]$ satisfies conditions (A') and (E), then ϱ satisfies (D), and hence $\varrho^\dagger \in \mathbb{Q}[[x, y]]$, and $f = \varrho^\dagger$ satisfies (A), (B) and (C).

A good part of Theorem 16 is proved directly in Lemma 17.

Lemma 17. *Let $f \in \mathbb{Q}[[x, y]]$.*

$$(i) \quad \varrho^\dagger = f \text{ for } \varrho = f^\sharp.$$

$$(ii) \quad \text{If } \varrho^\dagger = f \text{ for } \varrho \in \mathbb{Q}[[x, y]], \text{ then } \varrho = f^\sharp.$$

$$(iii) \quad \text{If } f \in \mathbb{Q}[[x, y]] \text{ satisfies (A) in Proposition 15, then } \varrho = f^\sharp \text{ satisfies (A')} \text{ in Theorem 16.}$$

On the other hand, if $f = \varrho^\dagger$ for a $\varrho \in \mathbb{Q}[[x, y]]$ that satisfies (A'), then f satisfies (A).

Proof. For (i), we observe that if $\varrho = f^\sharp$, then

$$\frac{e^y - e^x}{y - x} \cdot \varrho(y - x, x) - \frac{e^x - 1}{x} \cdot \varrho(x, y - x) = y f(x, y)$$

follows from substituting the formula for ϱ in terms of f coming from (29) into the left hand side and using the basic rules for the exponential power series.

For (ii), if $f = \varrho^\dagger \in \mathbb{Q}[[x, y]]$, then the argument is similar, only we substitute the formula for f in terms of ϱ (30) into the right hand side of (29).

To verify (iii), we may assume that $\varrho = f^\sharp$ by (i) and (ii), and then using (29), we deduce that

$$\begin{aligned} & (x+y)\varrho(x, y-x) - y\varrho(x, y) - x\varrho(y, x) \\ &= \frac{xy(x+y)}{(e^x-1)(e^y-1)} \cdot [f(x, y) + \exp(x) \cdot f(y-x, y)] \\ & \quad - \frac{xy(x+y)}{(e^x-1)(e^{x+y}-1)} \cdot [f(x, x+y) + \exp(x) \cdot f(y, x+y)] \\ & \quad - \frac{xy(x+y)}{(e^y-1)(e^{x+y}-1)} \cdot [f(y, x+y) + \exp(y) \cdot f(x, x+y)] \\ &= \frac{x}{e^x-1} \cdot \frac{y}{e^y-1} \cdot (x+y) \times \\ & \quad \times [f(x, y) + \exp(x) \cdot f(y-x, y) - f(x, x+y) - f(y, x+y)] \end{aligned}$$

where the elements of the field of fractions of $\mathbb{Q}[[x, y]]$ appearing in the above formulae belong actually to $\mathbb{Q}[[x, y]]$. $\square$

Given Lemma 17, we may assume that $\varrho \in \mathbb{Q}[[x, y]]$ satisfies (A').

Lemma 18. *If (A') holds for $\varrho \in \mathbb{Q}[[x, y]]$, then*

$$\rho(x, y-x) = \rho(y, x-y) \quad (31)$$

$$\rho(y, -x) = \rho(y-x, x) \quad (32)$$

$$\rho(x, y) = \rho(x+y, -y) \quad (33)$$

Proof. The right hand side of (A') is symmetric in x and y . Therefore if (A') holds for ϱ , then $(x+y)\rho(x, y-x)$ is symmetric in x and y , implying in turn that $\rho(x, y-x)$ is symmetric in x and y . That is, we have (31).

The equality (32) is obtained by making the linear substitution $x \mapsto y$, $y \mapsto y-x$ in (31), whereas (33) is obtained by making the substitution $x \mapsto x$, $y \mapsto x+y$ in (31). $\square$

Now we show that condition (D) for a $\varrho \in \mathbb{Q}[[x, y]]$ ensures that $\varrho^\dagger \in \mathbb{Q}[[x, y]]$.

Lemma 19. *For $\varrho \in \mathbb{Q}[[x, y]]$, $\varrho^\dagger$ is a formal power series if $\varrho \in \mathbb{Q}[[x, y]]$ satisfies $\varrho(-x, -y) = \varrho(x, y)$ (condition (D) preceding Theorem 16).*

Proof. Assume that the power series $\varrho(x, y) = \sum_{p, q \geq 0} a_{p, q} x^p y^q$ satisfies condition (D). Then $p+q$ is even for each term $a_{p, q} x^p y^q$ with $a_{p, q} \neq 0$. In particular, we have

$$\varrho(-x, x) - \varrho(x, -x) = 0.$$

It follows that $h(x, y) = \frac{e^y - e^x}{y - x} \cdot \varrho(y - x, x) - \frac{e^x - 1}{x} \cdot \varrho(x, y - x)$ satisfies

$$h(x, 0) = \frac{e^x - 1}{x} \cdot (\varrho(-x, x) - \varrho(x, -x)) = 0.$$

Therefore $h(x, y) = y \cdot g(x, y)$ for a formal power series $g(x, y) \in \mathbb{Q}[[x, y]]$, showing that $\varrho^\dagger(x, y) = \frac{1}{y} \cdot h(x, y) \in \mathbb{Q}[[x, y]]$. $\square$

