

ON SOME COVERING PROBLEMS IN GEOMETRY

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ABSTRACT. We present a method to obtain upper bounds on covering numbers. As applications of this method, we reprove and generalize results of Rogers on economically covering Euclidean n -space with translates of a convex body, or more generally, any measurable set. We obtain a bound for the density of covering the n -sphere by rotated copies of a spherically convex set (or, any measurable set). Using the same method, we sharpen an estimate by Artstein–Avidan and Slomka on covering a bounded set by translates of another.

The main novelty of our method is that it is not probabilistic. The key idea, which makes our proofs rather simple and uniform through different settings, is an algorithmic result of Lovász and Stein.

1. INTRODUCTION

Given two sets K and L in \mathbb{R}^n (resp. \mathbb{S}^n), we want to cover K by as few translates (resp. rotated copies) of L as possible. Upper bounds for these kinds of covering problems are often obtained by probabilistic methods, that is, by taking randomly chosen copies of L . We present a method that relies on an algorithmic result of Lovász and Stein, and yields proofs that are simple, non-probabilistic and quite uniform through different geometric settings.

For two Borel measurable sets K and L in \mathbb{R}^n , let $N(K, L)$ denote the *translative covering number* of K by L i.e. the minimum number of translates of L that cover K .

Definition 1.1. Let K and L be bounded Borel measurable sets in \mathbb{R}^n . A *fractional covering* of K by translates of L is a Borel measure μ on \mathbb{R}^n with $\mu(x - L) \geq 1$ for all $x \in K$. The *fractional covering number* of K by translates of L is

$$N^*(K, L) \\ = \inf \{ \mu(\mathbb{R}^n) : \mu \text{ is a fractional covering of } K \text{ by translates of } L \}.$$

Clearly, in Definition 3.1 we may assume that a fractional cover μ is supported on $\text{cl}(K - L)$. According to Theorem 1.7 of [AAS13], we have

$$(1) \quad \max \left\{ \frac{\text{vol}(K)}{\text{vol}(L)}, 1 \right\} \leq N^*(K, L) \leq \frac{\text{vol}(K - L)}{\text{vol}(L)}.$$

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Here, the upper bound is easy to see, as the Lebesgue measure restricted to $K - L$ with the following scaling $\mu = \text{vol} / \text{vol}(L)$ is a fractional cover of K by translates of L .

For two sets $K, T \subset \mathbb{R}^n$, we denote their *Minkowski difference* by $K \sim T = \{x \in \mathbb{R}^n : T + x \subseteq K\}$.

Theorem 1.2. *Let K, L and T be bounded Borel measurable sets in \mathbb{R}^n and let $\Lambda \subset \mathbb{R}^n$ be a finite set with $K \subseteq \Lambda + T$. Then*

$$(2) \quad N(K, L) \leq (1 + \ln(\max_{x \in K-L} \text{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K - T, L \sim T).$$

If $\Lambda \subset K$, then we have

$$(3) \quad N(K, L) \leq (1 + \ln(\max_{x \in K-L} \text{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K, L \sim T).$$

For a set $K \subset \mathbb{R}^n$ and $\delta > 0$, we denote the δ -inner parallel body of K by $K_{-\delta} := K \sim \mathbf{B}(o, \delta) = \{x \in K : \mathbf{B}(x, \delta) \subseteq K\}$, where $\mathbf{B}(x, \delta)$ denotes the Euclidean ball of radius δ centered at x . As an application of Theorem 1.2, we will obtain

Theorem 1.3. *Let $K \subseteq \mathbb{R}^n$ be a bounded measurable set. Then there is a covering of \mathbb{R}^n by translated copies of K of density at most*

$$\inf_{\delta > 0} \left[\frac{\text{vol}(K)}{\text{vol}(K_{-\delta})} \left(1 + \ln \frac{\text{vol}(K_{-\delta/2})}{\text{vol}(\mathbf{B}(o, \frac{\delta}{2}))} \right) \right].$$

The δ -inner parallel body could be defined with respect to a norm that is distinct from the Euclidean. As is easily seen from the proof, the theorem would still hold.

Now, we turn to coverings on the sphere. We denote the Haar probability measure on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by σ , the closed spherical cap of spherical radius φ centered at $u \in \mathbb{S}^n$ by $C(u, \varphi)$, and its measure by $\Omega(\varphi) = \sigma(C(u, \varphi))$. For a set $K \subset \mathbb{S}^n$ and $\delta > 0$, we denote the δ -inner parallel body of K by $K_{-\delta} = \{u \in K : C(u, \delta) \subseteq K\}$.

