LIST COLORINGS WITH DISTINCT LIST SIZES, THE CASE OF COMPLETE BIPARTITE GRAPHS

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List colorings with distinct list sizes, the case of complete bipartite graphs

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Abstract

Let $f : V \to \mathbb{N}$ be a function on the vertex set of the graph $G = (V, E)$. The graph G is *f-choosable* if for every collection of lists with list sizes specified by f there is a proper coloring using colors from the lists. The sum choice number, $\chi_{sc}(G)$, is the minimum of $\sum f(v)$, over all functions f such that G is f-choosable. It is known (Alon 1993, 2000) that if G has average degree d, then the usual choice number $\chi_\ell(G)$ is at least $\Omega(\log d)$, so they grow simultaneously.

In this paper we show that $\chi_{sc}(G)/|V(G)|$ can be bounded while the minimum degree $\delta_{\min}(G) \to \infty$. Our main tool is to give tight estimates for the sum choice number of the unbalanced complete bipartite graph $K_{a,q}$.

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1 Average list sizes and planar graphs

Given a graph G and a list of colors $L(v)$ for each vertex $v \in V(G)$, we say that G is *L-choosable* (or that L is *sufficient*) if it is possible to choose $c(v) \in L(v)$ for all v so that $c : V(G) \to \cup L(v)$ is a proper coloring of G. The *choice number* (or *list chromatic number*) χ_ℓ is the minimum t such that for every assignment L with $|L(v)| \geq t$ for all $v \in V$, the graph is L-choosable. It is well-known (Thomassen [8]) that

$$\chi_\ell(P) \leq 5 \tag{1}$$

for every planar graph P, and this is the best possible ([9]).

However, if we allow distinct list sizes, then the average size can be smaller. For example, Thomassen's beautiful proof for (1) gives that if P is an n-vertex planar graph, $v_1, \ldots, v_t$ are its external vertices (in this order) and the list sizes are

$$|L(v)| = \begin{cases} 1 & \text{for } v = v_1, \\ 2 & \text{for } v = v_2, \\ 3 & \text{for } v = v_3, \ldots, v_t, \\ 5 & \text{for the inner vertices,} \end{cases} \tag{2}$$

then P is L-choosable.

Consider a function $f : V(G) \to \mathbb{N}$. An *f-assignment* is an assignment of lists $L(v)$ to the vertices $v \in V(G)$ such that $|L(v)| = f(v)$ for all v. The function f is *sufficient* if G is L-choosable for all f-assignments L. We define the *sum choice number* of G, denoted by $\chi_{sc}(G)$, as the minimum of $\sum_{v \in V(G)} f(v)$ over all sufficient f.

Sum choice numbers were introduced by Isaak in [6] who proved that if G is the line-graph of $K_{2,q}$ then $\chi_{\mathrm{sc}}(G) = q^2 + \lceil 5q/3 \rceil$. Various classes of graphs were investigated by Isaak in [7], by Berliner, Bostelmann, Brualdi, and Deaett [3] and by Heinold in [4] and [5].

Thomassen's theorem (2) implies that $\chi_{sc}(P) \leq 5n - 9$ for planer P ($n \geq 2$). In fact, more is true. It is easy to show (see, e.g., [7]) that for every graph

$$\chi_{sc}(G) \leq |V(G)| + |E(G)| \tag{3}$$

holds. Hence $\chi_{sc}(P) \leq 4n - 6$. Our first result is a slight improvement.

Theorem 1. *Let P be an n-vertex planar graph. There exists an $f : V(P) \to \mathbb{N}$ such that $\sum f(v) \leq 4n - 6$, $\max f(v) \leq 6$, and P is f-choosable.* □

Proof. Consider a linear order of the vertices of P, and let $\hat{d}(v)$ be the number of neighbors of v that precede it. The function $f(v) = \hat{d}(v) + 1$ is a sufficient function, so $\sum(\hat{d}(v) + 1)$ yields an upper bound on $\chi_{sc}(P)$. Since every planar graph has a vertex of degree at most 5, it is possible to order the vertices so that $\hat{d}(v) \leq 5$ for all v. □