Lemma 20. For $\varrho \in \mathbb{Q}[[x, y]]$, ϱ satisfies the conditions (A') and (E) in Theorem 16 if and only if it satisfies the conditions (A'), (B') and (C') where

$$(B') \quad (x - y)\varrho(x, y - x) = x\varrho(y, -x) - y\varrho(x, -y) \text{ and}$$

$$(C') \quad (x - y)\varrho(-x, x - y) = x\varrho(y, -x) - y\varrho(x, -y).$$

In addition, the conditions (B') and (C') for ϱ yield (D) in Lemma 19.

Proof. Let us quickly show that the conditions (B') and (C') for $\varrho \in \mathbb{Q}[[x, y]]$ yield (D). Since the right hand sides of (B') and (C') coincide, we have $\varrho(x, y - x) = \varrho(-x, x - y)$. Thus the invertible change of variable $(x, y) \mapsto (x, x + y)$ implies that $\varrho(x, y) = \varrho(-x, -y)$.

First, let $\varrho \in \mathbb{Q}[[x, y]]$ satisfy (A'), (B') and (C'), and hence also (D). Substituting y by $-y$ in (B') leads to

$$(x + y) \cdot \varrho(x, -y - x) = x \cdot \varrho(-y, -x) + y \cdot \varrho(x, y).$$

The right hand side above equals the right hand side of (A') by (D), hence $(x + y) \cdot \varrho(x, -y - x) = (x + y) \cdot \varrho(x, y - x)$ by (A'), which in turn implies that

$$\varrho(x, y - x) = \varrho(x, -y - x). \quad (34)$$

Substituting y by $x + y$ in (34), we obtain (E).

We assume now that (A') and (E) hold for ϱ . Using the substitution $y \mapsto y - x$ in (E) shows that (34) holds, as well. It follows from Lemma 18 that ϱ also satisfies (33), and so we have both

$$\varrho(x, y) = \varrho(x + y, -y) \text{ and } \varrho(x, y) = \varrho(x, -2x - y).$$

Composing the above two linear substitutions, we get

$$\varrho(x, y) = \varrho(x + y, -2(x + y) - (-y)) = \varrho(x + y, -2x - y).$$

In addition, composing the linear substitution $x \mapsto x + y$, $y \mapsto -2x - y$ with itself, we obtain that

$$\varrho(x, y) = \varrho((x + y) + (-2x - y), -2(x + y) - (-2x - y)) = \varrho(-x, -y),$$

and so (D) holds. Now make the substitution $y \mapsto -y$ in (A') to get

$$(x - y)\rho(x, -y - x) = x\rho(-y, x) - y\rho(x, -y).$$

By (D), we can replace $\rho(-y, x)$ by $\rho(y, -x)$ on the right hand side of the above equality, and we can replace $\rho(x, -y - x)$ by $\rho(x, y - x)$ on the left hand side of the above equality by (34). This way we obtain (B').

Finally, the change of variable $y \mapsto y - x$ in (D) yields that $\varrho(x, y - x) = \varrho(-x, x - y)$, and hence (B') and (D) imply (C'), completing the proof of Lemma 20. $\square$

Proof of Theorem 16. According to Lemma 17, Lemma 19 and Lemma 20, it is equivalent to prove that $f \in \mathbb{Q}[[x, y]]$ satisfies (A), (B) and (C) in Proposition 15 if and only if $\varrho = f^\sharp \in \mathbb{Q}[[x, y]]$ satisfies (A'), (B') and (C') in Lemma 20.

According to Lemma 17 and Lemma 19, we may assume that $f \in \mathbb{Q}[[x, y]]$ satisfies (A), $\varrho = f^\sharp$ satisfies (A') and $f = \varrho^\dagger$. We deduce from Lemma 18 that ϱ satisfies (31) and (32), as well.

