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$\mathcal{PT}$ symmetry in Natanzon-class potentials

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Abstract The conditions under which $\mathcal{PT}$ symmetry can be applied to the general six-parameter Natanzon potential class are investigated. For this the transformation of the differential equation of the Jacobi polynomials to the Schrödinger equation is considered in its most general form. The parity and $\mathcal{PT}$-parity properties of the $y(x)$ function that is responsible for the transformation are studied in order to implement the $\mathcal{PT}$ symmetry of the potential $V(x)$. Situations in which the bound-state energy eigenvalues can or cannot become complex are identified. A number of known Natanzon-class potentials are analyzed. As a by-product, the relation of two variable transformation methods is clarified.

Keywords $\mathcal{PT}$ symmetry · Solvable potentials · Discrete energy eigenvalues

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1 Introduction

The introduction of the concept of $\mathcal{PT}$-symmetric quantum mechanics in 1998 [1] signaled the start of a remarkable and rapidly growing chapter of quantum mechanics. Born as an explanation of a humble mathematical physical curiosity, i.e. that an imaginary potential can support a fully or partly real discrete energy spectrum, $\mathcal{PT}$-symmetric quantum mechanics initiated renewed interest in non-hermitian quantum mechanical systems. $\mathcal{PT}$-symmetric quantum

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mechanics (for a recent review see [2]) was identified as a special case of pseudo-hermiticity [3]. The mapping of $\mathcal{PT}$-symmetric Hamiltonians into equivalent hermitian ones and describing the spontaneous breakdown of $\mathcal{PT}$ symmetry became central issues of theoretical investigations, which gradually extended to many fields of quantum pysics. Later experimental studies also started, leading to the the verification of the existence of $\mathcal{PT}$-symmetric systems in nature, as well as its spontaneous breakdown [4].

Considering non-relativistic quantum mechanics, $\mathcal{PT}$ symmetry implies that the potential appearing in the Schrödinger equation has to satisfy $V^*(-x) = V(x)$, i.e. it has to be invariant under the simultaneous action of the $\mathcal{P}$ space and the $\mathcal{T}$ time reflection operations. In practice this means that the real and the imaginary potential components have to be even and odd functions of x, respectively.

Although the first examples for $\mathcal{PT}$-symmetric potentials were solved by numerical techniques, the investigation of exactly solvable $\mathcal{PT}$-symmetric potentials started soon. It was noticed that techniques used in conventional quantum mechanics to generate exactly solvable potentials can be adapted to the $\mathcal{PT}$-symmetric setting too. Exact results have been obtained for bound states with real [5] and imaginary [6] energy eigenvalues, the mechanism of the spontaneous breakdown $\mathcal{PT}$ symmetry [7], the pseudo-norm of wavefunctions [8–11], the $\mathcal{C}$ operator [12], spectral singularities [13], supersymmetric [14,15], higher dimensional [16–18] and algebraic [19–21] aspects, etc.

The simplest textbook examples for exactly solvable potentials are shape-invariant potentials [22] (e.g. harmonic oscillator, Coulomb, Scarf, Pöschl–Teller, Morse, etc.). These form a subclass of the Natanzon and Natanzon confluent potential class [23,24], which contain the most general (six-parameter) potentials solvable in terms of the hypergeometric and confluent hypergeometric functions, respectively. There are further solvable potentials outside the Natanzon class, e.g. those solved in terms of exceptional orthogonal polynomials [25], Bessel functions [26], etc.

The first examples studied in terms of $\mathcal{PT}$-symmetric quantum mechanics were shape-invariant potentials [5,6]. The investigations were extended to some Natanzon-class potentials too [27–29] and even to potentials solved in terms of exceptional orthogonal polynomials [30]. It was found that some potentials have strictly real energy spectrum, while others support complex conjugate energy eigenvalues too. In fact, in all these potentials, with the exception of that in Ref. [29] the spontaneous breakdown of $\mathcal{PT}$ symmetry occurs via the *sudden* mechanism, i.e. all the real energy eigenvalues turn into complex at the same value of the parameter controling the odd imaginary potential term. It was also observed that in this mechanism levels with complex conjugate energy eigenvalues develop from the merging of two real energy eigenvalues carrying the same n principal quantum number, but different quasi-parity [15, 28,11,31,32].

The scarce and inconclusive results concerning $\mathcal{PT}$-symmetric Natanzon-class potentials call for a systematic study of these potentials. The aim of the present work is to discuss the properties of these potentials in a unified way,

based on the mathematical structure of the Jacobi polynomials (to which hypergeometric functions reduce in bound-state solutions) and the prescriptions enforced by $\mathcal{PT}$ symmetry on the parameters. Special attention is paid to parity and $\mathcal{PT}$-parity considerations not only of the potential $V(x)$, but also of other functions arising in the mathematical formulation.

The structure of the paper is as follows. In Section 2 an old method of constructing solvable potentials is outlined, it is specified for Natanzon-class potentials and its formulation is compared with that of Natanzon's original approach. Section 3 discusses the conditions under which these potentials or their terms can exhibit parity (i.e. $\mathcal{P}$ symmetry) and/or $\mathcal{PT}$ symmetry. The results are analyzed in Section 4 with special attention to their implications on the nature of the energy spectrum.

