Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2015-05. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Algorithmic aspects of covering supermodular functions under matroid constraints

Kristóf Bérczi, Tamás Király, and
Yusuke Kobayashi

May 2015

Algorithmic aspects of covering supermodular functions under matroid constraints

Kristóf Bérczi*, Tamás Király**, and Yusuke Kobayashi***

Abstract

A common generalization of earlier results on arborescence packing and the covering of intersecting bi-set families was presented by the authors in [2]. The present paper investigates the algorithmic aspects of that result and gives a polynomial-time algorithm for the corresponding optimization problem.

1 Introduction

In [2], we presented a result that joined two different directions of research on extensions of Edmonds' disjoint branchings theorem [5]. One direction, initiated by Frank [6] and pursued by Szegő [14] and by Bérczi and Frank [1], considered an abstract problem involving the covering of families of sets or bi-sets (see section 1.1 for definitions). The other direction of research, represented by the work of Katoh and Tanigawa [10], Durand de Gevigney, Nguyen, and Szigeti [4], and Cs. Király [11], involved a matroid given on the possible roots, with the requirement that roots of the branchings reaching a given node v should be a maximal independent set in the set of roots from which v is reachable. The common generalization in [2], described in detail in section 1.2, considers a digraph, bi-set families $\mathcal{F}_i, \mathcal{G}_i$ $(i = 1, \dots, t)$, and a matroid on ground set $[t] = \{1, \dots, t\}$. A succinct description of the problem is that the digraph should be partitioned into arc sets $A_1, \dots, A_t$ in such a way that for every $X \subseteq V$ the index set $\{i : A_i \text{ enters } X\} \cup \{j : X \in \mathcal{G}_j\}$ should span the index set $\{i : X \in \mathcal{F}_i \cup \mathcal{G}_i\}$ in the matroid. The paper proved that for bi-set families with certain properties the natural cut condition is sufficient, but the question of algorithmic complexity was left open.

In this paper we show that if the bi-set families are given by digraph representations, then there is a polynomial-time algorithm for finding a valid decomposition of the arc

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Supported by the Hungarian Scientific Research Fund - OTKA, K109240. Email:`berkri@cs.elte.hu`

**MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Supported by the Hungarian Scientific Research Fund - OTKA, K109240, and by the Bolyai Research Scholarship. Email:`tkiraly@cs.elte.hu`

***University of Tsukuba, Tsukuba, 305-8573, Japan. Supported by JST ERATO Kawarabayashi Project and by Grant-in-Aid for Scientific Research. Email: `kobayashi@sk.tsukuba.ac.jp`

set. Moreover, the problem of finding a minimum cost arc subset that has a valid decomposition is also in P. This result gives the first polynomial-time algorithm for the minimum-cost version of the problem in [11].

1.1 Notation

Let $D = (V, A)$ be a directed graph (digraph, for short). The **tail** and **head** nodes of an arc a are denoted by $t(a)$ and $h(a)$, respectively. A **bi-set** is a pair $X = (X_I, X_O)$ such that $X_I \subseteq X_O \subseteq V$ where X_I and X_O are called the **inner** and the **outer set** of X, respectively. The family of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** and **union** of bi-sets can be defined in a straightforward manner: for bi-sets X and Y, we define $X \cap Y = (X_I \cap Y_I, X_O \cap Y_O)$ and $X \cup Y = (X_I \cup Y_I, X_O \cup Y_O)$. An edge $a \in A$ **enters** or **covers** a bi-set X if $h(a) \in X_I$ and $t(a) \notin X_O$, and it **leaves** X if $h(a) \notin X_O$ and $t(a) \in X_I$. A subset of edges $A' \subseteq A$ **covers** a bi-set family $\mathcal{F}$ if each member of $\mathcal{F}$ is covered by at least one arc in A'. The set of arcs entering (leaving) a bi-set X are denoted by $\Delta^{in}(X)$ ($\Delta^{out}(X)$), while the number of arcs entering (leaving) a bi-set X are denoted by $\varrho(X)$ ($\delta(X)$). An arc is **contained** in a bi-set X if $t(a) \in X_O$ and $h(a) \in X_I$. We say that $X \subseteq Y$ if $X_I \subseteq Y_I$ and $X_O \subseteq Y_O$. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$. A family $\mathcal{F}$ of bi-sets is called **intersecting** if $X, Y \in \mathcal{F}, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$.