2 Unbalanced complete bipartite graphs

Erdős, Rubin and Taylor (see, e.g., [1]) showed for the complete bipartite graph that

$$\chi_\ell(K_{q,q}) = \Theta(\log q). \tag{4}$$

If one of the parts is substantially smaller than the other one, then allowing different list sizes results in smaller average lists. It is easy to show $\chi_{sc}(K_{1,q}) = 2q+1$ (as for every tree on $q+1$ vertices). Berliner, Bostelmann, Brualdi, and Deaettet [3] showed that for all $q \geq 1$ we have

$$\chi_{sc}(K_{2,q}) = 2q+1+\lfloor\sqrt{4q+1}\rfloor, \tag{5}$$

and Heinold [5] proved

$$\chi_{sc}(K_{3,q}) = 2q+1+\lfloor\sqrt{12q+4}\rfloor. \tag{6}$$

Our main result deals with the sum choice number of $K_{a,q}$ with arbitrary a.

Theorem 2. *There exist positive constants c_1 and c_2 such that for all $a \geq 2$ and $q \geq 4a^2 \log a$*

$$2q + c_1 a\sqrt{q\log a} \leq \chi_{sc}(K_{a,q}) \leq 2q + c_2 a\sqrt{q \log a}.$$

It is known that χ_ℓ is not independent of the average degree. Alon [1, 2] proved that for some constant $c > 0$, every graph G with average degree d has

$$\chi_\ell(G) \geq c\log d. \tag{7}$$

An easy corollary of our Theorem 2 is that if different list sizes are allowed, then such dependence does not exist. Indeed, we have

$$\lim_{\substack{a\to\infty,\\ q>>a^2\log a}} \frac{2|E(K_{a,q})|}{a+q} = \infty, \qquad \lim_{\substack{a\to\infty,\\ q>>a^2\log a}} \frac{\chi_{sc}(K_{a,q})}{a+q} = 2.$$

So the structure of the graph plays a more important role in determining the sum choice number than in the case of the list chromatic number.

3 Upper bound, there are sufficient short lists

Throughout this paper, the two parts of the complete bipartite graph $K_{a,q}$ will be denoted by A and Q, with $|A| = a$ and $|Q| = q$.

Theorem 3. *Suppose that $a, q \in \mathbb{N}$ with $q \geq a \geq 2$. Then*

$$\chi_{sc}(K_{a,q}) \leq 2q + a\lceil\sqrt{32q(1+\log a)}\rceil.$$

Proof. To prove the upper bound, we present a function f with $\sum_{v\in A\cup Q} f(v) = 2q + a\lceil\sqrt{32q(1+\log a)}\rceil$ such that every f-assignment is sufficient.

Define f as

$$f(v) = \begin{cases} r & \text{for } v \in A; \\ 2 & \text{for } v \in Q \end{cases}$$

where r will be defined later in (10) as any integer $r \geq \sqrt{32q(1+\log a)}$. Let L be an arbitrary f-assignment, i.e., $|L(v)| = f(v)$ for all v.

Consider $S := \bigcup_{v \in A \cup Q} L(v)$. The assignment L yields a (multi)hypergraph and a multigraph on the same vertex set S and with edge sets $\mathcal{L}_A := \{L(u) : u \in A\}$ and $\mathcal{L}_Q := \{L(v) : v \in Q\}$, respectively. Sufficiency of L means that one can find a set $T \subset S$ meeting all hyperedges of $\mathcal{L}_A$ such that $S \setminus T$ meets all edges of $\mathcal{L}_Q$, so T is an independent set in the graph $\mathcal{L}_Q$. Given T the choice function c can be defined as

$$c(u) \in L(u) \cap T, \text{ for } u \in A$$

and

$$c(v) \in L(v) \cap (S \setminus T), \text{ for } v \in Q.$$

We are going to construct such T by a 2-step random process.

Let us pick, randomly and independently, each element of S with probability p. Let B be the random set of all elements picked. Define a random variable X_u for each $u \in A$ as $X_u = |L(u) \cap B|$, and the random variable Y by

$$Y := |\{v \in Q : L(v) \subseteq B\}|,$$

so Y is the number of edges of $\mathcal{L}_Q$ spanned by B. Remove an element $c(v) \in L(v)$ for each edge of $\mathcal{L}_Q$ spanned by B, the remaining set $T \subset B$ is certainly independent in $\mathcal{L}_Q$, and if $Y < X_u$ for each $u \in A$, then T meets all $L(u) \in \mathcal{L}_A$ and we are done.