2 Generating exactly solvable potentials

The solutions of the most well-known potentials can be obtained by transforming the Schrödinger equation into the differential equation of some special functions of mathematical physics. The procedure presented here was first used to derive some simple potentials [33], but later it was developed further by Natanzon who applied it systematically to transform the Schrödinger equation into the differential equation of the hypergeometric and confluent hypergeometric functions [23]. Following the discussion of [35,36], let us consider transformation of the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \tag{1}$$

into the second-order differential equation of a special function $F(z)$,

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + Q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + R(z)F(z) = 0 \ . \tag{2}$$

For this, we search for the solutions in the form

$$\psi(x) = f(x)F(z(x)) \ , \tag{3}$$

At the moment we do not specify the domain of definition for the coordinate x itself. Later on we shall come back to this issue and its importance for $\mathcal{PT}$ symmetric problems.

Straightforward calculations lead to the equation

$$\begin{aligned} E - V(x) = {} & \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ & +(z'(x))^2\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right) . \end{aligned} \tag{4}$$

Besides $Q(z)$ and $R(z)$ defining the special function $F(z)$, (4) contains only the function representing a variable transformation, $z(x)$. This also applies to the solutions themselves:

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)\mathrm{d}z\right) F(z(x)) \ . \tag{5}$$

We are left with the task of finding a functional form of $z(x)$ which transforms the Schrödinger equation (4) into an exactly solvable problem.

In principle any randomly chosen $z(x)$ function satisfies this requirement for a particular potential $V(x)$ and energy E. However, it cannot be guaranteed in general that any other physical solution of the same physical problem can be found in the same manner as well. In this perspective, a useful way of finding reasonable $z(x)$ functions has been proposed by Bhattacharjie and Sudarshan [33]. According to them, if there is a constant (E) on the left-hand side of (4), then there must be one on the right-hand side too. In Ref. [35] this fact was exploited, and a systematic list of potentials was compiled by identifying certain terms found on the right-hand side of (4) with a constant C. This assignment leads to first-order differential equations for z of the type

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \Phi(z) = C \ , \tag{6}$$

where $\Phi(z)$ is a function of z originating from $R(z)$ or $Q(z)$. As we shall see later on, reasonable results arise generally for real values of C.

The general solution of the latter differential equation is given by formula

$$\int \Phi^{1/2}(z)\mathrm{d}z = C^{1/2}x + x_0 \ . \tag{7}$$

This defines an implicit function $x(z)$ and, in many cases of practical interest, also the explicit $z(x)$ function we need [35]. When only $x(z)$ is available one has the case of implicit potentials. Despite the lack of the explicit $z(x)$ function, the usual calculations (integrations, derivations) can be performed in this case too. Usually $x_0 = 0$ is considered in order to set $z(0) = 0$. The $x_0 \neq 0$ choice corresponds to a shift of the coordinate and reflects a trivial and also rarely relevant transformation for potentials defined on the real x axis, but we shall find it important in connection with $\mathcal{PT}$ symmetric potentials discussed later.

2.1 $F(y) = P_n^{(\alpha,\beta)}(y)$: Jacobi polynomials

Applying the method to Jacobi polynomials with $Q(y) = [(\beta-\alpha)-(\alpha+\beta+2)y]/(1-y^2)$ and $R(y) = n(n+\alpha+\beta+1)/(1-y^2)$ Eq. (4) takes the form [35]

$$E_n - V(x) = \frac{y'''(x)}{2y'(x)} - \frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2$$

$$
\begin{aligned}
&+\frac{(y'(x))^2}{1-y^2(x)}\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right)\\
&+\frac{(y'(x))^2}{(1-y^2(x))^2}\left[1-\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right]\\
&-\frac{2y(x)(y'(x))^2}{(1-y^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \qquad (8)
\end{aligned}
$$

The most general form of $\Phi(y)$ appearing in Eq. (7) is

$$
\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\Phi(z)\equiv\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\frac{\phi(y)}{(1-y^2(x))^2}\equiv\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\frac{p_{\mathrm{I}}(1-y^2)+p_{\mathrm{II}}+p_{\mathrm{III}}y}{(1-y^2(x))^2}=C\ , \qquad (9)
$$

as it allows selecting the most general combination of the three independent α- and β-dependent terms as a constant standing for the energy eigenvalue E_n. Combining Eqs. (8) and (9) and replacing the parameters α and β by $\omega=(\alpha+\beta)/2$ and $\rho=(\alpha-\beta)/2$ one obtains

$$
\begin{aligned}
E-V(x)=&\frac{y'''(x)}{2y'(x)}-\frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2\\
&+\frac{C}{\phi(y)}\left[(1-y^2(x))\left((n+\frac{1}{2}+\omega)^2-\frac{1}{4})\right)\right.\\
&\left.+\left(1-\omega^2-\rho^2\right)-2\omega\rho y(x)\right], \qquad (10)
\end{aligned}
$$

while Eq. (5) results in

$$
\begin{aligned}
\psi(x)&\simeq(y'(x))^{-\frac{1}{2}}(1+y(x))^{\frac{\beta+1}{2}}(1-y(x))^{\frac{\alpha+1}{2}}P_n^{(\alpha,\beta)}(y(x)) \qquad (11)\\
&\simeq(\phi(y(x)))^{\frac{1}{4}}(1+y(x))^{\frac{\beta}{2}}(1-y(x))^{\frac{\alpha}{2}}P_n^{(\alpha,\beta)}(y(x))\ . \qquad (12)
\end{aligned}
$$