A **bi-set function** is a function $p : \mathcal{P}_2 \to \mathbb{R}$. A bi-set function p is called **fully supermodular** (respectively, **intersecting supermodular**) if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

for every $X, Y \in \mathcal{P}_2$ (respectively, for every intersecting $X, Y \in \mathcal{P}_2$). If the reverse inequality holds, we call p **fully submodular**. A basic example for a submodular bi-set function is the in-degree function ϱ. We call p **positively intersecting supermodular** or **positively intersecting submodular** if the corresponding inequality holds whenever X and Y are intersecting and $p(X), p(Y) > 0$.

1.2 Problem definition

Motivated by results of de Gevigney, Nguyen and Szigeti [4] and Cs. Király [11] on packing arborescences under matroid constraints, and by results of Frank [6], Szegő [14], and Bérczi and Frank [1] on covering intersecting set families, we considered the following problem in [2].

Let $\mathcal{M} = ([t], r)$ be a matroid. The **closure** of $I \subseteq [t]$ is denoted by $\texttt{Span}(I)$, that is, $\texttt{Span}(I) = \{i : \ r(I + i) = r(I)\}$. A set $I \subseteq [t]$ is called **closed** if $\texttt{Span}(I) = I$. Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting, $\mathcal{G}_1, \dots, \mathcal{G}_t$ be arbitrary bi-set families over $\mathcal{P}_2$ with the property that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for each $i \in [t]$. We denote the family of bi-sets appearing in at least one of the families $\mathcal{F}_i$ by $\mathcal{F} = \bigcup \mathcal{F}_i$.

For a bi-set $X \in \mathcal{P}_2$, let

$$I_X = \{i : \ X \in \mathcal{F}_i\},$$
$$J_X = \{i : \ X \in \mathcal{G}_i\}.$$

The disjointness of $\mathcal{F}_i$ and $\mathcal{G}_i$ means that $I_X \cap J_X = \emptyset$ for each bi-set $X \in \mathcal{P}_2$. We introduce the following bi-set function defined on $\mathcal{P}_2$:

$$p(X) = \begin{cases} r(I_X \cup J_X) - r(J_X) & \text{if } X \in \mathcal{F}, \\ 0 & \text{otherwise.} \end{cases}$$

A bi-set X is said to be **active** if $p(X) > 0$, or equivalently $I_X \setminus \texttt{Span}(J_X) \neq \emptyset$. We denote the set $I_X \setminus \texttt{Span}(J_X)$ by I_X^{act} and say that X is **active for** i if $i \in I_X^{act}$. A bi-set X is called **tight** if $\varrho(X) = p(X) > 0$ and $I_X \neq [t]$. A bi-set is **tight for** i if it is tight and $i \notin I_X^{act}$. Note that every active or tight bi-set is in $\mathcal{F}$.

We say that $\mathcal{F}_1, \ldots, \mathcal{F}_t, \mathcal{G}_1, \ldots, \mathcal{G}_t$ satisfy the **active intersection property** if

$$\textbf{(AIP)} \quad X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X_I \cap Y_I \neq \emptyset, i \in I_X^{act} \Rightarrow i \in I_{X \cap Y}^{act}.$$

The main result of [2] is the following.

Theorem 1. *Let $\mathcal{M} = ([t], r)$ be a matroid and $D = (V, A)$ a digraph. Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be intersecting families, $\mathcal{G}_1, \ldots, \mathcal{G}_t$ arbitrary bi-set families over $\mathcal{P}_2$ such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

(a) $I_{X'} \cup J_{X'} = I_X \cup J_X$ for all $X, X' \in \mathcal{F}, X' \subseteq X$,

(b) $\varrho(X) \geq r(I_X \cup J_X) - r(J_X)$ for all $X \in \mathcal{F}$.