Under these conditions, the condition (B) for f is equivalent to (B') for ϱ , as using (31) and (32), we have

$$\begin{aligned} f(x, y) - f(y, x) &= \varrho^\dagger(x, y) - \varrho^\dagger(y, x) = \\ &= \frac{1}{xy(y-x)} \cdot \left[(e^y - e^x)x\varrho(y-x, x) + (e^x - 1)(x-y)\varrho(x, y-x) \right. \\ &\quad \left. + (e^x - e^y)y\varrho(x-y, y) + (e^y - 1)(y-x)\varrho(y, x-y) \right] \\ &= \frac{e^x - e^y}{xy(y-x)} \cdot \left[(x-y)\varrho(x, y-x) - x\varrho(y-x, x) + y\varrho(x-y, y) \right] \\ &= \frac{e^x - e^y}{xy(y-x)} \cdot \left[(x-y)\varrho(x, y-x) - x\varrho(y, -x) + y\varrho(x, -y) \right], \end{aligned}$$

where the calculations show that the power series in brackets always have the required divisibility properties. Therefore, we may also assume that f satisfies (B) and ϱ satisfies (B'). Under these additional conditions, we verify that $\varrho^\dagger = \varrho^\diamond$ where

$$\varrho^\diamond(x, y) = \frac{1}{x-y} \cdot \left[\frac{e^x - 1}{x} \cdot \varrho(x, -y) - \frac{e^y - 1}{y} \cdot \varrho(y, -x) \right]$$

where the power series in the brackets is anti-symmetric, and hence divisible by $y - x$.

Now as ϱ satisfies (B') and (32), the formula $\varrho^\dagger = \varrho^\diamond$ follows from

$$\begin{aligned} \varrho^\dagger - \varrho^\diamond &= \frac{e^y - e^x}{y(y-x)} \cdot \varrho(y-x, x) - \frac{e^x - 1}{xy} \cdot \varrho(x, y-x) - \\ &\quad - \frac{1}{x-y} \cdot \left[\frac{e^x - 1}{x} \cdot \varrho(x, -y) - \frac{e^y - 1}{y} \cdot \varrho(y, -x) \right] \\ &= \frac{1}{xy(y-x)} \cdot \left[(e^y - e^x) \cdot x \cdot \varrho(y, -x) + (e^x - 1)(x-y) \cdot \varrho(x, y-x) \right] \end{aligned}$$

$$\begin{aligned}
& + (e^x - 1) \cdot y \cdot \varrho(x, -y) - (e^y - 1) \cdot x \cdot \varrho(y, -x) \Big] \\
& = \frac{e^x - 1}{xy(x - y)} \cdot \left[(x - y) \cdot \varrho(x, y - x) - x \cdot \varrho(y, -x) + y \cdot \varrho(x, -y) \right].
\end{aligned}$$

In particular, we have $f = \varrho^\diamond$.

Finally, the condition (C) for f is equivalent to (C') for ϱ , as using (32), we have

$$\begin{aligned}
& f(y - x, -x) - \exp(-x) \cdot f(x, y) = \varrho^\diamond(y - x, -x) - e^{-x} \varrho^\diamond(x, y) = \\
& = \frac{1}{y} \cdot \left[\frac{e^{y-x} - 1}{y - x} \cdot \varrho(y - x, x) - \frac{e^{-x} - 1}{-x} \cdot \varrho(-x, x - y) \right] \\
& \quad - \frac{e^{-x}}{x - y} \cdot \left[\frac{e^x - 1}{x} \cdot \varrho(x, -y) - \frac{e^y - 1}{y} \cdot \varrho(y, -x) \right] \\
& = \frac{e^{-x}}{xy(x - y)} \cdot \left[x(e^x - e^y) \cdot \varrho(y, -x) - (x - y)(e^x - 1) \cdot \varrho(-x, x - y) - \right. \\
& \quad \left. - y(e^x - 1) \cdot \varrho(x, -y) + x(e^y - 1) \cdot \varrho(y, -x) \right] \\
& = \frac{e^{-x}(e^x - 1)}{xy(x - y)} \cdot \left[x \cdot \varrho(y, -x) - (x - y) \cdot \varrho(-x, x - y) - y \cdot \varrho(x, -y) \right],
\end{aligned}$$

completing the proof of Theorem 16. $\square$

We write D_4 to denote the subgroup of $\mathrm{GL}(2, \mathbb{Z}) \subset \mathrm{GL}(2, \mathbb{C})$ generated by the matrices $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$. In particular, this group D_4 is isomorphic to the dihedral group of 8 elements. We recall (cf. (3)) that a $\Phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ acts on a $h \in \mathbb{Q}[[x, y]]$ in a way such that

$$(\Phi \cdot h)(x, y) = h(ax + cy, bx + dy). \quad (35)$$

Now the subalgebra of $\mathbb{Q}[[x, y]]$ of D_4 invariant elements is (see Remark 21)

$$\begin{aligned}
\mathbb{Q}[[x, y]]^{D_4} &= \{h \in \mathbb{Q}[[x, y]] : \Phi \cdot h = h \text{ for } \Phi \in D_4\} \\
&= \{g(2x^2 + 2xy + y^2, 4x^2y^2 + 4xy^3 + y^4) \in \mathbb{Q}[[x, y]] : \\
& \quad g(a, b) \in \mathbb{Q}[[a, b]]\}
\end{aligned} \quad (36)$$

where the polynomials $2x^2 + 2xy + y^2$ and $4x^2y^2 + 4xy^3 + y^4$ are algebraically independent.

