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ON THE CONES OF CLASSICAL GROUPS*

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Dedicated to Vesselin Drensky on his 75th birthday

ABSTRACT. The cone of a classical group G is an affine $G \times G$ -variety. The aim of this note is to initiate its combinatorial study in the cases when G is the complex orthogonal or symplectic group (the case of the general linear group being well documented in the literature). The coordinate ring of the cone of G is a finitely generated commutative graded algebra. First the $G \times G$ -module structure of its homogeneous components is determined. This is used to compute the Hilbert series of this coordinate ring in the cases when G is the orthogonal group $O(3)$, $O(4)$, the special orthogonal group $SO(4)$, and when G is the symplectic group $Sp(4)$. It is concluded that the coordinate ring of the cone of $O(3)$ is not Koszul, hence the vanishing ideal of this cone has no quadratic Gröbner basis (although it is minimally generated by quadratic elements).

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1. Introduction. Quantum groups were originally defined as Hopf algebra deformations of universal enveloping algebras of complex semisimple Lie algebras. A dual approach was taken by Faddeev, Reshetikhin and Tahtajan [17], who defined quantized coordinate rings of classical groups by taking a noncommutative deformation of the coordinate ring of G . This is a two-step process: first they define a matrix bialgebra (called the FRT-bialgebra), and then specialize to 1 a central group-like element in it. Let G be any of the classical groups $\mathrm{SL}(n)$, $\mathrm{O}(n)$, or $\mathrm{Sp}(n)$ (for n even in the latter case). The FRT-bialgebra $\mathcal{A}_q(G)$ is an associative graded $\mathbb{C}$ -algebra generated by n^2 elements of degree 1 (thought of as the coordinate functions on the space of $n \times n$ “quantum matrices”) subject to a set of homogeneous quadratic relations depending on a non-zero parameter $q \in \mathbb{C}$. The algebra $\mathcal{A}_q(\mathrm{SL}(n))$ is the quantized coordinate ring of $n \times n$ matrices, and is thoroughly studied from combinatorial and ring theoretic aspects, see for example [15], [10], [2], [9]. It is a noncommutative deformation of the coordinate ring of the space of $n \times n$ matrices, a polynomial algebra in n^2 indeterminates, whose $\mathrm{GL}(n) \times \mathrm{GL}(n)$ -module structure is given by the Cauchy formula (see for example [16, Section 11.5.1]).

For $G = \mathrm{O}(n)$ or $G = \mathrm{Sp}(n)$, the algebra $\mathcal{A}_q(G)$ is a deformation of the coordinate ring $\mathcal{O}(\mathcal{C}(G))$ of the cone $\mathcal{C}(G)$ of G (see e.g. [6]). Therefore the Hilbert series of the graded algebra $\mathcal{A}_q(G)$ coincides with the Hilbert series of $\mathcal{O}(\mathcal{C}(G))$. Moreover, the problem of finding a normal form for the elements of $\mathcal{A}_q(G)$ (e.g. giving a monomial basis of $\mathcal{A}_q(G)$ together with the corresponding rewriting rules) is closely related to the similar task for $\mathcal{O}(\mathcal{C}(G))$. In this paper we initiate the combinatorial study of $\mathcal{O}(\mathcal{C}(G))$.

We shall work over the base field $\mathbb{C}$ of complex numbers, and throughout the paper we assume that $n \geq 3$ is a positive integer. Recall that the orthogonal group is

$$\mathrm{O}(n) = \{M \in \mathbb{C}^{n \times n} \mid M^T M = I\},$$

the set of $n \times n$ matrices over $\mathbb{C}$ whose transpose is their inverse. For $n = 2m \geq 4$ even, the symplectic group is

$$\mathrm{Sp}(n) = \{M \in \mathbb{C}^{n \times n} \mid M^T J M = J\},$$

where J is the $n \times n$ matrix obtained by glueing m copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the main diagonal. Now let G be any of $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$. Then G is a subset of $\mathbb{C}^{n \times n}$ defined by polynomial equations.

By the cone $\mathcal{C}(G)$ of G we mean the Zariski closure in $\mathbb{C}^{n \times n}$ of

$$\{cM \mid c \in \mathbb{C}, \quad M \in G\} \subset \mathbb{C}^{n \times n}.$$

Note that the subset of $\mathbb{C}^{n \times n}$ in the above line is not Zariski closed in general, see Example 6.3. We have the embeddings

$$G \hookrightarrow \mathcal{C}(G) \hookrightarrow \mathbb{C}^{n \times n}$$

of affine algebraic varieties. Restriction of functions induces the $\mathbb{C}$ -algebra surjections

$$(1) \quad \mathcal{O}(\mathbb{C}^{n \times n}) \xrightarrow{\varphi_1} \mathcal{O}(\mathcal{C}(G)) \xrightarrow{\varphi_2} \mathcal{O}(G)$$

between the corresponding coordinate rings. Here $\mathcal{O}(\mathbb{C}^{n \times n}) = \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq n]$ is an n^2 -variable polynomial algebra, where x_{ij} is the function assigning to $M \in \mathbb{C}^{n \times n}$ its (i, j) -entry. It is endowed with the standard grading. Since $\mathcal{C}(G)$ is a cone, the surjection φ_1 transports the grading to $\mathcal{O}(\mathcal{C}(G))$. So denoting by $\mathcal{O}(\mathcal{C}(G))_d$ the degree d homogeneous component of $\mathcal{O}(\mathcal{C}(G))$ we have $\mathcal{O}(\mathcal{C}(G)) = \bigoplus_{d=0}^{\infty} \mathcal{O}(\mathcal{C}(G))_d$. The group $G \times G$ acts on $\mathbb{C}^{n \times n}$ by

$$(g, h) \cdot M = gMh^{-1} \quad \text{for } g, h \in G \text{ and } M \in \mathbb{C}^{n \times n},$$

with matrix multiplication on the right hand side. The subvarieties $\mathcal{C}(G)$ and G are both $G \times G$ -stable, and we have the induced $G \times G$ -action on the corresponding coordinate rings. The $\mathbb{C}$ -algebra homomorphisms φ_1 and φ_2 are $G \times G$ -equivariant.

1.1. The content of the paper. In Section 2 we determine the structure of the representation $G \times G$ on $\mathcal{O}(\mathcal{C}(G))$ (see Theorem 2.3). Using some known formulae for the dimensions of the irreducible representations of G , this allows us to deduce in Section 3 the Hilbert series of the graded algebra $\mathcal{O}(\mathcal{C}(G))$ when G is $O(3)$, $O(4)$, or $Sp(4)$, and the same is done for $SO(4)$ in Section 4. Moreover, for the same groups G we compute in Section 5 also the Hilbert series of $\mathcal{O}(\mathcal{C}(G))^{U \times U}$, where U is the unipotent radical of a Borel subgroup in G . The vanishing ideal

$$\mathcal{I}(\mathcal{C}(G)) = \{f \in \mathcal{O}(\mathbb{C}^{n \times n}) \mid f(cg) = 0 \quad \forall g \in G, \quad \forall c \in \mathbb{C}\}$$

of $\mathcal{C}(G)$ for $G = O(n)$ or $G = Sp(n)$ (i.e. the kernel of φ_1 from (1)) was described by Weyl [20], who proved that these ideals are generated by explicitly given quadratic elements. In Section 6 we present in modern language the proof of this

result in the case of $O(n)$. As a consequence of the calculations in Section 3 we observe that the algebra $\mathcal{O}(\mathcal{C}(O(3)))$ is not Koszul, hence the ideal $\mathcal{I}(\mathcal{C}(O(3)))$ has no quadratic Gröbner basis. In Section 7 we present a Gröbner basis of $\mathcal{I}(\mathcal{C}(O(3)))$; in addition to the 10 quadratic generators it has 5 cubic elements.

Several calculations in the paper were performed using the online CoCalc platform [18].