Then there are pairwise disjoint arc-sets $A_1, \ldots, A_t \subseteq A$ such that

$$r(J_X \cup \{i \in I_X : \ \varrho_{A_i}(X) \geq 1\}) = r(J_X \cup I_X) \tag{1}$$

for every $X \in \mathcal{F}$.

Let us call a family of pairwise disjoint arc-sets $A_1, \ldots, A_t$ of A that satisfies (1) a **valid arc-cover**. In this note, we consider the algorithmic aspects of finding a valid arc-cover.

The rest of the paper is organized as follows. In Section 2, a polynomial-time algorithm is given for finding a valid arc-cover. In Section 3 we show that a minimum cost valid arc-cover can be found in polynomial time via the ellipsoid method; moreover, if $\mathcal{F}$ is an intersecting family, then there is a strongly polynomial algorithm by reduction to submodular flow.

2 Finding a valid arc-cover

Let V' be a copy of V, and identify $(X_I, X_O) \in \mathcal{P}_2(V)$ with $X_I \cup X'_O \in 2^{V \cup V'}$, where $X'_O \subseteq V'$ is the counterpart of $X_O \subseteq V$. Note that the union and intersection operations are consistent with this correspondence. Therefore, $\mathcal{F}_i \subseteq \mathcal{P}_2(V)$ can be regarded as an intersecting family of $2^{V \cup V'}$. In what follows in this section, we regard each bi-set in $\mathcal{P}_2(V)$ as a subset of $V \cup V'$. The definitions of $\mathcal{F}$, p, and active sets are the same as before.

To discuss polynomiality of the algorithm, we need a compact representation of each intersecting family $\mathcal{F}_i \subseteq 2^{V \cup V'}$. For $v \in V$, let $\mathcal{F}_i^v := \{X \mid v \in X \in \mathcal{F}_i\}$. Since $\mathcal{F}_i^v$ is closed under the union and intersection operations, it is a ring family (or a distributed lattice), which can be represented by a digraph whose vertex set is a subset of $V \cup V'$ by Birkhoff's Representation Theorem [3] (see also [12]). In what follows, we assume that we are given a digraph representation of $\mathcal{F}_i^v \subseteq 2^{V \cup V'}$ for $v \in V$, and $\mathcal{F}_i$ is given as $\bigcup_{v \in V} \mathcal{F}_i^v$. Note that the size of this representation is polynomial in $|V|$. We also assume the following:

1. $\mathcal{G}_i$ is given as an oracle. That is, for $X \subseteq V \cup V'$, we can check whether $X \in \mathcal{G}_i$ or not.

2. The rank function r of the matroid is given as an oracle. That is, for $S \subseteq [t]$, we can compute $r(S)$.

With these assumptions, we show the following.

Proposition 2. *In the statement of Theorem 1, suppose that $\mathcal{F}_i$, $\mathcal{G}_i$, and r are given as above. Assume that Condition* (a) *holds. Then, we can check whether Condition* (b) *holds or not in polynomial time.*

Proof. Let $\mathcal{F}_+ \subseteq \mathcal{F}$ be the set of all active sets, and for $v \in V$, let $\mathcal{F}_+^v := \{\bigcup X_i \mid v \in X_i \in \mathcal{F}_+ \text{ for each } i\}$.

Claim 3. *For each $v \in V$, $\mathcal{F}_+^v$ is a ring family, i.e., it is closed under the union and intersection operations. Furthermore, the digraph representation of $\mathcal{F}_+^v$ can be computed in polynomial time.*

Proof. It is obvious that $\mathcal{F}_+^v$ is closed under the union operation. For $X, Y \in \mathcal{F}_+^v$, suppose that $X = \bigcup X_i$ and $Y = \bigcup Y_j$, where $v \in X_i \in \mathcal{F}_+$ and $v \in Y_j \in \mathcal{F}_+$. By (AIP) (or [2, Proposition 2.2]), $X_i \cap Y_j \in \mathcal{F}_+$ for each i, j. Therefore,

$$X \cap Y = \bigcup_{i,j} (X_i \cap Y_j) \in \mathcal{F}_+^v,$$

which shows that $\mathcal{F}_+^v$ is closed under the intersection operation.