A set $K \subset \mathbb{S}^n$ is called *spherically convex* if it is contained in an open hemisphere and, for any two of its points, it contains the shorter great circular arc connecting them.

The *spherical circumradius* of a subset of an open hemisphere of \mathbb{S}^n is the spherical radius of the smallest spherical cap (the *circum-cap*) that contains the set.

Theorem 1.4. *Let $K \subseteq \mathbb{S}^n$ be a measurable set. Then there is a covering of \mathbb{S}^n by rotated copies of K of density at most*

$$\inf_{\delta > 0} \left[\frac{\sigma(K)}{\sigma(K_{-\delta})} \left(1 + \ln \frac{\sigma(K_{-\delta/2})}{\Omega(\frac{\delta}{2})} \right) \right].$$

Corollary 1.5. *Let $K \subseteq \mathbb{S}^n$ be a spherically convex set of spherical circumradius ρ . Then there is a covering of \mathbb{S}^n by rotated copies of K of density at most*

$$\inf_{\kappa > 0 : K_{-(\kappa\rho)} \neq \emptyset} \left[\frac{\sigma(K)}{\sigma(K) - \Omega(\rho)(1 - (1 - \kappa)^n)} \left(2n + n \ln \frac{1}{\kappa\rho} \right) \right].$$

We prove the Euclidean results in Section 4, and the spherical results in Section 5.

2. HISTORY

An important point in the theory of coverings in geometry is the following theorem of Rogers [Rog57]. For a definition of the covering density, cf. [Rog64].

Theorem 2.1 (Rogers, [Rog57]). *Let K be a bounded convex set in \mathbb{R}^n with non-empty interior. Then the covering density of K is at most*

$$(4) \quad \theta(K) \leq n \ln n + n \ln \ln n + 5n.$$

Earlier, exponential upper bounds for the covering density were obtained by Rogers, Bambah and Roth, and for the special case of the Euclidean ball by Davenport and Watson (cf. [Rog57] for references). The current best bound is due to G. Fejes Tóth [FT09], who replaced the last term $5n$ by $n + o(n)$.

We will obtain Theorem 2.1 as a corollary to our more general Theorem 1.3.

Another classical example of a geometric covering problem is the following. Estimate the minimum number of spherical caps of radius φ needed to cover the sphere \mathbb{S}^n in \mathbb{R}^{n+1} .

Theorem 2.2 (Böröczky Jr. and Wintsche, [BW03]). *Let $0 < \varphi < \frac{\pi}{2}$. Then there is a covering of \mathbb{S}^n by spherical caps of radius φ with density at most $n \ln n + n \ln \ln n + 5n$.*

This estimate was proved in [BW03] improving an earlier result of Rogers [Rog63]. The current best bound is better when $\varphi < \frac{\pi}{3}$: Dumer [Dum07] gave a covering in this case of density at most $\frac{n \ln n}{2}$.

We will obtain Theorem 2.2 as a corollary to our more general Theorem 1.4.

The fractional version of $N(K, \text{int } K)$ (see Definition 3.1) first appeared in [Nas09] and in general for $N(K, L)$ in [AAR11] and [AAS13].

A result very similar to our Theorem 1.2 appeared as Theorem 1.6 in the paper [AAS13] by Artstein-Avidan and Slomka. The main differences are the following. Quantitatively, our result is somewhat stronger by having $\max \text{card}(\dots)$ in the logarithm as opposed to $\text{card } \Lambda$. This allows us to obtain Theorems 2.1 and 1.3 as corollaries to Theorem 1.2. Furthermore, we have no minor term of order $\sqrt{\ln(\text{card } \Lambda)(N^* + 1)}$. The method of the proof in [AAS13] consists of two parts. One is to reduce the problem to a finite covering problem by replacing K by a sufficiently dense finite set (a δ -net). Next, a probabilistic argument is used to solve the finite covering problem. A similar route is followed in [FK08] where a variant of Theorem 2.1 (previously obtained in [ER61]) is proved (using Lovász's Local Lemma) according to which such a low density covering of \mathbb{R}^n by translates of K exists with the additional requirement that no point is covered too many times. An even earlier appearance of this method in the context of the illumination problem can be found in [Sch88]. A major contribution of [AAS13] is that they used this method to bridge the gap between N and N^* , that is, they noticed that the method works with any measure, not just the volume.

We also use the first part of the method (taking a δ -net), but then replace the second (probabilistic) part by a simple application of a non-probabilistic result, Lemma 3.2.