The expected value of Y is $p^2 q$, so Markov inequality gives

$$\text{Prob}\,(Y < 2p^2 q) \geq \frac{1}{2}. \tag{8}$$

The expected value of X_u is pr, so Chernoff inequality gives

$$\text{Prob}\,(X_u < EX_u - t) < e^{-t^2/2rp},$$

for any $t > 0$. Hence

$$\text{Prob}\,(X_u \geq pr - t \text{ for all } u \in A) > 1 - ae^{-t^2/2rp}, \tag{9}$$

and this is at least $1/2$ for $t^2 \geq 2rp\ln(2a)$. The sum of probabilities in (8) and (9) is larger than 1, so there is an appropriate choice of B (and then T) if $t^2 = 2rp(1+\log a)$ and $pr - t \geq 2p^2 q$. We can define, e.g.,

$$p := \sqrt{\frac{2(1+\log a)}{q}} \quad \text{and} \quad r \geq 4pq = \sqrt{32(1+\log a)q}. \tag{10}$$

□

4 Lower bound, much shorter lists are not sufficient

To prove that $\chi_{sc}(G) \geq k$ for a particular k, we need to show that for every f with $\sum_{v\in G} f(v) = k$, there exists an insufficient f-assignment. First, we show how to construct an insufficient assignment for some special f.

Lemma 1. *Let $t \geq 2$ and $\ell \geq 1$. For $a = 2^t$ and $q = t\ell^2$, there exists an insufficient assignment L with*

$$|L(v)| = \begin{cases} t\ell & \text{for } v \in A, \\ 2 & \text{for } v \in Q. \end{cases}$$

Proof. Take $2t$ pairwise disjoint sets X_i, Y_i of size ℓ, $Z = \cup(X_i \cup Y_i)$. Identify the elements of A by the set of 0-1 vectors of length t, $A = \{0,1\}^t$. For a vector $(\varepsilon_1, \ldots, \varepsilon_t) \in A$ define $L(v)$ as $(\bigcup_{\varepsilon_i=1} X_i) \cup (\bigcup_{\varepsilon_j=0} Y_j)$. So $L(v)$ contains either X_i or Y_i for all i. Let the graph G be the union of t complete bipartite graphs on the vertex set Z by setting $E(G) = \bigcup_{i=1}^t (X_i \times Y_i)$ and define the lists $L(v)$ for $v \in Q$ as the edges of G. The number of edges of G is $t\ell^2 = q$, so a one-to-one mapping can be done.

Every independent set T of G contains at most one vertex from each $X_i \cup Y_i$ so it cannot meet all hyperedges of $\mathcal{L}_A$, where $\mathcal{L}_A := \{L(v) : v \in A\}$. This means that this assignment L is not sufficient. □

Note that with this choice of a and q, we have $|L(v)| = \sqrt{q \log_2 a}$ for $v \in A$. Also notice that if we remove some elements from the lists in the above construction, the resulting list assignment is still insufficient.

Theorem 4. *If $a \geq 2$ and $q > 4a^2 \log a$, then*

$$\chi_{sc}(K_{a,q}) \geq 2q + 0.06a\sqrt{q \log a}$$

Proof. Suppose that $f : V(K_{a,q}) \to \mathbb{N}$ with $\sum_{v\in A\cup Q} f(v) = 2q + as$ where $s \leq 0.06\sqrt{q \log a}$. We will find an insufficient f-assignment.

Let q_1, q_2 and q_3 be the numbers of vertices $v \in Q$ with $f(v) = 1$, $f(v) = 2$ and $f(v) \geq 3$, respectively. If $f(u) \leq q_1$ for some $u \in A$, then f is obviously insufficient. From now on, we suppose that $f(u) > q_1$ for each $u \in A$. We obtain

$$2q + as = \sum_{v\in A\cup Q} f(v) \geq aq_1 + (q_1 + 2q_2 + 3q_3) \geq 2q + q_1 + q_3. \tag{11}$$

It follows that $q_1 + q_3 \leq as$ and Q has at least $q - as$ vertices with lists of size 2. Let $q^* = q_2$ and let a^* be the largest power of 2 not exceeding $\frac{1}{2}a$.