2. The $G \times G$ -module structure of $\mathcal{O}(\mathcal{C}(G))_d$. Throughout this section G stands for $O(n)$, or for $Sp(n)$ (where $n = 2m$ is even in the latter case). The finite dimensional irreducible representations of G (as a linear algebraic group) are naturally indexed by a set $\Lambda(G)$ of partitions. By a *partition* we mean a non-increasing finite sequence of integers. Denote by $\text{Par}(n) := \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{N}_0, \lambda_1 \geq \dots \geq \lambda_n\}$ the set of partitions with at most n non-zero parts. Recall from [16] or [20] that we have

$$\Lambda(O(n)) = \{\lambda \in \text{Par}(n) \mid |\{i \mid \lambda_i > 0\}| + |\{i \mid \lambda_i > 1\}| \leq n\},$$

whereas for $n = 2m$ even we have

$$\Lambda(Sp(n)) = \text{Par}(m).$$

Set $\Lambda(G)_d := \{\lambda \in \Lambda(G) \mid \lambda_1 + \dots + \lambda_n = d\}$. For $\lambda \in \Lambda(G)$ denote by V_λ the underlying vector space of the irreducible representation of G labeled by λ , endowed with the corresponding linear action of G . Note that $V_{(1,0,\dots,0)} \cong \mathbb{C}^n$, endowed with the defining representation of G (given by multiplication of column vectors by matrices). We shall use the following well-known fact (see for example [16, Section 11.3.1]):

Proposition 2.1. *For $\lambda \in \Lambda(G)_d$ the G -module V_λ occurs as a summand in the d th tensor power $T^d(\mathbb{C}^n)$ of $\mathbb{C}^n$, and V_λ is not a summand of $T^k(\mathbb{C}^n)$ for $k > d$ with $k - d$ odd, or for $k < d$.*

The representation of $G \times G$ on $\mathcal{O}(G)$ is described by the Peter-Weyl Theorem:

Proposition 2.2. *We have the following isomorphism of $G \times G$ -modules:*

$$\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda(G)} V_\lambda \otimes V_\lambda.$$

Proof. The Peter-Weyl Theorem says that $\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda(G)} V_\lambda \otimes V_\lambda^*$,

where V_λ^* stands for the dual of the representation V_λ (see for example [16, Theorems 8.3.2 and 8.7.2.3]). Taking into account the fact that all representations of G are self-dual, i.e. $V_\lambda \cong V_\lambda^*$ holds for all $\lambda \in \Lambda(G)$, we get the desired statement. $\square$

We mention that a basis of $\mathcal{O}(\mathrm{O}(n))$ consisting of so-called bideterminants is given in [3], extending for the orthogonal group the analogous theory for $\mathcal{O}(\mathbb{C}^{n \times n})$ (cf. [16, Section 13.4], [7], [5]). The case of $\mathcal{O}(\mathrm{Sp}(n))$ is treated in [4].

Theorem 2.3. *For $d = 0, 1, 2, \dots$ we have the following isomorphisms of $G \times G$ -modules:*

$$\mathcal{O}(\mathcal{C}(G))_d \cong \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \bigoplus_{\lambda \in \Lambda(G)_{d-2k}} V_\lambda \otimes V_\lambda,$$

where $\lfloor d/2 \rfloor$ is the lower integer part of $d/2$. In particular, for $d \geq 2$ we have

$$\mathcal{O}(\mathcal{C}(G))_d \cong \mathcal{O}(\mathcal{C}(G))_{d-2} \oplus \bigoplus_{\lambda \in \Lambda(G)_d} V_\lambda \otimes V_\lambda.$$

Proof. Note that $\varphi_2(\varphi_1(x_{ij}))$ are the matrix elements of the defining representation of G on $\mathbb{C}^n$ (the maps φ_1 and φ_2 are defined in (1)). Therefore $\varphi_2(\mathcal{O}(\mathcal{C}(G))_d)$ is the $\mathbb{C}$ -linear span of the length d products of the matrix elements of the defining representation of G , hence $\varphi_2(\mathcal{O}(\mathcal{C}(G))_d)$ is the space of matrix elements of the d -fold tensor power $T^d(\mathbb{C}^n)$ of the defining representation of G on $\mathbb{C}^n$. It follows by Proposition 2.2 that $\varphi_2(\mathcal{O}(\mathcal{C}(G))_d)$ is isomorphic to $\bigoplus_{\lambda \in \Lambda(G)_d} V_\lambda \otimes V_\lambda$, where the direct sum ranges over the $\lambda \in \Lambda(G)$ such that V_λ occurs as a direct summand in $T^d(\mathbb{C}^n)$. Hence by Proposition 2.1 we have $\varphi_2(\mathcal{O}(\mathcal{C}(G))_d) \cong \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \bigoplus_{\lambda \in \Lambda(G)_{d-2k}} V_\lambda \otimes V_\lambda$. The restriction to $\mathcal{O}(\mathcal{C}(G))_d$ of φ_2 is injective, since if

a homogeneous polynomial in $\mathcal{O}(\mathbb{C}^{n \times n})$ vanishes on G , then it vanishes on $\mathcal{C}(G)$ as well. Therefore we have $\mathcal{O}(\mathcal{C}(G))_d \cong \varphi_2(\mathcal{O}(\mathcal{C}(G))_d)$, finishing the proof of the first isomorphism. The second isomorphism follows from the first by induction on d . $\square$

3. Some Hilbert series. Recall that the *Hilbert series* of a graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$ (with finite dimensional homogeneous components A_d) is the

formal power series

$$H(A; t) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(A_d) t^d \in \mathbb{Z}[[t]].$$

For $G = O(n)$ or $G = Sp(n)$ introduce the following function $h_G : \mathbb{N}_0 \rightarrow \mathbb{N}_0$:

$$(2) \quad h_G(d) := \sum_{\lambda \in \Lambda(G)_d} \dim_{\mathbb{C}}(V_{\lambda})^2.$$

As a consequence of Theorem 2.3 we get the following:

Corollary 3.1. *For $G = O(n)$ or $G = Sp(n)$ we have*

$$H(\mathcal{O}(\mathcal{C}(G)); t) := \sum_{d=0}^{\infty} \sum_{k=0}^{\lfloor d/2 \rfloor} h_G(d-2k) t^d.$$

3.1. The case $G = O(3)$.

Theorem 3.2. *We have the equality*

$$H(\mathcal{O}(\mathcal{C}(O(3))); t) = -t + \sum_{d=0}^{\infty} \binom{2d+3}{3} t^{2d} = \frac{1 + 5t + 5t^2 - 6t^3 + 4t^4 - t^5}{(1-t)^4}.$$

Proof. We have $\Lambda(O(3))_d = \{(d, 0, 0), (d-1, 1, 0)\}$, except for $d = 3$ or $d = 1$, when $\Lambda(O(3))_1 = \{(1, 0, 0)\}$ and $\Lambda(O(3))_3 = \{(3, 0, 0), (2, 1, 0), (1, 1, 1)\}$. Moreover, $V_{(d,0,0)}$ can be identified with the space of 3-variable spherical harmonics of degree d (see for example [11, Section 5.2.3]), and so $\dim_{\mathbb{C}}(V_{(d,0,0)}) = \binom{d+2}{2} - \binom{d}{2} = 2d+1$. Moreover, the restriction to the special orthogonal group $SO(3)$ of the representation on $V_{(d-1,1,0)}$ is isomorphic to $V_{(d-1,0,0)}$, whence $\dim_{\mathbb{C}}(V_{(d-1,1,0)}) = \dim(V_{(d-1,0,0)}) = 2(d-1) + 1 = 2d-1$. Furthermore, $V_{(1,1,1)}$ can be identified with the third exterior power of $\mathbb{C}^3$, so $\dim_{\mathbb{C}}(V_{(1,1,1)}) = 1$. Thus we have

$$h_{O(3)}(d) = \begin{cases} (2d-1)^2 + (2d+1)^2 = 8d^2 + 2 & \text{for } d \notin \{0, 1, 3\}; \\ 1^2 & \text{for } d = 0; \\ 3^2 & \text{for } d = 1; \\ 1^2 + 5^2 + 7^2 & \text{for } d = 3. \end{cases}$$

It follows by Corollary 3.1 that that for $d \geq 2$ or $d = 0$ we have

$$\dim(\mathcal{O}(\mathcal{C}(\mathrm{O}(3)))_d) = \begin{cases} 1 + 3^2 + 5^2 + \cdots + (2d+1)^2 = \binom{2d+3}{3} & \text{for } d \neq 1; \\ 9 & \text{for } d = 1. \end{cases}$$

This shows $H(\mathcal{O}(\mathcal{C}(\mathrm{O}(3))))t = -t + \sum_{d=0}^{\infty} \binom{2d+3}{3} t^d$. Using that

$$\sum_{d=0}^{\infty} \binom{2d+3}{3} t^{2d} = \frac{1}{6} \left(\frac{d}{dt} \right)^3 \left(\sum_{d=0}^{\infty} t^{2d+3} \right) = \frac{1}{6} \left(\frac{d}{dt} \right)^3 \frac{t^3}{1-t^2} = \frac{1+6t^2+t^4}{(1-t^2)^4}$$

we get

$$\sum_{d=0}^{\infty} \binom{2d+3}{3} t^d = \frac{1+6t+t^2}{(1-t)^4},$$

whence

$$H(\mathcal{O}(\mathcal{C}(\mathrm{O}(3))))t = \frac{1+6t+t^2}{(1-t)^4} - t = \frac{1+5t+5t^2-6t^3+4t^4-t^5}{(1-t)^4}. \quad \square$$