In the simplest case only one of the p_i parameters is zero in $\phi(y)$. This is the case for the choices $p_{\mathrm{I}}\neq 0$, $p_{\mathrm{II}}=p_{\mathrm{III}}=0$ and $p_{\mathrm{II}}\neq 0$, $p_{\mathrm{I}}=p_{\mathrm{III}}=0$, which recover the potentials belonging to the PI and PII classes, respectively [35]. Both classes include three independent shape-invariant [22] potentials that originate from different solutions of the actual form of the differential equation (9). Besides these six potentials displayed in Table 1, sometimes two more PI-class potentials are also mentioned, however, the Pöschl–Teller I and II potentials can be obtained from the Scarf I and the generalized Pöschl–Teller potentials by the trivial $x\to 2x$ transformation [35]. Note that in the PI case the n quantum number appears only in the constant term corresponding to the energy E_n, and this implies that the remaining terms that contribute to the potential $V(x)$ will be free from n. This also means that ω and ρ (and thus α and β) are also independent of n. This is not the case for the PII class potentials [35]. The parameters defining the PI and PII class potentials are displayed in Table 1, along with some more general Natanzon-class potentials described earlier [37,38,34,39–42,29]. Some of these potentials are reduced to some shape-invariant potentials by tuning the parameters p_i in an appropriate

way. The potential in [29] even contains all six shape-invariant potentials as a secial limit, plus the DKV potential [39,40] as another non-shape-invariant limit. Such limits are not available when some of the p_i parameters take on a constant numerical value.

The most general potential in (8) contains the Schwartzian derivative and the three remaining terms with some coupling coefficients:

$$V(x) = -\frac{y'''(x)}{2y'(x)} + \frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2 + \frac{C}{\phi(y)}\left[s_I(1-y^2(x)) + s_{II} + s_{III}y(x)\right] . \quad (13)$$

Substituting (13) in Eq. (10) and multiplying by $\phi(y)$ the three independent terms each have to vanish, leading to the equations

$$(n+\frac{1}{2}+\omega)^2 - \frac{1}{4} + s_I - p_I\frac{E_n}{C} = 0 \ , \quad (14)$$

$$\left(1-\omega^2-\rho^2\right) + s_{II} - p_{II}\frac{E_n}{C} = 0 \ , \quad (15)$$

$$-2\omega\rho + s_{III} - p_{III}\frac{E_n}{C} = 0 \ . \quad (16)$$

These three equations have to be satisfied simultaneously. Solving them means that one obtains ω and ρ in terms of n and the p_i and s_i parameters, as well as expressing E_n in terms of all these quantities, plus the n principal quantum number. One can express E_n/C from one equation and substitute it into the remaining two. In this way the chosen equation will supply the formula for the energy eigenvalues, while from the other two ω and ρ can be determined. The method used in Ref. [33] corresponds to identifying a given combination of terms in (8) to obtain the constant E. The three equations can be rewritten to depend on α and β rather than on ω and ρ: for this Eqs. (15) and (16) have to be added and subtracted in order to get $(\omega \pm \rho)^2$, i.e. α^2 and β^2.

Before continuing it is worthwhile to mention that a similar treatment of the generalized Laguerre polynomials recovers three shape-invariant potential classes [35], the radial harmonic oscillator (LI), the Coulomb (LII) and the Morse (LIII) potentials, while the generalized Coulomb potential [43,44] contains both the radial harmonic oscillator and the Coulomb potentials as shape-invariant limits.

2.2 ${}_2F_1(z) = F(a,b;c;z)$: the hypergeometric function – Natanzon's approach

Here we discuss Natanzon's approach [23] to potentials the solutions of which are given in terms of to the hypergeometric function. These potentials are called Natanzon-class potentials and depend on six potential parameters. Here we review the essential formule and outline the connection of Natanzon's approach with that described in the preceding subsection. This is based on the close relation of the hypergeometric function and the Jacobi polynomials, i.e.