Proof of Theorem 2. For a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$, it is sufficient to characterize $Z(\{o\}) = f_0$ and $Z([o, e_1]) = f_1$ and $Z(T)$ according to Proposition 6.

It follows from Proposition 14 that $Z(\{o\}) = f_0$ is a constant power series, and

$$f_1(x, y) = Z([o, e_1])(x, y) = g(x^2) \cdot \exp\left(\frac{1}{2}x\right)$$

holds for some $g \in \mathbb{Q}[[x]]$; moreover, there exists a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_2 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that

$$Z(T)(x, y) = Z_2(T)(x, y) + \frac{1}{2}f_1(x, y) + \frac{1}{2}f_1(y, -x) + \frac{1}{2}\exp(x) \cdot f_1(-x + y, -x).$$

On the other hand, Proposition 14 also says that if $c \in \mathbb{Q}$ and $f_1(x, y) = g(x^2) \cdot \exp\left(\frac{1}{2}x\right)$ for some $g \in \mathbb{Q}[[x]]$, then there exists a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ such that $Z(\{o\}) = c$ and $Z([o, e_1]) = f_1$.

Therefore, the remaining task is to characterize $f_2 = Z_2(T)$ for a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_2 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$.

It follows from Proposition 15 and Theorem 16 that $f_2 = Z_2(T)$ for a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z_2 : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ if and only if $f = \varrho^\dagger$ for a $\varrho \in \mathbb{Q}[[x, y]]$ satisfying conditions (A') and (E) in Theorem 16.

In turn, Theorem 2 follows from the statement that $\varrho \in \mathbb{Q}[[x, y]]$ satisfies the conditions (A') and (E) in Theorem 16 if and only if $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ and (A') holds for ϱ .

Therefore, let $\varrho \in \mathbb{Q}[[x, y]]$ satisfy the condition (A'), and hence (33) holds for ϱ by Lemma 18. Since (33) and (E) states that ϱ is invariant under $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$, and (A') and (8) are readily equivalent, we conclude Theorem 2. $\square$

Remark 21. We obtain a more traditional representation of D_4 after the change of variable $s := 2x + y$, $t := y$. In addition, the substitution $s := 2x + y$, $t := y$ transforms the matrix group D_4 to the group (denoted also by D_4) generated by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, and then (see for example Hunziker [11, Section 4])

$$\mathbb{Q}[[s, t]]^{D_4} = \{g(s^2 + t^2, s^2t^2) : g(a, b) \in \mathbb{Q}[[a, b]]\}$$

where the polynomials $s^2 + t^2$ and s^2t^2 are algebraically independent. In terms of the original variables x, y , we deduce (36).

To understand the space of formal power series $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying condition (A') occurring in Theorem 2, for $\varrho \in \mathbb{Q}[[x, y]]$, we consider the $\sigma \in \mathbb{Q}[[s, t]]$ defined by $\sigma(s, t) = \varrho(x, y)$. Then $\rho(x, y) = \sigma(2x + y, y)$, hence $\rho(x, y - x) = \sigma(2x + y - x, y - x) = \sigma(x + y, y - x)$, and $\rho(y, x) = \sigma(x + 2y, x)$. Therefore, equation (A') for $\varrho \in \mathbb{Q}[[x, y]]$ translates to the functional equation

$$(s+t)\sigma(s+t, t-s) = s\sigma(s+2t, s) + t\sigma(2s+t, t) \quad (37)$$

for $\sigma \in \mathbb{Q}[[s, t]]$. In particular, for any $d > 0$, the $\mathbb{Q}$ -vector space $\mathcal{V}_d$ of d -homogeneous polynomials $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfying (A') is isomorphic to the $\mathbb{Q}$ -vector space of d -homogeneous polynomials $\sigma \in \mathbb{Q}[[s, t]]^{D_4}$ satisfying (37).

4. Dilative valuations

In this section we prove Theorem 4. We start with the first case, where $\delta \geq -2$ is an even and non-zero integer, which is the most involved case. Here, Theorem 3 says that any δ -dilative valuation is simple.

We recall (see (10) and the following remarks) that a simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is δ -dilative for $\delta \in \mathbb{Z}$ if

$$Z(mT)(x, y) = m^{-\delta} \cdot Z(T)(mx, my) \quad (38)$$

for any integer $m \geq 2$. In order to describe the left hand side of (38), if $m \in \mathbb{N}$, then we consider $g_m \in \mathbb{Q}[[x, y]]$ defined by

$$g_m(x, y) = \sum_{t, s \in \mathbb{N}, t+s \leq m} \exp(sx + ty) = \sum_{(s, t) \in (mT) \cap \mathbb{Z}^2} \exp(sx + ty). \quad (39)$$

In Lemma 22 (ii), we use our usual short hand notation for power series involving the exponential series.