Proposition 3.3. *The quadratic algebra $\mathcal{O}(\mathcal{C}(\mathrm{O}(3)))$ is not Koszul.*

Proof. For a graded $\mathbb{C}$ -algebra $A = \bigoplus_{d=0}^{\infty} A_d$, the *Poincaré series* of A is

$$P(A, t) = \sum_{d=0}^{\infty} \dim \mathrm{Tor}_d^A(\mathbb{C}, \mathbb{C}) t^d = \sum_{d=0}^{\infty} \dim \mathrm{Ext}_A^d(\mathbb{C}, \mathbb{C}) t^d.$$

It is well known that if A is Koszul, then the equality

$$(3) \quad H(\mathcal{O}(A), -t) \cdot P(\mathcal{O}(A), t) = 1$$

holds in the power series ring $\mathbb{Z}[[t]]$ (see [12], [8]). Up to degree 9 the terms of $1/H(\mathcal{O}(\mathcal{C}(\mathrm{O}(3))), -t)$ are the same as the corresponding terms of the polynomial

$$(1+t)^4 \left(1 + \sum_{d=1}^9 (5t - 5t^2 - 6t^3 - 4t^4 - t^5)^d \right)$$

which are

$$-7330t^9 + 17883t^8 + 11539t^7 + 5129t^6 + 1912t^5 + 628t^4 + 183t^3 + 46t^2 + 9t + 1.$$

Thus the coefficient of t^9 in $1/H(\mathcal{O}(\mathcal{C}(\mathcal{O}(3))), -t)$ is negative, hence it can not be a Poincaré series. Therefore the equality (3) can not hold for $A = \mathcal{O}(\mathcal{C}(\mathcal{O}(3)))$, whence $\mathcal{O}(\mathcal{C}(\mathcal{O}(3)))$ is not Koszul. $\square$

Remark 3.4. Our focus on the Koszul property is motivated by the connection to the construction of certain quantum groups (see the Introduction). In fact it was advertised in [13], [14] that quadratic (and in particular, Koszul) algebras should play a distinguished role in quantum group theory.

3.2. The case of $\mathcal{O}(4)$.

Theorem 3.5. *We have the equality*

$$H(\mathcal{O}(\mathcal{C}(\mathcal{O}(4)))); t) = \frac{1 + 9t + 27t^2 + 19t^3 - 30t^4 + 34t^5 - 35t^6 + 21t^7 - 7t^8 + t^9}{(1 - t)^7}.$$

Proof. For $d = 3$ or $d \geq 5$ we have $\Lambda(\mathcal{O}(4))_d = \{(d - 2, 1, 1, 0), (d - k, k, 0, 0) \text{ for } k = 0, \dots, \lfloor d/2 \rfloor\}$. For $d \leq 2$ we have to omit $(d - 2, 1, 1, 0)$, whereas for $d = 4$ the set $\Lambda(\mathcal{O}(4))_4$ contains also $(1, 1, 1, 1)$. The dimension of V_λ for $\lambda \in \Lambda(\mathcal{O}(4))_d$ is given in the following table. Note that the space $V_{(d,0,0,0)}$ can be identified with the space of spherical harmonics of degree d in 4 variables, hence its dimension is $\binom{d+3}{3} - \binom{d+1}{3}$. Moreover, $V_{(d-2,1,1,0)}$ is isomorphic to $V_{(d-2,0,0,0)}$ as an $\mathrm{SO}(4)$ -module, whereas $V_{(1,1,1,1)}$ can be identified with the 4th exterior power of $\mathbb{C}^4$. We read off the dimension of $V_{(d-k,k,0,0)}$ for $k > 0$ from [19, Formula 3.28].

λ	$(d, 0, 0, 0)$	$(d - k, k, 0, 0), 1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$	$(d - 2, 1, 1, 0)$	$(1, 1, 1, 1)$
$\dim_{\mathbb{C}}(V_\lambda)$	$(d + 1)^2$	$2(d + 1)(d - 2k + 1)$	$(d - 1)^2$	1

It follows that for $h_{\mathcal{O}(4)}(d)$ defined in (2) we have

$$h_{\mathcal{O}(4)}(d) = \begin{cases} (d+1)^4 + (d-1)^4 + 4 \sum_{k=1}^{\lfloor d/2 \rfloor} (d+1)^2 (d-2k+1)^2 & \text{for } d \notin \{0, 2, 4\} \\ 1 & \text{for } d = 0 \\ 117 & \text{for } d = 2 \\ 1707 & \text{for } d = 4. \end{cases}$$

So for $d \notin \{0, 2, 4\}$ we have

$$h_{\mathcal{O}(4)}(d) = \frac{2}{3}d^5 + \frac{10}{3}d^4 + \frac{32}{3}d^2 - \frac{2}{3}d + 2$$

by Corollary 3.1. A routine calculation yields that

$$\dim(\mathcal{O}(\mathcal{C}(\mathcal{O}(4)))_d) = \begin{cases} 1 & \text{for } d = 0 \\ 118 & \text{for } d = 2 \\ 1825 & \text{for } d = 4 \\ p(d) & \text{for } d \notin \{0, 2, 4\}, \end{cases}$$

where

$$p(d) := \frac{1}{18}d^6 + \frac{2}{3}d^5 + \frac{20}{9}d^4 + 4d^3 + \frac{85}{18}d^2 + \frac{10}{3}d + 1.$$

We have $p(0) = 1$, $p(2) = 119$, $p(4) = 1825$, so

$$(4) \quad \dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathcal{O}(4)))_d) = \begin{cases} p(d) & \text{for } d \neq 2 \\ p(d) - 1 & \text{for } d = 2. \end{cases}$$

We can rewrite $p(d)$ as a linear combination of binomial coefficients as follows:

$$(5) \quad p(d) = \binom{d+1}{1} - 14 \binom{d+2}{2} + 34 \binom{d+3}{3} - 60 \binom{d+5}{5} + 40 \binom{d+6}{6}.$$

It follows from (4) and (5) that

$$H(\mathcal{O}(\mathcal{C}(\mathcal{O}(4))))_t = -t^2 + \frac{1}{(1-t)^2} - \frac{14}{(1-t)^3} + \frac{34}{1-t^4} - \frac{60}{(1-t)^6} + \frac{40}{(1-t)^7},$$

which gives the formula in our statement. $\square$

3.3. The case of $\mathrm{Sp}(4)$.

Theorem 3.6. *We have the equality*

$$H(\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4))); t) = \frac{1 + 5t + 5t^2 + t^3}{(1 - t)^{11}}.$$

Proof. We have $\Lambda(\mathrm{Sp}(4))_d = \{(d - k, k) \mid k = 0, 1, \dots, \lfloor d/2 \rfloor\}$, and by [19, Formula 3.29],

$$\dim_{\mathbb{C}}(V_{(d-k,k)}) = \frac{1}{6}(d+3)(d-k+2)(k+1)(d-2k+1).$$

It follows that

$$\begin{aligned} h_{\mathrm{Sp}(4)}(d) = & \frac{1}{15120}d^9 + \frac{1}{560}d^8 + \frac{3}{140}d^7 + \frac{3}{20}d^6 + \frac{97}{144}d^5 + \frac{481}{240}d^4 \\ & + \frac{29683}{7560}d^3 + \frac{4069}{840}d^2 + \frac{473}{140}d + 1 \end{aligned}$$

(see (2) for the definition of $h_G(d)$). By Corollary 3.1 we obtain

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4)))_d) &= \frac{1}{302400}d^{10} + \frac{1}{7560}d^9 + \frac{47}{20160}d^8 + \frac{1}{42}d^7 + \frac{2243}{14400}d^6 \\ &+ \frac{49}{72}d^5 + \frac{121279}{60480}d^4 + \frac{2963}{756}d^3 + \frac{121883}{25200}d^2 + \frac{709}{210}d + 1 \\ &= -\binom{d+7}{7} + 8\binom{d+8}{8} - 18\binom{d+9}{9} + 12\binom{d+10}{10}. \end{aligned}$$