that the latter is obtained from the former if $a = -n$ (or $b = -n$) holds [45]. The most general Natanzon-class potential can be written as [23]

$$V(z(x)) = -\frac{1}{2}\frac{z'''}{z'} + \frac{3}{4}\left(\frac{z''}{z'}\right)^2 + \frac{fz(z-1) + h_0(1-z) + h_1 z}{\mathcal{R}(z)}, \tag{17}$$

where

$$\mathcal{R}(z) = a_1 z(z-1) + c_1 z + c_0(1-z). \tag{18}$$

$z(x)$ is then determined from the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2z(1-z)}{\mathcal{R}^{1/2}(z)}, \tag{19}$$

which is obtained from the current version of (9) after identifying the linear combination of three independent terms on its right-hand side with a constant. The three coefficients (a_1, c_1 and c_0) govern the behaviour of $z(x)$ and supply three of the six potential parameters. (The other three parameters are f, h_0 and h_1 in (17).) In general, there is no explicit expression for the energy eigenvalues, rather they can be determined from the implicit formula

$$\begin{aligned} 2n+1 &= (f+1-a_1E_n)^{1/2} - (h_0+1-c_0E_n)^{1/2} - (h_1+1-c_1E_n)^{1/2} \\ &\equiv \alpha_n - \beta_n - \delta_n, \end{aligned} \tag{20}$$

while the wavefunctions are written as

$$\psi_n(x) \simeq \mathcal{R}^{1/4}(z(x))(z(x))^{\beta_n/2}(1-z(x))^{\delta_n/2}\,{}_2F_1(-n, \alpha_n - n; \beta_n + 1; z(x)). \tag{21}$$

The similarity between Natanzon's approach and that originating from Bhattacharjie and Sudarshan [33] and outlined in Subsection 2.1 for the Jacobi polynomials is striking. Actually, the two methods can be rewritten into each other in this case using the following relations: $z(x) = (1-y(x))/2$, $\mathcal{R}(z(x)) = \phi(y(x))/C$, $a_1 = -4p_I/C$, $c_1 = (p_{II} - p_{III})/C$, $c_0 = (p_{II} + p_{III})/C$, $f = -4s_I$, $h_0 = s_{II} - s_{III}$, $h_1 = s_{II} + s_{III}$, $\beta_n = \alpha$ and $\delta_n = \beta$. The difference arises in the methods used in the two approaches to express the energy eigenvalue E_n. In [23] the square of δ_n, β_n and $2n+1+\delta_n+\beta_n$ are expressed in terms of the energy eigenvalue E_n, and this is how the implicit expression for E is obtained in Eq. (20). (The three expressions correspond to Eqs. (14), (15) and (16) in the present approach.) Alternatively, in Subsection 2.1 E was identified with a constant expression in the right handside of (10). As it was demonstrated there, the key formula of Natanzon's approach, Eq. (20) can be obtained in this way too. The two methods have their advantages in certain situations.

3 Parity and $\mathcal{PT}$-parity considerations for $y(x)$, $\phi(y(x))$ and $V(x)$

In this Section we discuss the conditions under which one can derive $\mathcal{PT}$-symmetric potentials. For this first we analyze the parity of the potential terms appearing in (10). It will be seen that the use of Jacobi polynomials is more suitable than that of the hypergeometric functions and Natanzon's approach, because the parity of the general expressions is easier to follow if $y(x)$ has definite parity. Although the two methods are pratically equivalent, in the Natanzon approach $z(x) = (1 - y(x))/2$ does not have definite parity in general.

3.1 Parity considerations

Let us see now under what conditions the potential (13) has terms that possess definite parity. From the derivations in the preceding section one may see that this potential can be separated into a constant, two terms that have independent coupling constant (s_i), and some further terms arising from the Schwartzian derivative $-y'''/2y'+3(y'')^{/}4(y')^2$ (that may coincide with the two potential terms mentioned before). These terms contain only the p_i parameters via (9). It can also be demonstrated after consecutive derivations of $y'(x)$ that the Schwartzian derivative can be expressed in terms of fractions containing up to quadratic expressions of $y(x)$ in the numerator and $\phi^k(y(x))$, $k = 1, 2, 3$ in the denominator.

Let us assume that $y(x)$ has definite parity, i.e. $\mathcal{P}y(x) \equiv y(-x) = \pm y(x)$. It is easy to see that the potential terms arising from the Schwartzian derivative contribute strictly to the even potential component. The parity of the terms proportional with the s_i parameters depends both on the parity of $y(x)$ and that of $\phi(y)$. The parity of this latter quantity is given by

$$\phi(y(x)) \text{ is } \begin{cases} \text{even} & \text{if } y(x) \text{ is even} \\ \text{even} & \text{if } y(x) \text{ is odd and } p_{III} = 0 \\ \text{odd} & \text{if } y(x) \text{ is odd and } p_I = p_{II} = 0 \\ \text{indefinite} & \text{if } y(x) \text{ has no definite parity} \end{cases}$$

Since the only case when $\phi(y(x))$ could be an odd function corresponds to the PIII potential [34], in which case $y(x)$ has no definite parity, we conclude that $\phi(y(x))$ has either even or indefinite parity.

The parities of $y(x)$ and that of $\phi(z(x))$, if they exist, are interrelated by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = C^{1/2}(1 - y^2(x))[\phi(y(x))]^{-1/2} \ , \tag{22}$$

which is derived from Eq. (6) for the Jacobi polyinomial case. This relation implies that $y'(x)$ and $[\phi(y(x))]^{1/2}$ have to possess the same parity, i.e. the parity of $y(x)$ and $[\phi(y(x))]^{1/2}$ have to be opposite. This requirement does not lead to conflict in general, except for $[\phi(y)]^{1/2} = y$, when it results in a

contradiction. This corresponds to $p_I = -1$, $p_{II} = 1$ and $p_{III} = 0$, i.e. the case of the DKV potential [39,40,27], when the parity of $y(x)$ is indefinite.