3. PRELIMINARIES

We start by introducing some combinatorial notions.

Definition 3.1. Let Y be a set, \mathcal{F} a family of subsets of Y and $X \subseteq Y$. A *covering* of X by \mathcal{F} is a subset of \mathcal{F} whose union contains X . The *covering number* $\tau(X, \mathcal{F})$ of X by \mathcal{F} is the minimum cardinality of its coverings by \mathcal{F} .

A *fractional covering* of X by \mathcal{F} is a measure μ on \mathcal{F} with

$$\mu(\{F \in \mathcal{F} : x \in F\}) \geq 1 \quad \text{for all } x \in X.$$

The *fractional covering number* of \mathcal{F} is

$$\tau^*(X, \mathcal{F}) = \inf \{ \mu(\mathcal{F}) : \mu \text{ is a fractional covering of } X \text{ by } \mathcal{F} \}.$$

When a group G acts on Y and \mathcal{F} is the set $\{g(A) : g \in G\}$ for some fixed subset A of Y , we will identify $F \in \mathcal{F}$ with $\{g \in G : g(A) = F\} \subseteq G$ and thus, we will call a measure μ on G a fractional covering of X by G if

$$\mu(\{g \in G : x \in g(A)\}) \geq 1 \quad \text{for all } x \in X.$$

For more on (fractional) coverings, cf. [Für88] in the abstract (combinatorial) setting and [Mat02] in the geometric setting.

The gap between τ and τ^* is bounded in the case of finite set families (hypergraphs) by the following result of Lovász [Lov75] and Stein[Ste74].

Lemma 3.2 (Lovász [Lov75], Stein[Ste74]). *For any finite Λ and $\mathcal{H} \subseteq 2^\Lambda$ we have*

$$(5) \quad \tau(\Lambda, \mathcal{H}) < (1 + \ln(\max_{H \in \mathcal{H}} \text{card } H))\tau^*(\Lambda, \mathcal{H}).$$

Furthermore, the greedy algorithm (always picking the set that covers the most number of uncovered points) yields a covering of cardinality less than the right hand side in (5).

The following straightforward corollary to Lemma 3.2 is a key element of our proofs.

Observation 3.3. Let Y be a set, \mathcal{F} a family of subsets of Y , and $X \subseteq Y$. Let Λ be a finite subset of Y and $\Lambda \subseteq U \subseteq Y$. Assume that for another family \mathcal{F}' of subsets of Y we have $\tau(X, \mathcal{F}) \leq \tau(\Lambda, \mathcal{F}')$. Then

$$(6) \quad \tau(X, \mathcal{F}) \leq \tau(\Lambda, \mathcal{F}') \leq (1 + \ln(\max_{F' \in \mathcal{F}'} \text{card}\{\Lambda \cap F'\})) \cdot \tau^*(U, \mathcal{F}').$$

We will rely on the following estimates of Ω by Böröczky Jr. and Wintsche [BW03].

Lemma 3.4 (Böröczky Jr. and Wintsche [BW03]). *Let $0 < \varphi < \pi/2$. Then*

$$(7) \quad \Omega(\varphi) > \frac{\sin^n \varphi}{\sqrt{2\pi(n+1)}},$$

$$(8) \quad \Omega(\varphi) < \frac{\sin^n \varphi}{\sqrt{2\pi n} \cos \varphi}, \quad \text{if } \varphi \leq \arccos \frac{1}{\sqrt{n+1}},$$

$$(9) \quad \Omega(t\varphi) < t^n \Omega(\varphi), \quad \text{if } 1 < t < \frac{\pi}{2\varphi}.$$

The following is known as Jordan’s inequality:

$$(10) \quad \frac{2x}{\pi} \leq \sin x \quad \text{for } x \in [0, \pi/2].$$

4. PROOF OF THE COVERING RESULTS IN \mathbb{R}^n

We present these proofs in the order of their difficulty. In this way, ideas and technicalities are –perhaps– easier to separate.

Proof of Theorem 1.2. The proof is simply a substitution into (6). We take $Y = \mathbb{R}^n$, $X = K$, $\mathcal{F} = \{L + x : x \in K - L\}$, $\mathcal{F}' = \{L \sim T + x : x \in K - L\}$. One can take $U = K - T$ as any member of Λ not in $K - T$ could be dropped from Λ and Λ would still have the property that $\Lambda + T \supseteq K$. That proves (2). To prove (3), we notice that in the case when $\Lambda \subset K$, one can take $U = K$. □

Proof of Theorem 1.3. Let C denote the cube $C = [-a, a]^n$, where $a > 0$ is large. Our goal is to cover C by translates of K economically.