If there are at least a^* vertices $u \in A$ with $f(u) \leq \sqrt{q^* \log_2 a^*}$, then we can use Lemma 1 to construct an insufficient assignment.

If this does not hold, then A has more than $\frac{a}{2}$ vertices with lists of size greater than $\sqrt{(q - as)\log_2 a^*}$. Using $a^* > a/4$ we obtain

$$\sum_{v\in A\cup Q} f(v) \geq \frac{1}{2}a\sqrt{(q - as)\log_2(a/4)} + 2q - as. \tag{12}$$

The rest is a little calculation to show that here the right hand side exceeds $2q+0.06a\sqrt{q\log a}$ for $a \geq 5$, $q > 4a^2 \log a$ and $s < 0.06\sqrt{q \log a}$. Finally, the case $a \leq 4$ (in fact $a \leq 30$) follow from (5) and (6), completing the proof. □

Let us remark that if we choose the constants in the proof of Theorem 3 more carefully, we can improve the constant $\sqrt{32}$ to 3.67. Using a randomized construction, it is possible to improve the constant 0.06 in Theorem 4 to 0.87.

5 For fixed a, a limit exists as $q \to \infty$

In this section we suppose that $a \geq 2$ is a fixed integer. We have proved bounds for $\alpha_q := (\chi_{sc}(K_{a,q}) - 2q)/\sqrt{q}$. Now we show that in fact the limit exists when q tends to ∞.

Theorem 5. *For fixed a, the limit* $\lim_{q\to\infty} \frac{\chi_{sc}(K_{a,q})-2q}{\sqrt{q}}$ *exists.*

First, we consider a simpler problem and consider only *type II* assignments of $K_{a,q}$ which means $f(v) = 2$ for all $v \in Q$. Define $\chi_{sc2}(K_{a,q})$ to be the minimum of $\sum_{A\cup Q} f(v)$ where f runs over all sufficient type II functions. Obviously $\chi_{sc2}(K_{a,q}) \geq \chi_{sc}(K_{a,q})$.

Theorem 6. *For fixed a, the limit* $\lim_{q\to\infty} \frac{\chi_{sc2}(K_{a,q})-2q}{\sqrt{q}}$ *exists.*

A type II f is not sufficient if and only if there exists a a hypergraph $\mathcal{L}$ with edges L_i satisfying $|L_i| = f_i$ for $i = 1,\ldots,a$ and a graph G on $V(\mathcal{L})$ with at most q edges, such that no transversal of $\mathcal{L}$ is an independent set in G.

For $I \subseteq [a]$, define $X_I = \cap_{i\in I} L_i$. An insufficient II assignment is *symmetric* if for all pairs $I \neq J$, the bipartite subgraph of G induced by X_I and X_J is either empty of complete, and for each I, X_I induces the empty graph. Without loss of generality we may assume that an insufficient type II assignment is symmetric, as the following lemma demonstrates. From now on, in this section, all assignments are of type II, except when stated otherwise.

Lemma 2. *Given a, q and f an insufficient type II assignment exists if and only if a symmetric insufficient assignment exists.*

Proof. Suppose that $(\mathcal{L}, G)$ is an insufficient assignment. If u and v belong to the same X_I, then no minimal transversal of $\mathcal{L}$ contains both of them. We can therefore delete all edges induced by X_I.

Now suppose that $u, v \in X_I$ and $|N(u)| \leq |N(v)|$. Replace the neighborhood of v by the neighborhood of u. It is still true that every transversal of $\mathcal{L}$ induces an edge of G. Repeated application of this procedure eventually produces a symmetric insufficient assignment. □

Proof of Theorem 6. Consider a symmetric insufficient assignment f for $K_{a,q}$. Let $\mathcal{L}, G$ and X_I (for $I \subseteq [a]$) be as before, $x_I = |X_I|$. Let $V := \{v_I : I \subseteq [a]\}$ be a 2^a-element set. Let R be the *reduced graph* of the symmetric insufficient assignment, i.e., the graph with $V(R) = \{v_I; x_I \neq 0\}$ and whose edges correspond to the complete bipartite subgraphs of G. Similarly, the hypergraph $\mathcal{L}$ turns into the reduced hypergraph on the same vertex set, $V(R)$.