Lemma 22. *Let $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ be a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation, and let $f = Z(T)$.*

(i) *If $m \geq 2$ for $m \in \mathbb{N}$, then*

$$Z(mT)(x, y) = g_{m-1}(x, y) \cdot f(x, y) + \exp(x + y) \cdot g_{m-2}(x, y) \cdot f(-x, -y).$$

(ii) *If $m \in \mathbb{N}$, then*

$$g_m(x, y) = \frac{e^{x+y} (e^{(m+1)x} - e^{(m+1)y}) - (e^{(m+2)x} - e^{(m+2)y}) + e^x - e^y}{(e^x - e^y)(e^x - 1)(e^y - 1)}.$$

Proof. The formula in (i) follows from the tiling of mT by translates of T and $-T$.

For (ii), we recall that $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and observe that the points of $\mathbb{Z}^2$ in $[(m+1)T] \setminus (e_i + mT)$ are of the form ℓe_j where $\{i, j\} = \{1, 2\}$ and $\ell = 0, \dots, m+1$. It follows that

$$\exp(x) \cdot g_m(x, y) + \sum_{\ell=0}^{m+1} \exp(\ell y) = g_{m+1}(x, y) = \exp(y) \cdot g_m(x, y) + \sum_{\ell=0}^{m+1} \exp(\ell x),$$

which in turn yields (ii). $\square$

Proposition 23. Let $\varrho \in \mathbb{Q}[[x, y]]^{D_4}$ satisfy the condition (A') in Theorem 16, and let $Z(T) = f = \varrho^\dagger$ for the simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ provided by Proposition 15 and Theorem 16. If ϱ is a homogeneous polynomial of degree d for $d \in 2\mathbb{N}$, then Z is $d - 2$ dilative.

Proof. According to (9) and Theorem 16, $f = Z(T)$ satisfies that

$$f(x, y) = \frac{e^x - e^y}{(x - y)y} \cdot \varrho(y - x, x) - \frac{e^x - 1}{xy} \cdot \varrho(x, y - x) \quad (40)$$

Since $\varrho(-x, -y) = \varrho(x, y)$, we also have

$$\begin{aligned} f(-x, -y) &= \frac{e^{-x} - e^{-y}}{(x - y)y} \cdot \varrho(y - x, x) - \frac{e^{-x} - 1}{xy} \cdot \varrho(x, y - x) \\ &= -\frac{e^x - e^y}{e^{x+y}(x - y)y} \cdot \varrho(y - x, x) + \frac{e^x - 1}{e^x xy} \cdot \varrho(x, y - x). \end{aligned} \quad (41)$$

In order to show that Z is $(d - 2)$ -dilative, we calculate the left hand side of (38) using Lemma 22, (40) and (41), and obtain that

$$Z(mT)(x, y) = G(x, y, m) \cdot \varrho(y - x, x) - H(x, y, m) \cdot \varrho(x, y - x)$$

where

$$\begin{aligned} G(x, y, m) &= \frac{e^{x+y}(e^{mx} - e^{my}) - (e^{(m+1)x} - e^{(m+1)y}) + e^x - e^y}{(e^x - e^y)(e^x - 1)(e^y - 1)} \cdot \frac{e^x - e^y}{(x - y)y} \\ &\quad - e^{x+y} \cdot \frac{e^{x+y}(e^{(m-1)x} - e^{(m-1)y}) - (e^{mx} - e^{my}) + e^x - e^y}{(e^x - e^y)(e^x - 1)(e^y - 1)} \cdot \frac{e^x - e^y}{e^{x+y}(x - y)y} \\ &= \frac{e^{mx} - e^{my}}{(x - y)y} \end{aligned}$$

and

$$\begin{aligned} H(x, y, m) &= \frac{e^{x+y}(e^{mx} - e^{my}) - (e^{(m+1)x} - e^{(m+1)y}) + e^x - e^y}{(e^x - e^y)(e^x - 1)(e^y - 1)} \cdot \frac{e^x - 1}{xy} \\ &\quad - e^{x+y} \cdot \frac{e^{x+y}(e^{(m-1)x} - e^{(m-1)y}) - (e^{mx} - e^{my}) + e^x - e^y}{(e^x - e^y)(e^x - 1)(e^y - 1)} \cdot \frac{e^x - 1}{e^x xy} \\ &= \frac{e^{mx} - 1}{xy}. \end{aligned}$$

Since ϱ is homogeneous of degree d , we deduce that

$$Z(mT)(x, y) = \frac{e^{mx} - e^{my}}{(x - y)y} \cdot \varrho(y - x, x) - \frac{e^{mx} - 1}{xy} \cdot \varrho(x, y - x) = m^{2-d} f(mx, my),$$

proving that Z is $(d - 2)$ -dilative. $\square$

In order to verify the converse of Proposition 23, we summarize basic properties of δ -dilative valuations based on Freyer, Ludwig, Rubey [10], and introduce various related notions. For $r \in \mathbb{N}$, we write $\mathbb{Q}[x, y]_r$ to denote the $(r + 1)$ -dimensional $\mathbb{Q}$ vector space of homogeneous polynomials of degree r , and for $f \in \mathbb{Q}[[x, y]]$, we write $\pi_r f \in \mathbb{Q}[x, y]_r$ to denote the r th degree term in f ; namely, f is the formal sum $\sum_{r \in \mathbb{N}} \pi_r f$.