The parity of the significant potential terms can be defined only in case both $y(x)$ and $\phi(x)$ have definite parity. As we have seen, if $y(x)$ is an even function, then $\phi(y(x))$ and all the terms in (13) will be even functions of x, so $V(x)$ will necessarily have even parity.

Odd-parity terms can appear in $V(x)$ only if $y(x)$ is odd. In this case terms proportional with s_I and s_{II} have the same parity as $\phi(y(x))$, while the parity of the term proportional with s_{III} is obtained as the product of the parity of $\phi(y(x))$ and that of $y(x)$.

A continuous odd $y(x)$ function satisfies $y(0) = y_0 = 0$. From (9) it is seen that $y'(x)$ must be finite at $x = 0$, except for the exceptional case of $p_I = -p_{II}$, when $\phi(y_0) = 0$, indicating a singularity of the potential there. Parity properties of $y(x)$ and $\phi(y)$ are displayed in Table 2 for the Natanzon-class potentials listed in Table 1.

3.2 $\mathcal{PT}$-parity considerations

Let us first assume that $y(x)$ has definite ($\mathcal{P}$) parity. In this case $\mathcal{PT}y(x) \equiv y^*(-x) = y(x)$, i.e. $y(x)$ is $\mathcal{PT}$-even if $y(x)$ is real and even, or if it is imaginary and odd. Similarly, $\mathcal{PT}y(x) \equiv y^*(-x) = -y(x)$ holds if $y(x)$ is real and odd, or if it is imaginary and even. However, $\mathcal{PT}$ parity does not require definite ($\mathcal{P}$) parity: an imaginary coordinate shift $y(x + \mathrm{i}\epsilon)$ cancels the parity, but leaves $\mathcal{PT}$-parity intact. This imaginary cordinate shift can be considered as the $x_0 = \mathrm{i}\epsilon C^{-1/2}$ constant of integration in Eq. (7). In the case of real potentials it can be set to zero without the loss of generality. In the $\mathcal{PT}$-symmetric setting, however, it can play an important role, as it cancels singularities at the origin. This means that radial potentials defined originally on the positive real axis can be extended to the domain of negative x values, making it possible to define the $\mathcal{PT}$-parity of such problems [5,6]. It has to be noted that although the shape of $V(x)$ may change significantly after implementing the imaginary coordinate shift, the energy spetrum remains the same, just as is the case for real coordinate shifts. It is also worthwhile to note that this transformation is a rather simple version of defining $V(x)$ on various trajectories in the complex x plane in order to secure the normalizability of the wave functions [1,11].

If $y(x)$ has definite $\mathcal{PT}$-parity, then the same holds for its derivatives. In particular, its even derivatives $y^{(2k)}(x)$ have the same $\mathcal{PT}$-parity, while its odd derivatives $y^{(2k+1)}(x)$ have opposite $\mathcal{PT}$-parity. $y^2(x)$ is always $\mathcal{PT}$-even. From these and Eq. (9) it follows that $C/\phi(y(x))$ also has to be $\mathcal{PT}$-even, i.e. if C is real, then $\phi(y(x))$ has to be $\mathcal{PT}$-even. Since $C^{1/2}$ appears in Eq. (7) as a parameter scaling the x coordinate, a general complex C is not compatible with $\mathcal{PT}$ symmetry. However, C can be negative, in which case the scaling factor becomes imaginary. Such a situation occurs for several potentials, and the formulas can generally be rewritten into alternative forms in which i is separated from x.

Assuming that $y(x)$ has definite $\mathcal{PT}$-parity, the $\mathcal{PT}$-even requirement of $\phi(y(x))$ implies

$$\begin{aligned}\mathcal{PT}\phi(y(x)) &= p_I^*(1-(y^*(-x))^2)+p_{II}^*+p_{III}^*y^*(-x)\\ &= p_I^*(1-y^2(x))+p_{II}^*\pm p_{III}^*y(x).\end{aligned} \tag{23}$$

From this it is seen that $\phi(y(x))$ can be $\mathcal{PT}$-symmetric (i.e. can have even $\mathcal{PT}$-parity): *i)* if $y(x)$ is $\mathcal{PT}$-even and all the p_i are real, and *ii)* if $y(x)$ is $\mathcal{PT}$-odd, p_{III} is imaginary, while p_I and p_{II} are real.

Given a $y(x)$ with even or odd $\mathcal{PT}$-parity and a $\mathcal{PT}$-even $\phi(y(x))$ function, the $\mathcal{PT}$ symmetry of $V(x)$ in (13) can be determined in a straightforward way. Recalling the arguments concerning the derivatives of $y(x)$ one can establish that the Schwartzian derivative contains only $\mathcal{PT}$-even potential terms. The three terms containing the s_i parameters also have this property if s_I and s_{II} are real and s_{III} is real or imaginary, depending on whether the $\mathcal{PT}$-parity of $y(x)$ is even or odd, respectively. The s_i coefficients in (13) thus have to obey the same conditions as the p_i in $\phi(y(x))$: this is the consequence that both $\phi(y(x))$ and $V(x)$ have to be $\mathcal{PT}$-even.