Fix $\delta > 0$, and let $\Lambda \subset \mathbb{R}^n$ be a finite set such that $\Lambda + \mathbf{B}(o, \delta/2)$ is a saturated (i.e. maximal) packing of $\mathbf{B}(o, \delta/2)$ in $C + \mathbf{B}(o, \delta/2)$. Thus, by the maximality, we have that Λ is a δ -net of C with respect to the Euclidean distance, i.e., $\Lambda + \mathbf{B}(o, \delta) \supseteq C$.

By considering volume, for any $x \in \mathbb{R}^n$ we have

$$(11) \quad \text{card}(\Lambda \cap (x + K_{-\delta})) \leq \frac{\text{vol}(K_{-\delta} + \mathbf{B}(o, \delta/2))}{\text{vol}(\mathbf{B}(o, \delta/2))} \leq \frac{\text{vol}(K_{-\delta/2})}{\text{vol}(\mathbf{B}(o, \delta/2))}.$$

Let $\varepsilon > 0$ be fixed. Clearly, if a is sufficiently large, then

$$(12) \quad N^*(C + \mathbf{B}(o, \delta/2), K_{-\delta}) \leq \frac{\text{vol}(C + \mathbf{B}(o, \delta/2) - K_{-\delta})}{\text{vol} K_{-\delta}} \leq (1 + \varepsilon) \frac{\text{vol} C}{\text{vol} K_{-\delta}}.$$

By (2), (11) and (12) we have

$$N(C, K) \leq (1 + \varepsilon) \left(1 + \ln \frac{\text{vol}(K_{-\delta/2})}{\text{vol}(\mathbf{B}(o, \delta/2))} \right) \frac{\text{vol} C}{\text{vol} K_{-\delta}}.$$

Finally,

$$\theta(K) \leq N(C, K) \text{vol}(K) / \text{vol}(C)$$

yields the promised bound. □

Proof of Theorem 2.1. Let C denote the cube $C = [-a, a]^n$, where $a > 0$ is large. Our goal is to cover C by translates of K economically. First, consider the case when $K = -K$.

Let $\delta > 0$ be fixed (to be chosen later) and let $\Lambda \subset \mathbb{R}^n$ be a finite set such that $\Lambda + \frac{\delta}{2}K$ is a saturated (i.e. maximal) packing of $\frac{\delta}{2}K$ in $C - \frac{\delta}{2}K$. Thus, by the maximality, we have that Λ is a δ -net of C with respect to K , i.e., $\Lambda + \delta K \supseteq C$. By considering volume, for any $x \in \mathbb{R}^n$ we have

$$(13) \quad \text{card}(\Lambda \cap (x + (1 - \delta)K)) \leq \frac{\text{vol}((1 - \delta)K + \frac{\delta}{2}K)}{\text{vol}(\frac{\delta}{2}K)} \leq \left(\frac{2}{\delta}\right)^n.$$

Let $\varepsilon > 0$ be fixed. Clearly, if a is sufficiently large, then

$$(14) \quad N^*(C - \delta K, (1 - \delta)K) \leq (1 + \varepsilon) \frac{\text{vol} C}{(1 - \delta)^n \text{vol} K}.$$

By (2), (13) and (14) we have

$$N(C, K) \leq \frac{1 + \varepsilon}{(1 - \delta)^n} \left(1 + n \ln \left(\frac{2}{\delta} \right) \right) \frac{\text{vol } C}{\text{vol } K}.$$

On the other hand,

$$(15) \quad \theta(K) \leq N(C, K) \text{vol}(K) / \text{vol}(C) \leq \frac{1 + \varepsilon}{(1 - \delta)^n} \left(1 + n \ln \left(\frac{2}{\delta} \right) \right).$$

We choose $\delta = \frac{1}{2n \ln n}$, and the following standard computation:

$$(16) \quad \begin{aligned} (1 + \varepsilon)^{-1} \theta(K) &\leq (1 + n \ln(4n \ln n)) \exp(1 / \ln n) \\ &\leq (1 + n \ln(4n \ln n)) (1 + 2 / \ln n) \leq (n \ln n + n \ln \ln n + 5n), \end{aligned}$$

yields the desired bound (as ε can be taken arbitrarily close to 0).