It follows that

$$H(\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4))); t) = \frac{-1}{(1-t)^8} + \frac{8}{(1-t)^9} - \frac{18}{(1-t)^{10}} + \frac{12}{(1-t)^{11}},$$

giving the formula in our statement. $\square$

4. The cone of the special orthogonal group. The special orthogonal group is the index two subgroup in $\mathrm{O}(n)$ defined as

$$\mathrm{SO}(n) = \{M \in \mathrm{O}(n) \mid \det(M) = 1\}.$$

For n odd the scalar matrix $-I$ belongs to $\mathrm{O}(n) \setminus \mathrm{SO}(n)$, hence the cone of $\mathrm{SO}(n)$ coincides with the cone of $\mathrm{O}(n)$. For $n = 2m$ even both scalar orthogonal matrices I and $-I$ have determinant 1, hence $\{cM \mid M \in \mathrm{SO}(2m), c \in \mathbb{C}^\times\}$ is disjoint

from $\{cM \mid M \in \mathrm{O}(2m) \setminus \mathrm{SO}(2m), c \in \mathbb{C}^\times\}$. In particular, the cone of $\mathrm{SO}(2m)$ is properly contained in the cone of $\mathrm{O}(2m)$. The irreducible representations of $\mathrm{SO}(2m)$ are labelled by

$$\Lambda(\mathrm{SO}(2m)) = \{\lambda \in \mathbb{Z}^m \mid (\lambda_1, \dots, \lambda_{m-1}, |\lambda_m|) \in \mathrm{Par}(m)\}.$$

For $\lambda \in \Lambda(\mathrm{SO}(2m))$ write W_λ for the corresponding irreducible representation of $\mathrm{SO}(2m)$. For $\lambda \in \Lambda(\mathrm{O}(2m))$ write $V_\lambda \downarrow_{\mathrm{SO}(2m)}^{\mathrm{O}(2m)}$ for the restriction of the $\mathrm{O}(2m)$ -module V_λ to the subgroup $\mathrm{SO}(2m)$. For $\lambda \in \mathrm{Par}(m)$

$$(6) \quad V_\lambda \downarrow_{\mathrm{SO}(2m)}^{\mathrm{O}(2m)} \cong \begin{cases} W_\lambda & \text{when } \lambda_m = 0 \\ W_\lambda \oplus W_{\lambda^\circ} & \text{when } \lambda_m > 0, \end{cases}$$

where $\lambda^\circ := (\lambda_1, \dots, \lambda_{m-1}, -\lambda_m)$ and we have $\dim_{\mathbb{C}}(W_\lambda) = \dim_{\mathbb{C}}(W_{\lambda^\circ})$ for all $\lambda \in \mathrm{Par}(m)$ (see e.g. [16, Page 422]).

Set $\Lambda(\mathrm{SO}(2m))_d := \{\lambda \in \Lambda(\mathrm{SO}(2m)) \mid \lambda_1 + \dots + \lambda_{m-1} + |\lambda_m| = d\}$.

Proposition 2.1 implies by (6) the following:

Proposition 4.1. *For $\lambda \in \Lambda(\mathrm{SO}(2m))_d$ the $\mathrm{SO}(2m)$ -module W_λ occurs as a summand in $\mathrm{T}^d(\mathbb{C}^{2m})$, and W_λ is not a summand of $\mathrm{T}^k(\mathbb{C}^{2m})$ for $k > d$ with $k - d$ odd, or for $k < d$.*

Repeating verbatim the proof of Theorem 2.3 and referring to Proposition 4.1 (instead of Proposition 2.1 we get the following:

Theorem 4.2. *For $d = 0, 1, 2, \dots$ we have the following isomorphisms of $\mathrm{SO}(2m) \times \mathrm{SO}(2m)$ -modules:*

$$\mathcal{O}(\mathcal{C}(\mathrm{SO}(2m)))_d \cong \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \bigoplus_{\lambda \in \Lambda(\mathrm{SO}(2m))_{d-2k}} W_\lambda \otimes W_\lambda.$$

In particular, for $d \geq 2$ we have

$$\mathcal{O}(\mathcal{C}(\mathrm{SO}(2m)))_d \cong \mathcal{O}(\mathcal{C}(\mathrm{SO}(2m)))_{d-2} \oplus \bigoplus_{\lambda \in \Lambda(\mathrm{SO}(2m))_d} W_\lambda \otimes W_\lambda.$$

As a consequence of Theorem 4.2 we get the following:

Corollary 4.3. *We have the equality*

$$\mathrm{H}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(2m))))(t) := \sum_{d=0}^{\infty} \sum_{k=0}^{\lfloor d/2 \rfloor} h_{\mathrm{SO}(2m)}(d-2k)t^d,$$

where

$$h_{\mathrm{SO}(2m)}(d) := \sum_{\lambda \in \Lambda(\mathrm{SO}(2m))_d} \dim_{\mathbb{C}}(W_{\lambda})^2.$$

4.1. The Hilbert series of $\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))$.

Theorem 4.4. *We have the equality*

$$H(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4))); t) = \frac{1 + 9t + 9t^2 + t^3}{(1 - t)^7}.$$

Proof. We have $\Lambda(\mathrm{SO}(4))_d = \{(d-k, k), (d-k, -k) \mid k = 0, 1, \dots, \lfloor d/2 \rfloor\}$ and

$$\dim_{\mathbb{C}}(W_{(d-k, k)}) = \dim_{\mathbb{C}}(W_{(d-k, -k)}) = (d+1)(d-2k+1).$$

Consequently,

$$\begin{aligned} h_{\mathrm{SO}(4)}(d) &= (d+1)^4 + 2 \sum_{k=1}^{\lfloor d/2 \rfloor} (d+1)^2 (d-2k+1)^2 \\ &= \frac{1}{3}d^5 + \frac{5}{3}d^4 + 4d^3 + \frac{16}{3}d^2 + \frac{11}{3}d + 1. \end{aligned}$$

By Corollary 4.3 we get

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))_d) &= \frac{1}{36}d^6 + \frac{1}{3}d^5 + \frac{29}{18}d^4 + 4d^3 + \frac{193}{36}d^2 + \frac{11}{3}d + 1 \\ &= -\binom{d+3}{3} + 12\binom{d+4}{4} - 30\binom{d+5}{5} + 20\binom{d+6}{6}. \end{aligned}$$

It follows that

$$H(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4))))t = \frac{-1}{(1-t)^4} + \frac{12}{(1-t)^5} - \frac{30}{(1-t)^6} + \frac{20}{(1-t)^7},$$

giving the formula in our statement. $\square$

5. $U \times U$ -invariants. For $G \in \{\mathrm{O}(3), \mathrm{O}(4), \mathrm{Sp}(4), \mathrm{SO}(4)\}$, denote by U the unipotent radical of a Borel subgroup of G . Recall that $\dim(\mathcal{O}(\mathcal{C}(G))_d^{U \times U})$ equals the number of irreducible $G^\circ \times G^\circ$ -module direct summands of $\mathcal{O}(\mathcal{C}(G))_d$, where G° denotes the identity component of G . Recall that the identity component of $\mathrm{O}(n)$ is $\mathrm{SO}(n)$, whereas $\mathrm{Sp}(n)$ is connected.

5.1. $U \times U$ -invariants in $\mathcal{O}(\mathcal{C}(\mathbf{O}(3)))$. By the remark above, by Theorem 2.3 and the description of $\Lambda_d(\mathbf{O}(3))$ in the first paragraph of Section 3.1 (recall that each irreducible $\mathbf{O}(3)$ -module restricts to an irreducible $\mathbf{SO}(3)$ -module) we have

$$\dim(\mathcal{O}(\mathcal{C}(\mathbf{O}(3)))_d^{U \times U}) = \begin{cases} d+1 & \text{for } d \geq 2 \text{ or } d = 0 \\ 1 & \text{for } d = 1 \end{cases}.$$

It follows that

$$H(\mathcal{O}(\mathcal{C}(\mathbf{O}(3)))^{U \times U}; t) = \frac{d}{dt} \left(\frac{1}{1-t} \right) - t = \frac{1-t+2t^2-t^3}{(1-t)^2} = \frac{1+t^2+t^3-t^4}{(1-t)(1-t^2)}.$$

By a linear change of coordinates we may switch to the orthogonal group preserving the non-degenerate quadratic form associated to the symmetric matrix

$$\beta := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ so consider}$$

$$\mathbf{O}(3, \beta) = \{M \in \mathbb{C}^{3 \times 3} : M^T \beta M = \beta\}, \quad \mathbf{SO}(3, \beta) = \{M \in \mathbf{O}(3, \beta) \mid \det(M) = 1\}.$$