The graph R is *blocking* i.e., every vertex cover of the reduced hypergraph contains an edge of R. The vector $x = (x_I)$ satisfies $\sum_{IJ \in E(R)} x_I x_J \le q$, and $x_I = 0$ whenever $v_I \notin V(R)$. The set A_R^q of all such x lies in the non-negative orthant of $\mathbb{R}^{2^{[a]}}$ and is bounded by a quadric surface which depends on R and q.

Define the linear map $\varphi : \mathbb{R}^{2^{[a]}} \to \mathbb{R}^a$ by $\varphi(x) = (f_1, \dots, f_a)$ where $f_i = \sum_{i \in I} x_I$. The function f is insufficient for this q if and only if f is the image of some integer point x that is in A_R^q for some blocking R.

If there exists an insufficient f-assignment for every integer vector f such that $\sum f_i = k$, then we have $\chi_{sc2}(K_{a,q}) - 2q > k$. We are therefore looking for the maximum k such that every integer point on the hyperplane $\sum f_i = k$ is the image (under φ) of some integer point in $\bigcup A_R^q$, where the union is taken over all (but finitely many) blocking R's.

Let us normalize everything by $\sqrt{q}$. For every blocking R, define

$$A_R := \{x : \sum_{IJ \in E(R)} x_I x_J \le 1, \text{ and } x_I = 0 \text{ for } v_I \notin V(R)\} \quad \text{and} \quad B_R := \varphi(A_R).$$

For every R we now have only one quadric surface, independent of q. We say that a vector v is a *q-grid point* if $\sqrt{q} \cdot v$ is an integer point.

For every q, define k_q to be the maximum k such that every q-grid point on the hyperplane $\sum f_i = k$ is the image of some q-grid point in $\bigcup A_R$. Also, define β to be the maximum k such that the simplex $C_k := \{f : \sum f_i \le k\}$ is a subset of $\bigcup B_R$.

We want to prove that the limit $\lim k_q$ exists and equals β. That is, we want to prove that for every ε, if q is large enough,

- every q-grid point in $C_{\beta-\varepsilon}$ is the image under φ of some q-grid point in $\bigcup A_R$, and
- there is a q-grid point in $C_{\beta+\varepsilon}$ which is not the image of any q-grid point in $\bigcup A_R$.

To prove the first claim, fix q and let f be a point on the hyperplane $\sum f_i = \beta$. The point f is in $\bigcup B_R$, so it is the image of some $x \in \bigcup A_R$. Each set A_R is a *downset* in the sense that with every x it also contains all points z such that $z_i \le x_i$ for all i. It follows that $y := \frac{\lfloor \sqrt{q} \cdot x \rfloor}{\sqrt{q}}$ is a q-grid point in $\bigcup A_R$. Each entry of y differs by at most $\frac{1}{\sqrt{q}}$ from the corresponding entry of x, and a simple computation suffices to show that the distance of $\varphi(y)$ and f is at most $\frac{c}{\sqrt{q}}$, where c is a constant dependent only on a. That is, for each point f on the hyperplane $\sum f_i = \beta$ we have found, in distance at most $\varepsilon_q := \frac{c}{\sqrt{q}}$, an image of a q-grid point from $\bigcup A_R$. Call this point f'.

Note that, by definition of χ_{sc}, whenever a q-grid point f is the image of a q-grid point in $\bigcup A_R$, the same is true for all q-grid points in the box $D_f := \{g : g_i \le f_i \text{ for all } i\}$.

Let h be a q-grid point such that $\sum h_i \le \beta - \varepsilon_q \cdot \sqrt{a}$. Let f be its perpendicular projection on the hyperplane $\sum f_i = \beta$ and find the corresponding f'. Since the distance of f and f' is at most ε_q, the point h belongs to $D_{f'}$, and hence it is the image of a q-grid point in $\bigcup A_R$. Choosing q large enough so that $\varepsilon_q \cdot \sqrt{a} \le \varepsilon$ for our given ε concludes the proof.

Now we prove the second claim. Let f be a point outside $\cup B_R$, but within the distance ε from C_β. Take a bounded cube $Q \subseteq \mathbb{R}^a$ that contains f. Now take a bounded cube S in $\mathbb{R}^{2^a}$ which contains all points x such that $\varphi(x) \in Q$. Then $T := S \cap (\bigcup A_R)$ is a compact set, so φ maps it to a compact set. The complement of $\varphi(T)$ in $\varphi(S)$ is open, and contains f. Note that $(\bigcup B_R) \cap Q \subseteq \varphi(T)$.