Since each irreducible element in $\mathcal{F}_+^v$ is irreducible also in $\mathcal{F}_i^v$ for some i, by checking whether each irreducible element in $\mathcal{F}_i^v$ is in $\mathcal{F}_+$ or not, we can enumerate all irreducible elements in $\mathcal{F}_+^v$ in polynomial time. Then we can obtain the digraph representation of $\mathcal{F}_+$. □

Claim 4. *For each $v \in V$, p is supermodular on the ring family $\mathcal{F}_+^v$.*

Proof. Let $X, Y \in \mathcal{F}_+^v$. We consider the following three cases.

1. If $p(X) > 0$ and $p(Y) > 0$, then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ by [2, Proposition 2.3].

2. If $p(X) = p(Y) = 0$, then it is obvious that $p(X) + p(Y) = 0 \leq p(X \cap Y) + p(X \cup Y)$.

3. Suppose that $p(X) > 0$ and $p(Y) = 0$. Since $X \cap Y \in \mathcal{F}_+^v$ by Claim 3, $X \cap Y$ can be represented as $\bigcup Z_j$, where $v \in Z_j \in \mathcal{F}_+$. By (AIP), Z_j is active for any $i \in I_X^{act}$, which implies that $X \cap Y = \bigcup Z_j \in \mathcal{F}_i \subseteq \mathcal{F}$. Then, by [2, Proposition 2.2] and Condition (a), we obtain $p(X) \leq p(X \cap Y)$. By combining this with $p(Y) = 0 \leq p(X \cup Y)$, we have $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$.

This completes the proof. □

The digraph D can also be regarded as a bipartite digraph on node set $V \cup V'$, where every arc has its head in V and its tail in V'. It is easy to check that this is consistent with the notion of covering. Condition (b) holds if and only if the condition

$$\varrho(X) - p(X) \geq 0 \text{ for every } X \in \mathcal{F}_+^v$$

holds for every $v \in V$, and this is equivalent to $\min_{X \in \mathcal{F}_+^v}\{\varrho(X) - p(X)\} \geq 0$ for every $v \in V$. Since $\varrho - p$ is a submodular function on $\mathcal{F}_+^v$, we can compute this minimum value in polynomial time by submodular function minimization algorithms (see [12, 13]). Therefore, Condition (b) can be checked in polynomial time. □

Corollary 5. *In the statement of Theorem 1, suppose that $\mathcal{F}_i$, $\mathcal{G}_i$, and r are given as above. Assume that Conditions* (a) *and* (b) *hold. Then, in polynomial time, we can find arc-sets $A_1, \dots A_t \subseteq A$ satisfying the conditions in Theorem 1.*

Proof. For any arc $a = uv \in A$ and for any index $i \in [t]$, we can check using the digraph representation of $\mathcal{F}_i^v$ whether the unique smallest bi-set in $\mathcal{F}_i$ that contains a is active for i. If it is not, then we try the following procedure:

- adding a to A_i and removing it from A;
- adding bi-sets in $\mathcal{F}_i$ covered by a to $\mathcal{G}_i$;
- deleting bi-sets covered by a from $\mathcal{F}_i$.

The new instance obtained by this procedure still satisfies the active intersection property because a is not contained in any bi-set active for i. The new digraph representations for $\mathcal{F}_i$ and the oracle for $\mathcal{G}_i$ are easy to construct using the original ones.

It is possible that the new instance does not satisfy Condition (b). However, by the results of [2], there exists a pair (a, i) such that Condition (b) still holds after this procedure, and by Proposition 2 we can check it in polynomial time. Thus we can check all pairs (a, i) and choose one where Condition (b) is satisfied in the new instance. By repeating this procedure, A becomes empty and the obtained arc sets $A_1, \dots A_t$ satisfy the conditions in Theorem 1. □

Remark 6. *Note that in the case of [2, Theorem 3.2], the arcs can be chosen in such an order that each A_i is an arborescence at any given point in the algorithm. This gives an algorithmic proof of the theorem of Cs. Király.*

3 Finding a minimum cost arc-cover

The main tool used in this section is the following theorem of Frank [7], see also [8, Theorem 17.1.11].