For $r \in \mathbb{N}$, we say that a valuation $\Psi : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_r$ is *translatively polynomial* if there exist functions $\Psi_{(i)} : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_i$ for $i = 0, \dots, r$ such that if $v = (\alpha, \beta) \in \mathbb{Z}^2$ and $P \in \mathcal{P}(\mathbb{Z}^2)$, then

$$\Psi(P + v)(x, y) = \sum_{j=0}^r \Psi_{(r-j)}(P)(x, y) \cdot (\alpha x + \beta y)^j; \quad (42)$$

or in the more compact tensor notation, $\Psi(P + v) = \sum_{j=0}^r \Psi_{(r-j)}(P) \otimes v^j$. In this case, each $\Psi_{(i)}$ in (42) is a translatively polynomial valuation. The crucial fact proved originally by McMullen [21] if $r \leq 1$ and by Khovanskii, Pukhlikov [14] if $r \in \mathbb{N}$ is that there exists a j -homogeneous translatively polynomial valuation $\Psi_j : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_r$ for $j = 0, \dots, r + 2$ such that

$$\Psi = \sum_{j=0}^{r+2} \Psi_j. \quad (43)$$

It was observed by Freyer, Ludwig, Rubey in [10, Section 6] that a valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is $\mathcal{G}(\mathbb{Z}^2)$ equivariant, if and only if each summand $\pi_r Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_r$ is translatively polynomial and $\mathrm{GL}(2, \mathbb{Z})$ equivariant, i.e.,

$$\pi_r Z \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P \right) (x, y) = \pi_r Z(P)(ax + cy, bx + dy),$$

holds for all $a, b, c, d \in \mathbb{Z}$ with $|ad - bc| = 1$.

For $r \in \mathbb{N}$, Freyer, Ludwig, Rubey [10] consider the $\mathbb{Q}$ vector space Val^r of translatively polynomial and $\mathrm{GL}(2, \mathbb{Z})$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_r$, and hence

$$\mathrm{Val}^r = \oplus_{j=0}^{r+2} \mathrm{Val}_j^r \quad (44)$$

where Val_j^r is the $\mathbb{Q}$ vector space Val^r of j -homogeneous translatively polynomial and $\mathrm{GL}(2, \mathbb{Z})$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]_r$ for $j = 0, \dots, r$ (cf. (43)). In [10, Lemma 25] it is shown that a $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ is δ -dilative for an integer $\delta \geq -2$ if and only if for each $r \in \mathbb{N}$, we have

$$\pi_r Z \in \text{Val}_{r-\delta}^r. \quad (45)$$

Proof of Theorem 4, Case 1. Recall that $\mathcal{V} \subset \mathbb{Q}[[x, y]]$ denotes the vector space of power series in x and y that satisfy (8) and that $\mathcal{V}_d$ is its degree d part, i.e., $\mathcal{V}_d = \pi_d \mathcal{V}$.

According to Theorem 2, there exists a vector space isomorphism Φ from $\mathcal{V}$ to the $\mathbb{Q}$ vector space of all simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ where

$$Z(T)(x, y) = \frac{e^x - e^y}{(x - y)y} \cdot \varrho(y - x, x) - \frac{e^x - 1}{xy} \cdot \varrho(x, y - x) \quad (46)$$

for $\varrho \in \mathcal{V}$ and $Z = \Phi(\varrho)$. We deduce from Proposition 23 that if $\varrho \in \mathcal{V}_d$ for some $d \in 2\mathbb{N}$, then $\Phi(\varrho)$ is $(d - 2)$ -dilative, and hence (44) and (45) yield that then

$$\begin{aligned} \pi_r \Phi(\varrho) &= 0 & \text{if } r < d - 2 \\ \pi_r \Phi(\varrho) &\in \text{Val}_{r-d+2}^r & \text{if } r \geq d - 2. \end{aligned} \quad (47)$$