Therefore, for some small δ, the δ-ball around f is outside $\bigcup B_R$. If q is large enough, the ball contains some q-grid point. This point not only has no q-grid preimages in $\cup A_R$, it has no preimages in $\bigcup A_R$ whatsoever, and the claim is proven. $\square$

Proof of Theorem 5. Define sequences $\{\alpha_q\}_{q=1}^\infty$ and $\{\beta_q\}_{q=1}^\infty$ as follows

$$\alpha_q = \frac{\chi_{sc}(a,q) - 2q}{\sqrt{q}} \quad \text{and} \quad \beta_q = \frac{\chi_{sc2}(a,q) - 2q}{\sqrt{q}}.$$

It was already mentioned that $\alpha_q \le \beta_q$ for all q.

The argument in the proof of Theorem 4 shows that whenever we have an insufficient function f for $K_{a,q}$, we can delete at most $d(q) = O(\sqrt{q})$ vertices of Q and get an insufficient function for $K_{a,q-d(q)}$ where $f(v) = 2$ for $v \in Q$. We therefore have $\alpha_q\sqrt{q} = \chi_{sc}(a,q) - 2q \ge \chi_{sc2}(q, q-d(q)) - 2(q-d(q)) = \beta_{q-d(q)}\sqrt{q-d(q)}$. We get the following relationship between α_q and β_q:

$$\beta_q \ge \alpha_q \ge \frac{\sqrt{q-d(q)}}{\sqrt{q}}\beta_{q-d(q)}.$$

The limit $\lim_{q\to\infty} \beta_q$ exits by Theorem 6. Since $d(q) = O(\sqrt{q})$, we have

$$\lim_{q\to\infty} \frac{\sqrt{q-d(q)}}{\sqrt{q}}\beta_{q-d(q)} = \lim_{q\to\infty} \beta_q,$$

which proves the claim. $\square$

6 Graphs with large independent sets and a generalization of Turán's theorem

Let $G_{a,q}$ be the graph that we get from $K_{a,q}$ by inserting an edge $\{u, v\}$ for every pair of distinct $u, v \in A$.

Theorem 7. *There exist positive constants c_1 and c_2, independent of q and a, such that for $q > a \ge 2$ we have*

$$2q + c_1 a\sqrt{q(a-1)} \le \chi_{sc}(G_{a,q}) \le 2q + c_2 a\sqrt{q(a-1)}.$$

We will use the following generalization of Turán theorem. For given positive integers s and k, let $t(s,k) = \min \sum_{1\le i\le k} \binom{d_i}{2}$, where the minimum is taken over all non-negative integer sequences $(x_1, \dots, x_k)$ such that $\sum d_i = s$.

Theorem 8. *Let $s, a \geq 2$ be integers and let G be a graph with less than $t(s, a-1)$ edges. Let $L_1, \ldots, L_a \subset V(G)$ be sets of size s. Then there exists a system of distinct representatives $\{u_1, \ldots, u_a\}$ of $L_1, \ldots, L_a$, (i.e., $u_i \in L_i$) which is an independent a-element set in G.*

The case $V(G) = L_1 = \cdots = L_a$ gives (the dual form of) Turán's theorem.

Let us also remark that the result concerning $|E(G)|$ is sharp: taking $L_1 = \cdots = L_a = V(G)$, $|V(G)| = s$ with G being the disjoint union of $a-1$ cliques of almost equal sizes provide a graph of $t(s, a-1)$ edges and a family without any system of distinct representatives which is independent in G (because G has no any independent set of size a).

Proof. We define the sequence of distinct vertices $u_1, \ldots, u_a$ one by one by an algorithm, such that $u_k \in L_{i_k}$, where $\{i_1, i_2, \ldots, i_a\}$ is a permutation of $\{1, 2, \ldots, a\}$ and also the set $\{u_1, \ldots, u_a\}$ is independent in G.

Let $V_1 = L_1 \cup \cdots \cup L_a$ and $G_1 = G[V_1]$ (the restriction of G to V_1). Let u_1 be the vertex of minimum degree in G_1, D_1 the closed neighborhood of u_1 in G_1, and $d_1 = |D_1|$. Let L_{i_1} be one of the hyperedges containing u_1.