Theorem 7 (Frank)**.** *Let $D = (V, A)$ be a digraph, q a positively intersecting supermodular bi-set function on V, and $g \in \mathbb{Z}_+^A$ a nonnegative upper bound on the arcs. Then the system*

$$\{x \in \mathbb{R}^A : 0 \leq x_a \leq g_a \ \forall a \in A,\ \varrho_x(Z) \geq q(Z) \ \forall Z \in \mathcal{P}_2(V)\}$$

is total dual integral. If q is the nonnegative part of an intersecting supermodular bi-set function, then the system defines a submodular flow polyhedron.

We first describe a polynomial-time algorithm that uses the ellipsoid method.

Theorem 8. *Let $\mathcal{M} = ([t], r)$ be a matroid, $D = (V, A)$ a digraph and $c : A \to \mathbb{R}$ a cost function. Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be intersecting families, $\mathcal{G}_1, \ldots, \mathcal{G}_t$ arbitrary bi-set families over $\mathcal{P}_2$ such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

(a) $I_{X'} \cup J_{X'} = I_X \cup J_X$ for all $X, X' \in \mathcal{F}, X' \subseteq X$.

We can find a minimum cost arc set $A' \subseteq A$ that satisfies

$$\varrho(X) \geq p(X) \text{ for every } X \in \mathcal{F} \tag{2}$$

in polynomial time.

Proof. Let

$$P = \{x \in \mathbb{R}^A : 0 \leq x_a \leq 1 \ \forall a \in A,\ \varrho_x(Z) \geq p(Z) \ \forall Z \in \mathcal{F}\}.$$

The set function p is positively intersecting supermodular by the results of [2], so by Theorem 7 the system of inequalities defining P is TDI. As a consequence, the polyhedron P is integer, so finding a minimum cost arc-cover amounts to optimization over P. By Proposition 2, there is a polynomial-time separation algorithm for the linear system defining P, thus we can optimize on P in polynomial time using the ellipsoid method by the results of Grötschel, Lovász and Schrijver [9]. □

It is left as an open question whether there is a strongly polynomial combinatorial algorithm for finding the minimum cost arc-cover. However, in the special case when $\mathcal{F}$ is an intersecting family, this is possible using submodular flows.

Theorem 9. *Let $\mathcal{M} = ([t], r)$ be a matroid, $D = (V, A)$ a digraph and $c : A \to \mathbb{R}$ a cost function. Let $\mathcal{F}$ be an intersecting family over $\mathcal{P}_2$, $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be intersecting subfamilies of $\mathcal{F}$, $\mathcal{G}_1, \ldots, \mathcal{G}_t$ arbitrary bi-set families over $\mathcal{P}_2$ such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

(a) $I_{X'} \cup J_{X'} = I_X \cup J_X$ for $X, X' \in \mathcal{F}, X' \subseteq X$.

We can find a minimum cost arc set $A' \subseteq A$ that satisfies

$$\varrho(X) \geq p(X) \text{ for every } X \in \mathcal{F} \tag{3}$$

in strongly polynomial time.

Proof. The following proposition was proved in [2] (Proposition 2.2).

Proposition 10. *If $X, X' \in \mathcal{F}$ and $X' \subseteq X$, then $J_{X'} \subseteq \mathtt{Span}(J_X)$.*

Let

$$p'(X) = \begin{cases} r(I_X \cup J_X) - r(J_X) & \text{if } X \in \mathcal{F}, \\ -\infty & \text{otherwise.} \end{cases}$$

The main advantage of $\mathcal{F}$ being intersecting is that it implies the intersecting supermodularity of p'.

Proposition 11. *The bi-set function p' is intersecting supermodular on $\mathcal{P}_2(V)$.*

Proof. Let X and Y be intersecting bi-sets. If $p'(X)$ or $p'(Y)$ is $-\infty$ then the supermodular inequality trivially holds.