4 Implications on the discrete energy spectrum

Now let us use the results of Section 3 to discuss the general features of $\mathcal{PT}$-symmetric Natanzon-class potentials and their discrete energy eigenvalues. These potentials depend on six essential parameters, which are strictly separated into two types, both having their characteristic role in determining the character of the potential. The s_i parameters play the role of cupling coefficients in the independent terms of the potential $V(x)$, while the p_i parameters define the variable transformation $y(x)$. Besides them there are also C and x_0: the former one, $C=\pm a^2$ merely scales the energy (as a does the same for the coordinate), while x_0 corresponds to a coordinate shift, which rarely plays an important role for real potentials, whereas it can have an important role in the case of $\mathcal{PT}$-symmetric potentials. Let us now discuss the importance of the s_i and p_i parameters separately.

From (13) it is seen that the three s_i parameters correspond to three significant potential terms, however, due to the structure of $\phi(y(x))$ only up to two such terms appear in $V(x)$, and one parameter gives a constant term. This latter term is formally a part of $V(x)$, but it could also be defined as part of the energy eigenvalue E after modifying the zero point of the energy scale. In the case of $\mathcal{PT}$-symmetric potentials it has to be real. If $y(x)$ has definite parity, then one of the two independent potential terms is always an even function of x, while the other one carries the parity of $y(x)$. The potential may contain further even potential terms originating from the Schwartzian derivative, however, these have fixed coupling coefficients, so they cannot be considered significant (tunable) independent terms.

The p_i parameters define the $y(x)$ variable transformation via (9). They may have fixed values or can act as tunable parameters, furthermore, they

can be correlated with each other. These circumstances have influence on the structure of the inverse $x(y)$ function, which can be obtained by direct integration from Eq. (7). In the case of shape-invariant potentials only p_I or p_{II} is non-zero, while for more general Natanzon-class potentials either p_{III} or at least two of the p_i are non-zero.

The p_i parameters also appear in Eqs. (14), (15) and (16), where they occur together with E_n in the $p_i E_n/C$ combination. More non-zero p_i parameter obviously complicates the determination of the energy eigenvalues in terms of the parameters and the other variables. Looking at the same problem from the perspective of Natanzon's approach in Subsection 2.2 one may also conclude that fewer or more correlated parameters may lead to a simpler determination of E_n from Eq. (20).

In the case of $\mathcal{PT}$-symmetric potentials complex conjugate energy eigenvalues E_n can also occur besides the real ones. Equations (14), (15) and (16) are indicative from this point of view too: a complex E_n hints at the presence of futher coplex quantities in the equations. It was established before that the $\mathcal{PT}$ symmetry of $V(x)$ requires real values of these parameters in general: only p_{III} and s_{III} can be imaginary in case $y(x)$ is $\mathcal{PT}$-odd. It is seen from Table 2 that this is a rather exceptional situation. Besides p_{III} and s_{III} only ω and ρ remain as possible complex quantities. Taking the complex conjugate of Eq. (14) one finds that the complex conjugate energy eigenvalue E^* can originate from ω^* (at least when $p_I \neq 0$). This also holds for Eq. (15) if ρ^2 is real, i.e. if ρ is real or imaginary. Equation (16) deserves more attention, because there p_{III} and s_{III} can be imaginary if $y(x)$ is $\mathcal{PT}$-odd. Dividing Eq. (16) with $p_{III} \neq 0$ we find that s_{III}/p_{III} is always real, so the nature of ρ/p_{III} is crucial. If this quantity is real, then again, E^* originates from ω^*.

The situation is simplified significantly if $p_{III} = 0$. Then from Eq. (16) it follows that $\rho = -(2s_{III}\omega)^{-1}$, so Eq. (15) leads to an equation from which ω can be determined as the function of n. This equation is a quartic algebraic equation with real coefficients (and missing linear term) and was derived in Ref. [29]. It has to be solved numerically, and its roots supply the ω_n variables necessary to evaluete the energy eigenvalues from (14). It was found that in a $\mathcal{PT}$-symmetric version of this potential complex conjugate roots appear starting with $n = 0$, which lead to complex conjugate energy eigenvalues, i.e. to the spontaneous breakdown of $\mathcal{PT}$ symmetry occurs.

There are also situations when ω, and thus E_n has to be real. This is the case, for example, for the PII-type potentials, when $p_I = p_{III} = 0$ and $1/4 > s_I$ holds. In this situation from Eq. (14) it follows that $(\omega + n + 1/2)^2 > 0$, so ω has to be real. Combining this with $\rho = -(2s_{III}\omega)^{-1}$ following from (15) it turns out that ρ^2 is real, and with it all the terms in (15), including E_n. This explains why the spontaneous breakdown of $\mathcal{PT}$ symmetry could not be observed for the Rosen–Morse I [10] and II [46] potentials.

In summary, the spontaneous breakdown of $\mathcal{PT}$ symmetry, i.e. the occurrence of complex conjugate energy eigenvalues is possible when the ω variable that is expressed from Eq. (15) and (16) with E substituted from (14) is com-

plex. In the case of $\mathcal{PT}$-even $y(x)$ functions all the s_i and p_i are real, while in the case of $\mathcal{PT}$-odd $y(x)$ functions p_{III} and s_{III} can be imaginary.