If u_j, D_j and L_{i_j} are already defined for $j = 1, \ldots, k$, consider $V_{k+1} = (\bigcup_{i \notin \{i_1, \ldots, i_k\}} L_i) \setminus (D_1 \cup \cdots \cup D_k)$, $G_{k+1} = G[V_{k+1}]$, and let u_{k+1} be a vertex of minimum degree in G_{k+1}, D_{k+1} its closed neighborhood in G_{k+1}, $d_{k+1} = |D_{k+1}|$, and $L_{i_{k+1}}$ one of the hyperedges containing u_{k+1}, different from $L_{i_1}, \ldots, L_{i_k}$.

We claim that this algorithm only stops after a steps thus supplying the desired independent set $\{u_1, \ldots, u_a\}$. If we cannot define u_{k+1} for a $k < a$, then V_{k+1} is empty, and $|D_1| + \cdots + |D_k| \geq s$. As $D_1, \ldots, D_k$ are non-empty, disjoint sets we get the contradiction

$$2|E(G)| \geq \sum_{u \in D_1 \cup \cdots \cup D_k} \deg(u) \geq d_1(d_1 - 1) + \cdots + d_k(d_k - 1) \geq 2t(s, k) \geq 2t(s, a-1). \quad \square$$

Proof of Theorem 7. Again, we will prove the upper bound (with $c_2 = 3$) by presenting a sufficient function f. Let

$$f(v) = \begin{cases} s & \text{for } v \in A, \\ 2 & \text{for } v \in Q, \end{cases}$$

where $s := \lfloor 3\sqrt{(a-1)q} \rfloor$. Note that $s \geq a$ and $t(s, a-1) > q$. Consider any f-assignment, it provides sets $L_1, \ldots, L_a$ and a graph G with $|E(G)| = q$. This assignment is sufficient by Theorem 8.

For the lower bound, we proceed as in Section 4 and first consider f's with $f(v) = 2$ for all $v \in Q$. Let $\{v_1, \ldots, v_a\}$ be the vertices of A and let

$$f(v) = \begin{cases} s_i & \text{for } v = v_i \in A \text{ and} \\ 2 & \text{for } v \in Q \end{cases}$$

such that $s_1 \leq \cdots \leq s_a$. We claim that if f is sufficient, then $q < t(s_i, i-1)$ for all $i \geq 2$. These inequalities imply that $s_i \geq \sqrt{2(i-1)q}$ so

$$\sum f(v) \geq 2q + \left(\sum_{1 \leq i \leq a} \sqrt{i-1} \right) \sqrt{2q} \geq 2q + \frac{1}{2} a \sqrt{(a-1)q}.$$

Suppose, on the contrary, that $q \geq t(s_i, i-1)$ for some $i \geq 2$. Let G be the graph consisting of $i-1$ disjoint cliques with sizes as equal as possible. Assign the pairs corresponding to the edges of G as the lists for the vertices of Q and let $L(v_1) \subset \cdots \subset L(v_i) = V(G)$. This assignment is not sufficient.

To finish the proof of the lower bound for an arbitrary sufficient f with $\sum f(v) = 2q + as$ we use the inequality (11) to obtain that $f(v) = 2$ for all but at most $q - as$ vertices $v \in Q$. Then we conclude the proof with an argument analogous to (12). The details are omitted. □

Problem 9. *Suppose that*

$$f(v) = \begin{cases} s_i & \textit{for } v = v_i \in A \textit{ and} \\ 2 & \textit{for } v \in Q \end{cases}$$

such that $s_1 \leq \cdots \leq s_a$.

What conditions are sufficient and necessary for f being $G_{a,q}$-sufficient?

We already have seen that $q < t(s_i, i-1)$ for all $i \geq 2$ are necessary. It is easy to see that $q \geq \binom{s_1}{2} + s_1(s_2 - s_1)$ is also necessary, let G be the graph that we get by taking a clique of order s_2 and deleting edges of a clique of order $s_2 - s_1$. Assign its edges as the lists of Q, and let $L(v_1) \subset L(v_2) = V(G)$. One is tempted to conjecture that these conditions altogether are already sufficient.

This is the same problem as to ask that how much the sizes of L_i's can be decreased in Theorem 8.

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