Now let $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$ be a non-trivial δ -dilative $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation, where $\delta \geq -2$ is a non-zero integer. Then Z is simple (cf. Theorem 3). In particular, $Z = \Phi(\varrho)$ for a $\varrho \in \mathcal{V}$, and let $\varrho_d = \pi_d \varrho$ for $d \in \mathbb{N}$. We suppose that there exists a $\bar{d} \in 2\mathbb{N}$ with $\bar{d} \neq \delta + 2$ such that $\varrho_{\bar{d}} \neq 0$, and seek a contradiction. As Φ is an isomorphism and $\Phi(\varrho_{\bar{d}})$ is $(\bar{d} - 2)$ -dilative, (45) yields the existence of a $\bar{r} \geq \bar{d} - 2$ such that

$$\pi_{\bar{r}} \Phi(\varrho_{\bar{d}}) \in \text{Val}_{\bar{r}-\bar{d}+2}^{\bar{r}} \setminus \{0\}. \quad (48)$$

According to (44), there exists a natural linear projection map $\Pi : \text{Val}^{\bar{r}} \rightarrow \text{Val}_{\bar{r}-\bar{d}+2}^{\bar{r}}$, which then satisfies by (47), (48) and $\bar{d} \neq \delta + 2$ that

$$\Pi \circ \pi_{\bar{r}} \circ \Phi(\varrho_{\bar{d}}) \neq 0 \quad \text{and} \quad \Pi \circ \pi_{\bar{r}} \circ \Phi(\varrho) = 0. \quad (49)$$

Let $\theta = \varrho - \sum_{d=0}^{\bar{r}+2} \varrho_d$, and hence θ may contain only non-trivial terms of degree at least $\bar{r} + 3$. It follows from (46) that $\Phi(\theta)$ may contain only non-trivial terms of degree at least $\bar{r} + 1$; therefore, $\pi_{\bar{r}} \Phi(\theta) = 0$. We deduce from $\bar{r} + 2 \geq \bar{d}$, (47) and (49) that

$$0 = \Pi \circ \pi_{\bar{r}} \circ \Phi(\varrho) = \Pi \circ \pi_{\bar{r}} \circ \Phi \left(\theta + \sum_{d=0}^{\bar{r}+2} \varrho_d \right) = \Pi \circ \pi_{\bar{r}} \circ \Phi(\varrho_{\bar{d}}) \neq 0,$$

which is a contradiction and proves that ϱ is a homogeneous polynomial of degree $\delta + 2$.

In summary, we have proved so far that for any $d \in 2\mathbb{N}$, the restriction of Φ to $\mathcal{V}_d$ is an isomorphism to the $\mathbb{Q}$ vector space $\overline{\text{Val}}_{(d-2)}$ of $(d - 2)$ -dilative simple $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[[x, y]]$. This completes the proof of the first case in Theorem 4. $\square$

The proofs of the other two cases in Theorem 4 are shorter, since we can make use of the fact that the respective spaces of δ -dilative valuations $\overline{\text{Val}}_{\delta}$ are 1-dimensional.

Proof of Theorem 4, Case 2. Let $\delta > -2$ be an odd integer. We start by showing that the valuation Z obtained from Theorem 2 by choosing $c = 0$, $g(x) = x^{\delta/2} \sinh(\frac{\sqrt{x}}{2})$, $f_2 = 0$ is δ -dilative. Note that Z vanishes on points, since $c = 0$. We show that Z is δ -dilative for the standard segment, i.e.,

$$Z([0, me_1])(x, y) = m^{-\delta} Z([0, e_1])(mx, my) \quad (50)$$

holds for all $m \in \mathbb{N}$. We write $[0, me_1] = \bigcup_{0 \leq k < m} ke_i + [0, e_1]$, where the union is disjoint up to points. Thus, by Theorem 2, we have

$$\begin{aligned} Z([0, me_1]) &= \left(\sum_{k=0}^{m-1} \exp(kx) \right) Z([0, e_1]) = \frac{1 - \exp(mx)}{1 - \exp(x)} x^{\delta} \sinh\left(\frac{x}{2}\right) \exp\left(\frac{x}{2}\right) \\ &= x^{\delta} \frac{\exp(mx) - 1}{2}. \end{aligned}$$

Regarding the right hand side of (50), we have

$$m^{-\delta} Z([0, e_1])(mx, my) = m^{-\delta} (mx)^{\delta} \sinh\left(\frac{mx}{2}\right) \exp\left(\frac{mx}{2}\right) = x^{\delta} \frac{\exp(mx) - 1}{2},$$

so (50) is verified. It follows from the affine equivariance that Z is δ -dilative for all segments.

In order to see that Z is δ -dilative for 2-dimensional polygons, we observe that $Z(P) = \frac{1}{2} \sum_{e \in P} Z(e)$ holds for any proper polygon, where the sum ranges over the edges of P : For a unimodular triangle, this follows from (6) together with the $\mathcal{G}(\mathbb{Z}^2)$ equivariance. For an arbitrary polygon, we see that the interior edges of a unimodular triangulation $\mathcal{T}$ of P do not contribute to the sum

$$Z(P) = \sum_S \sum_{e \in S} \frac{1}{2} Z(e) - \sum_{e'} Z(e'), \quad (51)$$

where S ranges over the unimodular triangles in $\mathcal{T}$, e over the edges of S for a given S , and e' over the edges in $\mathcal{T}$. Hence, the only contributions to (51) come from the boundary edges of $\mathcal{T}$ which sum up to the edges of P . This, together with the δ -dilative property for segments, proves that Z is indeed a δ -dilative valuation.