The Lie algebra of a Borel subgroup B in $\mathbf{SO}(3, \beta)$ is spanned by $E_{11} - E_{33}$, $E_{12} - E_{23}$, where E_{ij} stands for the matrix unit having entry 1 in the (i, j) position and zero in all other positions. The element $E_{11} - E_{33}$ spans the Lie algebra of the maximal torus

$$\mathbb{T} = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\times \right\}$$

in $\mathbf{SO}(3, \beta)$, and the Lie algebra of the unipotent radical U is spanned over $\mathbb{C}$ by $E_{12} - E_{23}$. Write x_{ij} for the standard coordinate functions on $\mathbb{C}^{3 \times 3}$ (and also for their restriction to the cone of $\mathbf{O}(3, \beta)$), and set

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

For a square matrix M with entries in a commutative ring, we write $\det(M)$ and $\text{Tr}(M)$ for the determinant of M and for the trace (the sum of the diagonal

entries) of M . Explicit computation yields the following bases in $\mathcal{O}(\mathcal{C}(G))_d^{U \times U}$ for $d = 1, 2, 3$:

$$\begin{aligned}\mathcal{O}(\mathcal{C}(\mathrm{O}(3, \beta)))_1^{U \times U} &= \mathrm{Span}_{\mathbb{C}}\{x_{31}\} \\ \mathcal{O}(\mathcal{C}(\mathrm{O}(3, \beta)))_2^{U \times U} &= \mathrm{Span}_{\mathbb{C}}\{x_{31}^2, x_{31}x_{22} - x_{21}x_{32}, \mathrm{Tr}(X^T \beta X \beta)\} \\ \mathcal{O}(\mathcal{C}(\mathrm{O}(3, \beta)))_3^{U \times U} &= \mathrm{Span}_{\mathbb{C}}\{x_{31}^3, x_{31}(x_{31}x_{22} - x_{21}x_{32}), x_{31}\mathrm{Tr}(X^T \beta X \beta), \det(X)\}\end{aligned}$$

So up to degree 3, the algebra $\mathcal{O}(\mathcal{C}(\mathrm{O}(3)))^{U \times U}$ is generated by $x_{31}, x_{31}x_{22} - x_{21}x_{32}, \mathrm{Tr}(X^T \beta X \beta), \det(X)$. The above $U \times U$ -invariants are in fact invariant as elements in $\mathcal{O}(\mathbb{C}^{n \times n})$. They are $\mathbb{T}$ -eigenvectors, so they are highest weight vectors in the $\mathrm{SO}(3, \beta) \times \mathrm{SO}(3, \beta)$ -module $\mathcal{O}(\mathbb{C}^{n \times n})$.

5.2. $U \times U$ -invariants in $\mathcal{O}(\mathcal{C}(\mathrm{O}(4)))$.

Proposition 5.1. *We have*

$$H(\mathcal{O}(\mathcal{C}(\mathrm{O}(4)))^{U \times U}; t) = \frac{1 + 3t^2 + t^3 + t^4 - 2t^5 - t^6 + t^7}{(1 - t)(1 - t^2)^2}.$$

Proof. For $\lambda \in \Lambda(\mathrm{O}(4))$ denote by $\epsilon(\lambda) \in \{1, 2\}$ the number of irreducible $\mathrm{SO}(4)$ -module direct summands in the irreducible $\mathrm{O}(4)$ -module V_λ , hence $\epsilon(\lambda)^2$ is the number of irreducible $\mathrm{SO}(4) \times \mathrm{SO}(4)$ -module direct summands in the irreducible $\mathrm{O}(4) \times \mathrm{O}(4)$ -module $V_\lambda \otimes V_\lambda$. Set $N_{\mathrm{O}(4)}(d) := \sum_{\lambda \in \Lambda(\mathrm{O}(4))_d} \epsilon(\lambda)^2$. By

Theorem 2.3 we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{O}(4)))_d^{U \times U}) = \sum_{k=0}^{\lfloor d \rfloor} N_{\mathrm{O}(4)}(d - 2k).$$

Now

$$N_{\mathrm{O}(4)}(d) = \begin{cases} 2 + 4\lfloor d/2 \rfloor & \text{if } d \geq 5 \text{ or } d = 3 \\ 1 & \text{if } d = 0, 1 \\ 5 & \text{if } d = 2 \\ 11 & \text{if } d = 4. \end{cases}$$

We conclude that for $d = 2f$ or $d = 2f + 1$ with $d \geq 5$ we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(G))_d^{U \times U}) = 2f^2 + 4f + 1 = -\binom{f}{0} - 2\binom{f+1}{1} + 4\binom{f+2}{2},$$

and one can check that in fact the same holds except for $d = 2$, for which $2f^2 + 4f + 1|_{f=1} - \dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(G))_d^{U \times U}) = 7 - 6 = 1$. It follows that

$$H(\mathcal{O}(\mathcal{C}(\mathrm{O}(4)))^{U \times U}; t) = -t^2 + \left(\frac{-1}{1-t^2} - \frac{2}{(1-t^2)^2} + \frac{4}{(1-t^2)^3} \right) (1+t),$$

from which our statement easily follows. $\square$

5.3. $U \times U$ -invariants in $\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))$.

Proposition 5.2. *We have*

$$H(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))^{U \times U}; t) = \frac{1+t^2}{(1-t)(1-t^2)^2}.$$

Proof. By Theorem 4.2 we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))_d^{U \times U}) = \sum_{k=0}^{\lfloor d/2 \rfloor} |\Lambda(\mathrm{SO}(4))_{d-2k}|.$$

Moreover, $|\Lambda(\mathrm{SO}(4))_d| = 1 + 2\lfloor d/2 \rfloor$. It follows that for $d = 2f$ or $2f + 1$ we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))_d^{U \times U}) = f^2 + 2f + 1 = -\binom{f+1}{1} + 2\binom{f+2}{2},$$

implying in turn that

$$H(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))^{U \times U}; t) = \left(\frac{-1}{(1-t^2)^2} + \frac{2}{(1-t^2)^3} \right) (1+t). \quad \square$$

5.4. $U \times U$ -invariants in $\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4)))$.

Proposition 5.3. *We have*

$$H(\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4)))^{U \times U}; t) = \frac{1}{(1-t)(1-t^2)^2}.$$

Proof. By Theorem 2.3 we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{Sp}(4)))_d^{U \times U}) = \sum_{k=0}^{\lfloor d/2 \rfloor} |\Lambda(\mathrm{Sp}(4))_{d-2k}|.$$

Moreover, $|\Lambda(\mathrm{Sp}(4))_d| = 1 + \lfloor d/2 \rfloor$. It follows that for $d = 2f$ or $2f + 1$ we have

$$\dim_{\mathbb{C}}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))_d^{U \times U}) = \frac{1}{2}f^2 + \frac{3}{2}f + 1 = \binom{f+2}{2},$$

implying in turn that

$$\mathrm{H}(\mathcal{O}(\mathcal{C}(\mathrm{SO}(4)))^{U \times U}; t) = \frac{1+t}{(1-t^2)^3}. \quad \square$$

6. The vanishing ideal of the cone of the orthogonal group.

Recall that for $1 \leq i, j \leq n$ we denote by $x_{ij} \in \mathcal{O}(\mathbb{C}^{n \times n})$ the coordinate function mapping a matrix $M \in \mathbb{C}^{n \times n}$ to its (i, j) -entry, so $\mathcal{O}(\mathbb{C}^{n \times n})$ is the polynomial algebra $\mathbb{C}[x_{ij} \mid 1 \leq i, j \leq n]$. Denote by X the $n \times n$ generic matrix whose (i, j) -entry is x_{ij} . Write X^T for the transpose of X , and for an $n \times n$ matrix M over $\mathcal{O}(\mathbb{C}^{n \times n})$ write M_{ij} for the (i, j) -entry of M . The following statement is due to H. Weyl:

Theorem 6.1. [20, Chapter V.A.4] *The ideal $\mathcal{I}(\mathcal{C}(\mathrm{O}(n)))$ is generated by the following $n^2 + n - 2$ quadratic polynomials:*

$$\begin{aligned} & (X^T X)_{ij}, \quad (X X^T)_{ij} \quad \text{for } 1 \leq i < j \leq n, \\ & (X^T X)_{11} - (X^T X)_{kk}, \quad (X X^T)_{11} - (X X^T)_{kk} \quad \text{for } k = 2, \dots, n \end{aligned}$$

Remark 6.2. (i) Clearly, a minimal homogeneous generating set of the ideal $\mathcal{I}(\mathcal{C}(\mathrm{O}(n)))$ spans an $\mathrm{O}(n) \times \mathrm{O}(n)$ -invariant subspace in $\mathcal{O}(\mathbb{C}^{n \times n})$. The $\mathrm{O}(n) \times \mathrm{O}(n)$ -module structure of the degree 2 homogeneous component of $\mathcal{I}(\mathcal{C}(\mathrm{O}(n)))$ is given (using the notation of Section 2) as follows:

$$\mathcal{I}(\mathcal{C}(\mathrm{O}(n)))_2 \cong V_{(2,0,\dots,0)} \otimes V_{(0,\dots,0)} \oplus V_{(0,\dots,0)} \otimes V_{(2,0,\dots,0)}$$

(recall that $V_{(0,\dots,0)}$ is the 1-dimensional trivial $\mathrm{O}(n)$ -module, and $V_{(2,0,\dots,0)}$ can be identified with the space of n -variable spherical harmonics of degree 2). This shows that the generating set of $\mathcal{I}(\mathcal{C}(\mathrm{O}(n)))$ provided by Theorem 6.1 is minimal.