Although a number of conclusions can be drawn from the gereral considerations presented here, there are features that can be inspected only individually for each potential. For example, to determine the normalizability conditions of the wave functions (12) the concrete behavior (singularities, asymptotics) of the $y(x)$ function has to be known, and this is not always apparent from Eq. (9). This question is also related to the presence or lack of quasi-parity, which plays an essential role in the mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry. It is, however, instructive that it often appears via the complex conjugation of the α and β parameters, i.e $\omega \pm \rho$. As it was discussed above, the real or complex nature of these variables is determined by the key equations of our study, Eqs. (14), (15) and (16).

It is worthwhile to inspect all the potentials displayed in Table 2 individually. In the case of PI class (shape-invariant) potentials the energy specrum is quadratic in n, as expected from Eq. (14), furthermore, ω and ρ (i.e. α and β) are independent of n. In the case of PII class (also shape-invariant) potentials the energy spectrum is obtained from Eq. (15), an the dependence of ω and ρ is set determined by Eqs. (14) and (16). As we have mentioned before, $\mathcal{PT}$ symmetry restricts the potential parameters such that only real energy eigenvalues can occur. All these shape-invariant potentials have two independent terms with with two of the s_i parameters being non-zero. The $y(x)$ function is parameter-free in this case, apart from a trivial scaling of the coordinate with a, where $C = \pm a^2$.

A rather general 2+2 parameter Natanzon-class potential was described in Ref. [29], which contains all the six shape-invariant potentials as special limits. Two parameters C and δ appear in $y(x)$ (which is an implicit $x(y)$ function, while two others act as coupling coefficients of two independent potential terms. (Actually, there are more potential terms originating from the Schwartzian derivative, but these do not carry independently variable coupling coefficients. In the case of the shape-invariant potentials these terms have the same form as the even potential component, so they do not appear as separate terms.) Depending on the sign of C and δ, a PI and a PII class potential can be connected continuously from the same potential. A detailed analysis was given in Ref. [29] for the case of $C < 0$, $\delta \geq 0$, from wich the Scarf II and the Rosen-Morse I potentials can be reached in the $\delta \to 0$ and the $\delta \to \infty$ limits (in the latter case considering also the condition $\delta/C = \tilde{C} = const.$). It was shown that the energy eigenvalues can be determined from a quartic algebraic equation on ω, which, in the $\mathcal{PT}$-symmetric setting can have complex conjugate roots, so the spontaneous breakdown can occur in this case. Furthermore, the complex energy eigenvalues appear continuously, starting with $n = 0$ as the magnitude of the imaginary potential component is increased. It was also sown that the Ginocchio potential [37] can be obtained from this potential if the odd potential component is switched off. This also means that the Ginocchio potential can be made $\mathcal{PT}$-symmetric only by using an imaginary coordinate

shift, so its energy spectrum remains real. Nevertheless the energy eigenvalues can be obtained in a closed form in this case.

Another special limit is obtained for $\delta = -1$: the Dutt-Khare-Varshni potential [39,40]. The $y(x)$ function can be expressed explicitly in this case, while the energy eigenvalue have to be determined from a cubic algebraic equation, to which the quartic algebraic equation of Ref. [29] reduces for $\delta = -1$. The x_0 parameter of Eq. (7) plays a role here, as it generates the c coefficient displayed in Table 2. Chosing $x_0 = \pm \mathrm{i}\pi/2$ the $c = -1$ choice can also be made. The $\mathcal{PT}$-symmetric versio nof the DKV potential was also constructed [27]. As we have discussed in Subsection 3.1, the structure of $\phi(y)$ is such that $y(x)$ cannot have definite parity, nevertheless, in can have definite (even) $\mathcal{PT}$ parity in its $\mathcal{PT}$-symmetric version.

The generalized Ginocchio potential was originally defined on the positive half x axis as a radial s-wave potential [38], so its $\mathcal{PT}$-symmetrization [28] is possible only applying an imaginary coordinate shift that cancels the singularity at the origin. It is interesting that its ingredients ($y(x)$ and $\phi(y)$) exhibit indefinite parity, still they possess definite $\mathcal{PT}$-parity, allowing the whole potential to become $\mathcal{PT}$ symmetric.

There are further Natanzon-class potential in the literature, which are not suitable for converting them into a $\mathcal{PT}$-symmetric form. The corresponding $y(x)$ and $\phi(y)$ functions typically do not have definite parity and $\mathcal{PT}$-parity [34,41,42]. Actually, they are all implicit potentials, i.e only the implicit $x(y)$ function is available after the integration in Eq. (7). Their energy eigenvalues are, however, determined in an explicit form, typically as a quadratic expression of n. Another potential (not mentioned in Tables 1 and 2) is described in Ref. [47], and it can be obtained from the potential [42] taking $\gamma = -1$.