The reverse implication now follows from the fact that $\overline{\text{Val}}_{\delta}$ is a 1-dimensional space of non-simple valuations for odd $\delta > -2$. $\square$

Proof of Theorem 4, Case 3. As in the previous proof, it suffices to show that the valuation Z from Theorem 2 with $c = 1$, $g(x) = \cosh\left(\frac{\sqrt{x}}{2}\right)$ and $f_2 = 0$ is 0-dilative. For a point $p = (p_1, p_2) \in \mathbb{Z}^2$ we have

$$Z(p) = Z(\{0\} + p) = \exp(p_1 x + p_2 y) Z(\{0\}) = \exp(p_1 x + p_2 y)$$

and for $m \in \mathbb{N}$ it follows that

$$\begin{aligned} Z(mp)(x, y) &= Z(p + (m-1)p)(x, y) \\ &= \exp((m-1)(p_1x + p_2y)) \exp(p_1x + p_2y) \\ &= \exp(p_1mx + p_2my) = Z(p)(mx, my). \end{aligned}$$

Hence, Z is 0-dilative for points. Next, we show that Z is 0-dilative for segments, i.e., we aim to verify (50) for $\delta = 0$. This time, we have to take into account that Z does not vanish on points, so we compute

$$\begin{aligned} Z([0, me_1])(x, y) &= \left(\sum_{k=0}^{m-1} \exp(kx) \right) Z([0, e_1]) - \sum_{\ell=1}^{m-1} \exp(\ell x) Z(\{0\}) \\ &= \frac{1 - \exp(mx)}{1 - \exp(x)} \cosh\left(\frac{x}{2}\right) \exp\left(\frac{x}{2}\right) - \frac{\exp(x) - \exp(mx)}{1 - \exp(x)} \\ &= \frac{1}{2} \frac{(1 - \exp(mx))(\exp(x) + 1) - 2(\exp(x) - \exp(mx))}{1 - \exp(x)} \\ &= \frac{1}{2} \frac{1 - \exp(x) + \exp(mx) - \exp(mx + 1)}{1 - \exp(x)} \\ &= \frac{1}{2} \frac{(1 - \exp(x))(1 + \exp(mx))}{1 - \exp(x)} \\ &= \frac{\exp(\frac{mx}{2}) + \exp(-\frac{mx}{2})}{2} \exp(\frac{mx}{2}) \\ &= \cosh\left(\frac{(\sqrt{mx})^2}{2}\right) \exp(\frac{mx}{2}) = Z([0, e_1])(mx, my). \end{aligned}$$

This shows that Z is also 0-dilative on segments. In order to see that Z is 0-dilative on full-dimensional polygons, we can proceed as in the previous proof, exploiting the fact that interior lattice points of a polygon are contained in as many triangles as edges for any unimodular triangulation. Hence, the formula for $Z(P)$ is the same as in (51) and we see that Z is indeed 0-dilative.

As in Case 2, the reverse implication follows from $\dim \overline{\text{Val}}_0 = 1$ (cf. Theorem 3). $\square$

As a by-product of the proofs we obtain that if δ is odd or zero, any δ -dilative valuation acts as an equivariant surface area measure.

Corollary 24. *Let $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{Q}[x, y]$ be a δ -dilative $\mathcal{G}(\mathbb{Z}^2)$ equivariant valuation, where $\delta > -2$ is odd or zero. Then, $Z(P) = \frac{1}{2} \sum_{e \subset P} Z(e)$, where e ranges over the edges of P .*

The conceptual difference of Case 1 to Cases 2 and 3 is that in the latter cases we knew a priori that the space of δ -dilative valuations $\overline{\text{Val}}_\delta$ is of the same dimension than the valuations described by the parameters in the respective case, whereas in the first

case it was *not* a priori clear that the dimensions of $\overline{\text{Val}}_\delta$ and $\mathcal{V}_{\delta+2}$ agree. A posteriori we see that they indeed agree and together with Theorem 3 we obtain:

Corollary 25. *Let $d \in \mathbb{N}$ and let $\mathcal{V}_d$ be the vector space of d -homogeneous polynomials that satisfy (8). Then,*

$$\dim \mathcal{V}_d = \begin{cases} 0, & d \text{ odd}, \\ \lfloor \frac{d}{12} \rfloor + 1, & d \text{ even and } \delta \bmod 12 \neq 2, \\ \lfloor \frac{d}{12} \rfloor, & d \text{ even and } \delta \bmod 12 = 2. \end{cases}$$

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