(ii) A statement similar to Theorem 6.1 holds for the vanishing ideal of the cone of the symplectic group, see [20, Chapter X.5].

Example 6.3. Denoting by i the imaginary complex unit with $i^2 = -1$, the matrix $A := \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfies $AA^T = 0 = A^T A$, hence all the generators of $\mathcal{I}(\mathcal{C}(\mathrm{O}(3)))$ given in Theorem 6.1 vanish on A . It follows that A belongs to the cone $\mathcal{C}(\mathrm{O}(3))$. However, A is not a scalar multiple of an element of $\mathrm{O}(3)$.

For sake of completeness we provide a proof of Theorem 6.1 in modern language. Write $V = \mathbb{C}^n$, and let $e_1, \dots, e_n$ denote the standard basis vectors. Identify $\mathbb{C}^{n \times n}$ with $\mathrm{End}_{\mathbb{C}}(V)$ in the standard way, then $G = \mathrm{O}(n)$ is identified with the subgroup of the general linear group $\mathrm{GL}(V)$ consisting of the linear transformations that preserve the non-degenerate symmetric bilinear form $\langle - | - \rangle$ on V given by

$$\langle e_i | e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For a finite dimensional $\mathbb{C}$ -vector space W we write W^* for the dual space of W , and for a positive integer d we write $T^d(W) = W \otimes \dots \otimes W$ for the d -fold tensor power of W . We shall make use of the standard identifications $W \otimes W^* \cong \mathrm{End}_{\mathbb{C}}(W)$ and $T^d(W^*) \cong T^d(W)^*$. These induce an identification

$$T^d(\mathrm{End}_{\mathbb{C}}(V)) \cong \mathrm{End}_{\mathbb{C}}(T^d(V))$$

via

$$T^d(V \otimes V^*) \cong V \otimes V^* \otimes \dots \otimes V \otimes V^* \cong T^d(V) \otimes T^d(V^*) \cong T^d(V) \otimes T^d(V)^*.$$

Write $\mathcal{U}_d$ for the $\mathbb{C}$ -subspace

$$\mathcal{U}_d = \mathrm{Span}_{\mathbb{C}}\{M \otimes \dots \otimes M \in \mathrm{End}_{\mathbb{C}}(T^d(V)) \mid M \in \mathrm{End}_{\mathbb{C}}(V)\}.$$

Clearly $\mathcal{U}_d$ is a subalgebra of the noncommutative $\mathbb{C}$ -algebra $\mathrm{End}_{\mathbb{C}}(T^d(V))$, called a *Schur algebra*. It contains the subalgebra

$$\mathcal{U}(G)_d = \mathrm{Span}_{\mathbb{C}}\{g \otimes \dots \otimes g \in \mathrm{End}_{\mathbb{C}}(T^d(V)) \mid g \in \mathrm{O}(n)\}.$$

Recall that the polynomial ring $\mathcal{O}(\mathbb{C}^{n \times n})$ is graded in the usual way, and $\mathcal{O}(\mathbb{C}^{n \times n})_d$ stands for the degree d homogeneous component.

Lemma 6.4. *There exists a unique $\mathbb{C}$ -vector space isomorphism*

$$\mathcal{O}(\mathbb{C}^{n \times n})_d \rightarrow \mathcal{U}_d^*, \quad f \mapsto \tilde{f}$$

such that for any $f \in \mathcal{O}(\mathbb{C}^{n \times n})_d$ and $M \in \mathbb{C}^{n \times n}$ we have

$$\tilde{f}(M \otimes \cdots \otimes M) = d!f(M).$$

Proof. Consider the isomorphisms

$$\mathcal{O}(\mathbb{C}^{n \times n})_d \cong (\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^*)^{S_d} \cong (\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^{S_d})^*.$$

The first of the above isomorphisms is given by the standard *multilinearization process*: it assigns to the homogeneous form $f \in \mathcal{O}(\mathbb{C}^{n \times n})_d$ the linear functional $\hat{f} \in \mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^*$, characterized by the property that for $M_1, \dots, M_d \in \mathrm{End}_{\mathbb{C}}(V)$, $\hat{f}(M_1 \otimes \cdots \otimes M_d)$ is the value at $(M_1, \dots, M_d)$ of the multilinear component of the function $\mathrm{End}_{\mathbb{C}}(V) \times \cdots \times \mathrm{End}_{\mathbb{C}}(V) \rightarrow \mathbb{C}$, $(M_1, \dots, M_d) \mapsto f(M_1 + \cdots + M_d)$. The symmetric group S_d acts on $\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))$ by permuting the tensor components. Clearly $\hat{f}(M_1 \otimes \cdots \otimes M_d) = \hat{f}(M_{s(1)} \otimes \cdots \otimes M_{s(d)})$ holds for any permutation $s \in S_d$, so $\hat{f}$ belongs to the subspace $(\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^*)^{S_d}$. Moreover, $f \mapsto \hat{f}$ is an isomorphism, because for any $h \in (\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^*)^{S_d}$ we have that $h = \hat{f}$ where $f(M) = \frac{1}{d!}h(M \otimes \cdots \otimes M)$. The second isomorphism is given by restriction of an S_d -invariant linear functional on $\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))$ to the subspace $\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^{S_d}$ of symmetric tensors. This sends $\hat{f} \mapsto \tilde{f}$. Observe finally $\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^{S_d}$ is spanned by $M \otimes \cdots \otimes M$ as M ranges over $\mathrm{End}_{\mathbb{C}}(V)$, see for example [16, Lemma 9.1.1]. So we have $\mathrm{T}^d(\mathrm{End}_{\mathbb{C}}(V))^{S_d} = \mathcal{U}_d$ by definition of the latter. $\square$

For a subset $S \subset \mathrm{End}_{\mathbb{C}}(\mathrm{T}^d(V))$ denote by $Z(S)$ the centralizer of S in the algebra $\mathrm{End}_{\mathbb{C}}(\mathrm{T}^d(V))$; clearly $Z(S)$ is a subalgebra in $\mathrm{End}_{\mathbb{C}}(\mathrm{T}^d(V))$. The endomorphism algebra $\mathrm{End}_G(\mathrm{T}^d(V)) = Z(\mathcal{U}(G)_d)$ is generated by the algebra $\mathrm{End}_{GL(V)}(\mathrm{T}^d(V)) = Z(\mathcal{U}_d)$ and a single linear transformation $B \in \mathrm{End}_{\mathbb{C}}(\mathrm{T}^d(V))$ given by

$$B(v_1 \otimes \cdots \otimes v_d) = \langle v_1 \mid v_2 \rangle \sum_{i=1}^n e_i \otimes e_i \otimes v_3 \otimes \cdots \otimes v_d,$$

see for example [16, Section 11.3.2], where this is deduced from the first fundamental theorem for vector invariants of the orthogonal group (the algebra $Z(\mathcal{U}(G)_d)$ is called *Brauer centralizer algebra*). Since $\mathcal{U}_d$ and $\mathcal{U}(G)_d$ are semisimple subalgebras of $\mathrm{End}_{\mathbb{C}}(\mathrm{T}^d(V))$, by the double centralizing theorem for semisimple algebras (see for example [16, Theorem 6.2.5]), one recovers them as the centralizer of their centralizer. Thus we have

$$\mathcal{U}(G)_d = Z(Z(\mathcal{U}(G)_d)) = Z(\{B\} \cup Z(\mathcal{U}_d))$$

$$= Z(\{B\}) \cap Z(Z(\mathcal{U}_d)) = \mathcal{U}_d \cap Z(\{B\}).$$

This can be reformulated as follows:

Proposition 6.5. *The kernel of the natural surjection $\mathcal{U}_d^* \twoheadrightarrow \mathcal{U}(G)_d^*$ (given by restriction of linear functionals on $\mathcal{U}_d$ to the subspace $\mathcal{U}(G)_d$) is spanned by the restrictions to $\mathcal{U}_d$ of the coordinate functions of the map*

$$\text{End}_{\mathbb{C}}(\text{T}^d(V)) \rightarrow \text{End}_{\mathbb{C}}(\text{T}^d(V)), \quad Y \mapsto BY - YB.$$

Proof of Theorem 6.1. Denote by Q the ideal in $\mathcal{O}(\mathbb{C}^{n \times n})$ generated by the $n^2 + n - 2$ quadratic polynomials in the statement. By definition of G they vanish on G , and since they are homogeneous, they vanish on $\mathcal{C}(G)$ as well. So $Q \subseteq \mathcal{I}(\mathcal{C}(G))$. The ideal Q is generated by homogeneous elements. Being the vanishing ideal of a cone, $\mathcal{I}(\mathcal{C}(G))$ is also a homogeneous ideal. Denote by $Q_d, \mathcal{I}(\mathcal{C}(G))_d$ their degree d homogeneous components. It remains to prove that $Q_d = \mathcal{I}(\mathcal{C}(G))_d$ for all positive integers d . By Lemma 6.4 the isomorphism $\mu : \mathcal{O}(\mathbb{C}^{n \times n})_d \rightarrow \mathcal{U}_d^*, f \rightarrow \tilde{f}$ maps $\mathcal{I}(\mathcal{C}(G))_d$ to the kernel of the natural surjection $\mathcal{U}_d^* \twoheadrightarrow \mathcal{U}(G)_d^*$. Therefore by Proposition 6.5 it is sufficient to show that the restriction to $\mathcal{U}_d$ of the coordinate functions of the map $\kappa : \text{End}_{\mathbb{C}}(\text{T}^d(V)) \rightarrow \text{End}_{\mathbb{C}}(\text{T}^d(V)), Y \mapsto BY - YB$ are contained in the space $\mu(Q_d)$. Now we compute the restrictions to $\mathcal{U}_d$ of the coordinate functions of κ with respect to the basis $E_{i_1 j_1} \otimes \cdots \otimes E_{i_d j_d}$ of $\text{End}_{\mathbb{C}}(\text{T}^d(V)) \cong \text{T}^d(\text{End}_{\mathbb{C}}(V))$, where

$$E_{ij}(e_k) = \begin{cases} e_i & \text{for } k = j \\ 0 & \text{for } k \neq j. \end{cases}$$

In order to do so we need to point out that under the identification $\text{End}_{\mathbb{C}}(\text{T}^d(V)) \cong \text{T}^d(\text{End}_{\mathbb{C}}(V))$ the element B corresponds to

$$B = \sum_{i,j=1}^n E_{ij} \otimes E_{ij} \otimes \text{id}_V \otimes \cdots \otimes \text{id}_V.$$

For $M = \sum_{i,j=1}^n M_{ij} E_{ij} \in \text{End}_{\mathbb{C}}(V)$ we have

$$M \otimes M \sum_{k,l=1}^n E_{kl} \otimes E_{kl} = \sum_{i,j,s,t,k,l} M_{ij} M_{st} E_{ij} E_{kl} \otimes E_{st} E_{kl}$$

$$= \sum_{i,k,s,l} M_{ik} M_{sk} E_{il} \otimes E_{sl} = \sum_{i,s,l} \left(\sum_{k=1}^n M_{ik} M_{sk} \right) E_{il} \otimes E_{sl}$$

and

$$\begin{aligned} \left(\sum_{k,l=1}^n E_{lk} \otimes E_{lk} \right) M \otimes M &= \sum_{i,j,s,t,k,l} M_{ji} M_{ts} E_{lk} E_{ji} \otimes E_{lk} E_{ts} \\ &= \sum_{i,k,s,l} M_{ki} M_{ks} E_{li} \otimes E_{ls} = \sum_{i,s,l} \left(\sum_{k=1}^n M_{ki} M_{ks} \right) E_{li} \otimes E_{ls} \end{aligned}$$

This shows that the restriction to $\mathcal{U}_d$ of the coordinate function of the map κ corresponding to $E_{il} \otimes E_{sl} \otimes E_{i_3j_3} \otimes \cdots \otimes E_{i_dj_d}$ is the image under μ of

$$\frac{1}{d!} \cdot \begin{cases} x_{i_3j_3} \cdots x_{i_dj_d} \sum_{k=1}^n x_{ik} x_{sk} & \text{when } i \neq s \\ x_{i_3j_3} \cdots x_{i_dj_d} \sum_{k=1}^n (x_{ik}^2 - x_{kl}^2) & \text{when } i = s \end{cases},$$

the restriction to $\mathcal{U}_d$ of the coordinate function of the map κ corresponding to $E_{li} \otimes E_{ls} \otimes E_{i_3j_3} \otimes \cdots \otimes E_{i_dj_d}$ for $i \neq s$ is the image under μ of

$$\frac{1}{d!} x_{i_3j_3} \cdots x_{i_dj_d} \sum_{k=1}^n x_{ki} x_{ks},$$

and the remaining coordinate functions of κ restrict to the zero map on $\mathcal{U}_d$. Since $\sum_{k=1}^n x_{ik} x_{sk}$, $\sum_{k=1}^n x_{ki} x_{ks}$ for $i \neq s$ and $\sum_{k=1}^n (x_{ik}^2 - x_{kl}^2)$ belong to Q , this shows that the restriction to $\mathcal{U}_d$ of each of the coordinate functions of κ is contained in $\mu(Q_d)$, and we are done. $\square$

7. Gröbner basis for the vanishing ideal of $\mathcal{C}(G)$.

7.1. Gröbner basis for $\mathcal{I}(\mathcal{C}(\mathbf{O}(3)))$. A Gröbner basis of the vanishing ideal of the cone of $\mathbf{O}(3, \beta)$ with respect to the degree reverse lexicographic ordering of the monomials, induced by the ordering $x_{11} > x_{12} > x_{13} > x_{21} > x_{22} > x_{23} > x_{31} > x_{32} > x_{33}$ of the variables is the following:

$$\begin{aligned} x_{12}x_{21}x_{31} - 2x_{11}x_{22}x_{31} + x_{11}x_{21}x_{32}, \\ x_{12}x_{21}x_{32} - 2x_{11}x_{22}x_{32} - 2x_{11}x_{21}x_{33}, \end{aligned}$$

$$\begin{aligned}
& x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33}, \\
& x_{12}x_{23}x_{32} - 2x_{12}x_{22}x_{33} - 2x_{11}x_{23}x_{33}, \\
& x_{13}x_{23}x_{32} - 2x_{13}x_{22}x_{33} + x_{12}x_{23}x_{33}, \\
& x_{12}^2 + 2x_{11}x_{13}, \\
& x_{13}x_{21} + x_{12}x_{22} + x_{11}x_{23}, \\
& x_{21}^2 + 2x_{11}x_{31}, \\
& x_{21}x_{22} + x_{12}x_{31} + x_{11}x_{32}, \\
& x_{22}^2 - x_{13}x_{31} + x_{12}x_{32} - x_{11}x_{33}, \\
& x_{21}x_{23} - x_{12}x_{32}, \\
& x_{22}x_{23} + x_{13}x_{32} + x_{12}x_{33}, \\
& x_{23}^2 + 2x_{13}x_{33}, \\
& x_{23}x_{31} + x_{22}x_{32} + x_{21}x_{33}, \\
& x_{32}^2 + 2x_{31}x_{33}
\end{aligned}$$

The leading monomials of the elements in the above Gröbner basis are

$$\begin{aligned}
& x_{32}^2, x_{23}x_{31}, x_{23}^2, x_{22}x_{23}, x_{21}x_{23}, x_{22}^2, x_{21}x_{22}, x_{21}^2, x_{13}x_{21}, x_{12}^2, \\
& x_{13}x_{23}x_{32}, x_{12}x_{23}x_{32}, x_{11}x_{23}x_{32}, x_{12}x_{21}x_{32}, x_{12}x_{21}x_{31}.
\end{aligned}$$

For a set $\{a, b, \dots\}$ of variables denote by $\langle a, b, \dots \rangle$ the set of monomials involving only the variables $a, b, \dots$. The standard monomials are the following:

$$\begin{aligned}
& \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \sqcup x_{12} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \sqcup x_{22} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \\
& \sqcup x_{32} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle x_{12}x_{32} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \\
& \sqcup x_{12}x_{22} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \sqcup x_{22}x_{32} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \\
& \sqcup x_{12}x_{22}x_{32} \langle x_{11}, x_{13}, x_{31}, x_{33} \rangle \sqcup x_{21} \langle x_{11}, x_{31}, x_{33} \rangle \sqcup x_{23} \langle x_{11}, x_{13}, x_{33} \rangle \\
& \sqcup x_{12}x_{23} \langle x_{11}, x_{13}, x_{33} \rangle \sqcup x_{21}x_{32} \langle x_{13}, x_{31}, x_{33} \rangle \sqcup x_{12}x_{21} \langle x_{11}, x_{33} \rangle \sqcup x_{23}x_{32} \langle x_{33} \rangle
\end{aligned}$$