5 Summary and outlook

We employed a known variable transformation method of generating exactly solvable potentials to the Jacobi polynomials in order to construct the most general six-parameter Natanzon-class potentials. We implemented the method for the $\mathcal{PT}$-symmetric version of these potentials. It turned out that the structure of the differential equation of the Jacobi polynomials (i.e. the $Q(y)$ and $R(y)$ functions appearing in it) restrict the possible potential forms considerably. The $\mathcal{PT}$ invariance requirement for the potential introduces further constraints on the parameters appearing in the formulas. We investigated the parity and $\mathcal{PT}$-parity properties of the $y(x)$ transformation function and the $\phi(y)$ function in order to construct the $V(x)$ potential in a $\mathcal{PT}$-symmetric form. The results were used to determine in which cases the bound-state energy eigenvalues can take on complex values, i.e. when the spontaneous breakdown of $\mathcal{PT}$ symmetry can occur. These findings demonstrated why this phenomenon cannot take place in the case of the Rosen–Morse I and II potentials, answering an long-standing problem.

Table 1 Parameters of Natanzon-class potentials. In many cases $C = \pm a^2$, where a scales the coordinate x, but here we chose $a = 1$ for the sake of simplicity. $s^{(t)}$ and $s^{(f)}$ stand for the number of independently tunable and fixed non-zero s_i parameters.

	C	p_I	p_{II}	p_{III}	$s^{(t)}$	$s^{(f)}$
Scarf II	-1	1	0	0	2	0
Gen. Pöschl–Teller	-1	1	0	0	2	0
Scarf I	1	-1	0	0	2	0
Rosen–Morse II	1	0	1	0	2	0
Eckart	1	0	1	0	2	0
Rosen–Morse I	-1	0	1	0	2	0
PIII	1	0	0	1	2	0
Ref. [29]	-1	1	δ	0	2	0
Ginocchio	$\frac{\lambda^4}{\lambda^2-1}$	1	$(\lambda^2-1)^{-1}$	0	1	0
Gen. Ginocchio	$4\lambda^4$	$\frac{\gamma^2-1}{4\lambda^4}$	$(2\lambda^2)^{-1}$	$(2\lambda^2)^{-1}$	2	0
DKV	1	-1	1	0	2	0
Ref. [41]	-1	1	-1	-1	1	1
Ref. [42]	1	-1	γ^2+1	2γ	1	1

Known Natanzon-class potentials were analyzed from the point of view of $\mathcal{PT}$ symmetry, and it was demonstrated in several instances that tehir $\mathcal{PT}$-symmetrization is not possible.

As a by-product the relation of the methods described in Refs. [33] and [23] was clarified. It was found that the two methods differ only in the strategy of expressing the energy eigenvalues of a system of three equations.

These investigations can be extended to further potential classes. One obvious target is the confluent hypergeometric potential class, in which case the method has to be applied to the confluent hypergeometric functions and the generalized Laguerre polynomials, to which they reduce in the case of bound-state solutions. Another possibility is considering exceptional orthogonal polynomials in the same systematic way. Due to the more complicated structure of the $Q(y)$ and $R(y)$ functions one may expect a more limited range of $\mathcal{PT}$-symmetric potentials in that case.

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Table 2 The properties of $y(x)$, $\phi(y)$ with respect to $\mathcal{P}$ and $\mathcal{PT}$ parity. $\phi(y)$ can be constructed by substitutinging the p_i parameters from Table 1 into Eq. (9). The signs indicate the respective parity in each case. $\mathcal{P}_y\phi(y)$ stands for parity defined for the y variable. It is different from $\mathcal{P}\phi(y(x))$, corresponding to the $x \to -x$ operation. (+) indicates that the given item can be made $\mathcal{PT}$-even with appropriate choice of the parameters. Implicit potentials are indicated with $x(y)$ in the second column. For [41] $X = \pi(1 - 2^{-1/2})$.

	$y(x)$	$x \in$	$\mathcal{P}y(x)$	$\mathcal{P}_y\phi(y)$	$\mathcal{PT}y(x)$	$\mathcal{PT}\phi(y)$
Scarf II	$\mathrm{i}\sinh(x)$	$(-\infty,\infty)$	–	+	+	+
Gen. Pöschl–Teller	$\cosh(x)$	$(-\infty,\infty)$	+	+	+	+
Scarf I	$\cos(x)$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	+	+	+	+
Rosen–Morse II	$\tanh(x)$	$(-\infty,\infty)$	–	+	–	+
Eckart	$\coth(x)$	$[0,\infty)$	–	+	–	+
Rosen–Morse I	$\mathrm{i}\tan(x)$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	–	+	+	–
PIII	$x(y)$	$[0,\infty)$	indef.	–	–	indef.
Ref. [29]	$x(y)$	$(-\infty,\infty)$	–	+	+	+
Ginocchio	$x(y)$	$(-\infty,\infty)$	–	+	+	+
Gen. Ginocchio	$x(y)$	$[0,\infty)$	indef.	indef.	+	+
DKV	$[1+\mathrm{c}\mathrm{e}^{-2x}]^{1/2}$	$(-\infty,\infty)$	indef.	+	(+)	(+)
Ref. [41]	$x(y)$	$[0,X]$	indef.	indef.	indef.	indef.
Ref. [42]	$x(y)$	$(-\infty,\infty)$	indef.	indef.	indef.	indef.

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