If $p'(X), p'(Y) \geq 0$ then $X, Y \in \mathcal{F}$. As $\mathcal{F}$ is intersecting, $X \cap Y$ and $X \cup Y$ are also in $\mathcal{F}$. Note that Proposition 10 implies $J_{X \cap Y} \subseteq \mathtt{Span}(J_X) \cap \mathtt{Span}(J_Y)$. As the families $\mathcal{F}_i$ are also intersecting, we have $I_{X \cup Y} \supseteq I_X \cap I_Y$. This, together with (a), implies $J_{X \cup Y} \subseteq J_X \cup J_Y$. Thus

$$\begin{aligned} r(J_X) + r(J_Y) &= r(\mathtt{Span}(J_X)) + r(\mathtt{Span}(J_Y)) \\ &\geq r(\mathtt{Span}(J_X) \cap \mathtt{Span}(J_Y)) + r(\mathtt{Span}(J_X) \cup \mathtt{Span}(J_Y)) \\ &\geq r(J_{X \cap Y}) + r(J_{X \cup Y}). \end{aligned}$$

Using this and (a) we get

$$\begin{aligned} p'(X) + p'(Y) &= r(I_X \cup J_X) - r(J_X) + r(I_Y \cup J_Y) - r(J_Y) \\ &\leq r(I_{X \cap Y} \cup J_{X \cap Y}) - r(J_{X \cap Y}) + r(I_{X \cup Y} \cup J_{X \cup Y}) - r(J_{X \cup Y}) \\ &= p'(X \cap Y) + p'(X \cup Y). \end{aligned}$$

□

As p is the nonnegative part of p', Theorem 7 of Frank implies that the system

$$\{x \in \mathbb{R}^A : 0 \leq x_a \leq 1 \ \forall a \in A, \ \varrho_x(Z) \geq p(Z) \ \forall Z \in \mathcal{P}_2(V)\}$$

describes a submodular flow polyhedron. Moreover, the proof of Frank gives a construction of a corresponding submodular flow problem that in our case can be solved in strongly polynomial time, since the support of p' is given by the digraph representations. □

References

[1] K. Bérczi and A. Frank. Variations for Lovász' submodular ideas. *Building Bridges*, pages 137–164, 2008.

[2] K. Bérczi, T. Király, and Y. Kobayashi. Covering intersecting bi-set families under matroid constraints. Technical Report TR-2013-06, Egerváry Research Group, Budapest, 2013. `www.cs.elte.hu/egres`.

[3] G. Birkhoff. *Lattice theory*, volume 25. American Mathematical Soc., 1967.

[4] O. Durand de Gevigney, V. H. Nguyen, and Z. Szigeti. Basic packing of arborescences. *SIAM Journal of Discrete mathematics*, 27(1):567–574, 2013.

[5] J. Edmonds. Edge-disjoint branchings. *Combinatorial Algorithms*, 9:91–96, 1973.

[6] A. Frank. Kernel systems of directed graphs. *Acta Scientiarum Mathematicarum (Szeged)*, 41(1-2):63–76, 1979.

[7] A. Frank. Kernel systems of directed graphs. *Actu Sci. Math. Szeged*, 41(l-2):63–76, 1979.

[8] A. Frank. *Connections in combinatorial optimization*, volume 38. OUP Oxford, 2011.

[9] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[10] N. Katoh and S. ichi Tanigawa. Rooted-tree decompositions with matroid constraints and the infinitesimal rigidity of frameworks with boundaries. *SIAM Journal on Discrete Mathematics*, 27(1):155–185, 2013.

[11] Cs. Király. On maximal independent arborescence-packing. Technical Report TR-2013-03, Egerváry Research Group, March 2013.

[12] S. T. McCormick. Submodular function minimization. *Handbooks in operations research and management science*, 12:321–391, 2005.

[13] A. Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory, Series B*, 80(2):346–355, 2000.

[14] L. Szegő. On covering intersecting set-systems by digraphs. *Discrete Mathematics*, 234(1):187–189, 2001.