A form of the Hilbert series $H(\mathcal{O}(\mathcal{C}(G)); t)$ reflecting the above list is

$$H(\mathcal{O}(\mathcal{C}(G)); t) = \frac{1 + 3t + 3t^2 + t^3}{(1-t)^4} + \frac{2t + 2t^2}{(1-t)^3} + \frac{t^2}{(1-t)^2} + \frac{t^2}{1-t}.$$

The quadratic elements in this Gröbner basis form a minimal generating set of the ideal $\mathcal{I}(\mathcal{C}(\mathrm{O}(3)))$, since (up to our coordinate change) they form a $\mathbb{C}$ -vector space basis in the linear span of the generators of $\mathcal{I}(\mathcal{C}(\mathrm{O}(3)))$ given in Theorem 6.1.

Proposition 7.1. *Regardless of the chosen term order in $\mathcal{O}(\mathbb{C}^{n \times n})$, the ideal $\mathcal{I}(\mathcal{C}(\mathcal{O}(3)))$ has no quadratic Gröbner basis.*

Proof. Suppose to the contrary that there is a term order such that $\mathcal{I}(\mathcal{C}(\mathcal{O}(3)))$ has a quadratic Gröbner basis. Then the graded algebra $\mathcal{O}(\mathcal{C}(\mathcal{O}(3)))$ is Koszul, see [8], [1]. This contradicts Proposition 3.3. $\square$

7.2. Gröbner basis for $\mathcal{I}(\mathcal{C}(\mathrm{Sp}(4)))$. We do not know if $\mathcal{O}(\mathcal{C}(G))$ for $G = \mathrm{O}(4)$ or $G = \mathrm{Sp}(4)$ is Koszul or not. In contrast with $\mathcal{O}(\mathcal{C}(\mathcal{O}(3)))$, for these groups G the first 50 coefficients of $1/\mathrm{H}(\mathcal{O}(\mathcal{C}(G)); -t)$ are non-negative. On the other hand, the quadratic generators for $\mathcal{I}(\mathcal{C}(G))$ do not constitute a Gröbner basis with respect to the degree reverse lexicographic monomial ordering induced by the ordering $x_{11} > x_{12} > x_{13} > x_{14} > x_{21} > x_{22} > x_{23} > x_{24} > x_{31} > x_{32} > x_{33} > x_{34} > x_{41} > x_{42} > x_{43} > x_{44}$ of the variables. For example, the Gröbner basis for $\mathcal{I}(\mathcal{C}(\mathrm{Sp}(4)))$ has two cubic elements, in addition to 10 quadratic elements:

$$\begin{aligned} & x_{11}x_{32}x_{41} - x_{14}x_{33}x_{41} + x_{13}x_{34}x_{41} - x_{11}x_{31}x_{42} + x_{14}x_{31}x_{43} - x_{13}x_{31}x_{44}, \\ & x_{21}x_{32}x_{41} - x_{24}x_{33}x_{41} + x_{23}x_{34}x_{41} - x_{21}x_{31}x_{42} + x_{24}x_{31}x_{43} - x_{23}x_{31}x_{44}, \\ & x_{12}x_{21} - x_{11}x_{22} - x_{34}x_{43} + x_{33}x_{44}, \quad x_{13}x_{21} - x_{11}x_{23} + x_{33}x_{41} - x_{31}x_{43}, \\ & x_{14}x_{21} - x_{11}x_{24} + x_{34}x_{41} - x_{31}x_{44}, \quad x_{13}x_{22} - x_{12}x_{23} + x_{33}x_{42} - x_{32}x_{43}, \\ & x_{14}x_{22} - x_{12}x_{24} + x_{34}x_{42} - x_{32}x_{44}, \quad x_{14}x_{23} - x_{13}x_{24} - x_{32}x_{41} + x_{31}x_{42}, \\ & x_{12}x_{31} - x_{11}x_{32} + x_{14}x_{33} - x_{13}x_{34}, \quad x_{22}x_{31} - x_{21}x_{32} + x_{24}x_{33} - x_{23}x_{34}, \\ & x_{12}x_{41} - x_{11}x_{42} + x_{14}x_{43} - x_{13}x_{44}, \quad x_{22}x_{41} - x_{21}x_{42} + x_{24}x_{43} - x_{23}x_{44} \end{aligned}$$

REFERENCES

- [1] D. J. ANICK. On the homology of associative algebras. *Trans. Amer. Math. Soc.* **296**, 2 (1986), 641–659.
- [2] K. A. BROWN, K. R. GOODEARL. Lectures on Algebraic Quantum Groups. *Adv. Courses Math. CRM Barcelona*. Basel, Birkhäuser Verlag, 2002.
- [3] G. CLIFF. A basis of bideterminants for the coordinate ring of the orthogonal group. *Comm. Algebra* **36**, 7 (2008), 2719–2749.
- [4] C. DE CONCINI. Characteristic free “decomposition” of the coordinate ring of the symplectic group. *Quad. “Ricerca Sci.”* [Publications of “Scientific Research”], vol. **109**. Consiglio Nazionale delle Ricerche (CNR), 1981, 121–128.

- [5] C. DE CONCINI, D. EISENBUD, C. PROCESI. Young diagrams and determinantal varieties. *Invent. Math.* **56**, 2 (1980), 129–165.
- [6] M. DOMOKOS, T. H. LENAGAN. Representation rings of quantum groups. *J. Algebra* **282**, 1 (2004), 103–128.
- [7] P. DOUBILET, G.-C. ROTA, J. STEIN. On the foundations of combinatorial theory. IX: Combinatorial methods in invariant theory. *Studies in Appl. Math.* **53** (1974), 185–216.
- [8] R. FRÖBERG. Koszul algebras. Advances in commutative ring theory (Fez, 1997), 337–350, Lecture Notes in Pure and Appl. Math., vol. **205**, New York, Marcel Dekker, Inc., 1999.
- [9] K. R. GOODEARL. Commutation relations for arbitrary quantum minors. *Pacific J. Math.* **228**, 1 (2006), 63–102.
- [10] K. R. GOODEARL, T. H. LENAGAN. Quantum determinantal ideals, *Duke Math. J.* **103**, 1 (2000), 165–190.
- [11] R. GOODMAN, N. R. WALLACH. Representations and Invariants of the Classical Groups. *Encyclopedia Math. Appl.*, vol. **68**, Cambridge, Cambridge University Press, 1998.
- [12] C. LÖFWALL. On the subalgebra generated by the one-dimensional elements in the Yoneda ext-algebra. In: *Algebra, Algebraic Topology and their Interactions* (Ed. J.-E. Roos). Lecture Notes in Mathematics, vol. **1183**. Berlin, Heidelberg, Springer, <https://doi.org/10.1007/BFb0075468>.
- [13] YU. I. MANIN. Some remarks on Koszul algebras and quantum groups. *Ann. Inst. Fourier (Grenoble)* **37**, 4 (1987), 191–205.
- [14] YU. I. MANIN. Quantum groups and non-commutative geometry. Université de Montreal, Centre de Recherches Mathématiques, Montreal, QC, 1988.
- [15] B. PARSHALL, J. P. WANG. Quantum linear groups. *Mem. Amer. Math. Soc.* **89**, 439 (1991), vi+157 pp.
- [16] C. PROCESI. Lie Groups (An approach through invariants and representations). Universitext. New York, Springer, 2007.
- [17] N. YU. RESHETIKHIN, L. A. TAKHTADZHIAN, L. D. FADDEEV. Quantization of Lie groups and Lie algebras. *Algebra i Analiz* **1**, 1 (1989), 178–206 (in Russian); English translation in: *Leningrad Math. J.* **1**, 1 (1990), 193–225.

- [18] Sagemath, Inc., CoCalc – Collaborative Calculation and Data Science, 2020, <https://cocalc.com>.
- [19] N. EL SAMRA, R. C. KING. Dimensions of irreducible representations of the classical Lie groups. *J. Phys. A* **12**, 12 (1979), 2317–2328.
- [20] H. WEYL. The Classical Groups, Second edition with supplement. Princeton, Princeton University Press, 1953.

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