Next, consider the general case, that is, when K is not necessarily symmetric about the origin. We need to make the following modifications. Milman and Pajor (cf. Corollary 3 of [MP00]) showed that, if the centroid (that is, the center of mass) of K is the origin, then $\text{vol}(K \cap -K) \geq \frac{\text{vol } K}{2^n}$. (Note that the existence of a translate of K for which this inequality holds was proved by Stein [Ste56] using a probabilistic argument.) We define Λ as a saturated packing of translates of $\frac{\delta}{2}(K \cap -K)$ in $C - \frac{\delta}{2}(K \cap -K)$. Thus, we have $C \subseteq \Lambda + \delta(K \cap -K) \subseteq \Lambda + \delta K$. Instead of (13), we now have

$$\text{card}(\Lambda \cap (x + (1 - \delta)K)) \leq \left(\frac{4}{\delta} \right)^n$$

for any $x \in \mathbb{R}^n$. Rolling this change through the proof, at the end in place of (15), we obtain

$$\theta(K) \leq \frac{1 + \varepsilon}{(1 - \delta)^n} \left(1 + n \ln \left(\frac{4}{\delta} \right) \right),$$

which, however, is still less than $(1 + \varepsilon)(n \ln n + n \ln \ln n + 5n)$ with the same choice of $\delta = \frac{1}{2n \ln n}$. □

5. PROOF OF THE SPHERICAL RESULTS

Proof of Theorem 1.4. Let Λ be the set of centers of a saturated (i.e. maximal) packing of caps of radius $\delta/2$. Clearly, Λ is a δ -net of \mathbb{S}^n , and thus, if we cover Λ by rotated copies of radius $K_{-\delta}$, then the same rotations yield a covering of \mathbb{S}^n by copies of K .

Let $\bar{\sigma}$ denote the probability Haar measure on $SO(n + 1)$. Let $H \subset \mathbb{S}^n$ be a measurable set, and denote the family of rotated copies of H by $\mathcal{F}(H) = \{AH : A \in SO(n + 1)\}$. Recall that for any fixed $u \in \mathbb{S}^n$ we have

$$\begin{aligned} &\bar{\sigma}(\{A \in SO(n + 1) : u \in AH\}) \\ &= \bar{\sigma}(\{A \in SO(n + 1) : u \in A^{-1}H\}) \\ &= \bar{\sigma}(\{A \in SO(n + 1) : Au \in H\}) = \sigma(H). \end{aligned}$$

It follows that the measure $\frac{\bar{\sigma}}{\sigma(H)}$ on $SO(n + 1)$ is a fractional cover of \mathbb{S}^n by $\mathcal{F}(H)$ and thus, $\tau^*(\mathbb{S}^n, \mathcal{F}(H)) \leq \frac{1}{\sigma(H)}$.

Thus by (6), we obtain the following for the density of a covering by rotated images of K :

$$\begin{aligned} \text{density} &\leq \sigma(K)\tau(\mathbb{S}^n, \mathcal{F}(K)) \leq \sigma(K)\tau(\Lambda, \mathcal{F}(K_{-\delta})) \\ &\leq (1 + \ln(\max_{A \in SO(n+1)} \text{card}\{\Lambda \cap AK_{-\delta}\})) \cdot \frac{\sigma(K)}{\sigma(K_{-\delta})} \\ &\leq \frac{\sigma(K)}{\sigma(K_{-\delta})} \left(1 + \ln \frac{\sigma(K_{-\delta/2})}{\Omega\left(\frac{\delta}{2}\right)} \right). \end{aligned}$$

Since it holds for any $\delta > 0$, the theorem follows. \square

Proof of Theorem 2.2. We will apply Theorem 1.4 with K being a cap of spherical radius φ . We set $\delta = \eta\varphi$, where η will be specified later. By Theorem 1.4 and (9), we obtain for the density of a covering of \mathbb{S}^n by caps of radius φ :

$$\text{density} \leq \left(1 + n \ln \left(\frac{2}{\eta} \right) \right) \cdot \left(\frac{1}{1 - \eta} \right)^n.$$

We choose $\eta = \frac{1}{2n \ln n}$, and the same computation as in (16) yields the desired bound. \square

Proof of Corollary 1.5. We set $\delta = \kappa\rho$. First, observe that the measure of the belt-like region $K \setminus K_{-\delta}$ at the boundary of K is at most as large as the measure of the belt-like region $C(v, \rho) \setminus C(c, \rho - \delta)$ at the boundary of the circum-cap $C(v, \rho)$ of K .

Next, combine $\ln \frac{\sigma(K_{-\delta/2})}{\Omega\left(\frac{\delta}{2}\right)} \leq \ln \frac{1}{\Omega\left(\frac{\delta}{2}\right)}$ with (9) and (10) to obtain the statement. \square

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