Acta Oeconomica, Vol. 54 (1) pp. 1–2 (2004)

FOREWORD

to the special issue about John von Neumann

The interplay of the forces of history produced a remarkable economic and cultural flourishing in the Hungarian part of the short-lived (1867–1918) and ill-fated Austro-Hungarian Monarchy. The remarkable cultural atmosphere following the 1867 Compromise between Hungary and Austria allowed for and induced many bright members of the cohort born in the period, especially between 1885 and 1905, to unfold their exceptional artistic and/or scientific talents. The famous "generation of 1900" was far from being homogeneous but had at least two crucial assets in common: outstanding schooling (especially in the grammar schools) and devotion to modernisation.

John Lukacs in his book, Budapest 1900 (New York: Weidenfeld & Nicolson, 1988) portrayed and documented in details the historical and cultural components of this remarkable period. He provided a long list of writers, painters, composers, conductors, philosophers and scientists, known and famous not only in Hungary but in most other parts of the world, too. Suffice to refer here, as a point of illustration, to the well-known names of artists such as Béla Bartók, Zoltán Kodály, Ernst von Dohnányi, Ferenc Molnár and Sándor Márai, scientists such as Theodore von Karman, Albert Szent-Györgyi, Leo Szilárd, Edward Teller and Georg Békésy, mathematicians such as Frigyes Riesz and Lipót Fejér, a sociologist such as Karl Mannheim, the economic historian Karl Polanyi, and economists such as Thomas Balogh and Nicholas Kaldor. Some of them chose, others were forced to leave Hungary in various waves of emigration in the turbulent times following the two world wars with devastating consequences for Hungary. That was certainly a great loss to Hungary but a gain for mankind in most cases.

John von Neumann, born on 28 December 1903 in Budapest, was also an outstanding representative of that famous generation. He was a versatile scholar, who – with his path-breaking ideas – made contributions of great importance to various disciplines (see his biographical sketch in this issue). To celebrate the centenary of his birth the Hungarian Government announced a *"von Neumann memorial year"* and trusted the Minister of Informatics and Communication to organise a series of

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events in his honour. As part of that series a one-day conference was held at the Hungarian Academy of Sciences on 15 November 2003 to commemorate von Neumann's contributions to economics (see a report on the conference also in this issue). The papers presented at the conference reviewed and extended his improvement/extension to general equilibrium theory, optimal economic growth and game theory. Some papers presented at the conference were refereed and four of them selected for publication in this special issue of Acta Oeconomica. We believe they give an excellent overview of von Neumann's original contributions to economics.

Prof. Ernő Zalai
Member of HAS
Guest Editor

Prof. Ádám Török
Member of HAS
Editor-in-Chief

Acta Oeconomica, Vol. 54 (1) pp. 3–38 (2004)

THE VON NEUMANN MODEL AND THE EARLY MODELS OF GENERAL EQUILIBRIUM

E. ZALAI*

(Received: 16 November 2003; revision received: 17 December 2003; accepted: 5 January 2004)

The paper reconstructs the von Neumann model, comments on its salient features and critically reviews some of its generalisations. The issues related to the treatment of consumption, decomposability and uniqueness of the rate of growth and interest will be especially scrutinised. The most prominent models of general equilibrium that appeared before or roughly at the same time as von Neumann's model will be also reviewed in the paper and compared with it. It will be demonstrated that none of them had any noticeable influence on von Neumann's model, which is genuinely distinct, ideologically free and methodologically fresh and forward-looking. It will be argued that the model can be viewed as a brilliant mathematical metaphor of some deep-rooted old vision, pertaining to the core issues of commodity production.

Keywords: John von Neumann, Walras, Cassel, Wald, Leontief, general equilibrium models, mathematical economics, economic methodology, history of economic thought

JEL classification index: A1, B3, C0, D5

1. INTRODUCTION

John von Neumann was a versatile scholar, whose path-breaking ideas have enriched various disciplines. He has also made contributions of great importance to economics, although his only paper that was directly concerned with economics was his article on balanced economic growth, presented first at a Princeton seminar in 1932.[1] We hasten to add that his influence on the later development of eco-

* I would like to acknowledge and thank P. G. Hare, H. Kurz, J. Móczár, T. Révész, A. Simonovits, A. Steenge and J. Varga for their valuable comments on earlier versions of the paper. The usual caveat applies.

[1] Published in German in 1937, under the title "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes" (On an Economic System of

Correspondence: E. Zalai, Department of Mathematical Economics and Economic Analysis, Budapest University of Economic Sciences and Public Administration, H-1093 Budapest, Fővám tér 8, Hungary. E-mail: erno.zalai@bkae.hu

nomics became equally important through his ground-breaking work on game theory. He was the first to prove the existence of equilibrium for two-person zero-sum games in 1928, and their book (with Oskar Morgenstern) on game theory, published in 1944, was path-breaking in that field. Game theory, which studies the rules of rational human behaviour, is however an independent methodological discipline in itself and its domain is wider than economics.

Von Neumann's model of general equilibrium can be linked to some common and critical points of departure of different competing schools. Von Neumann's model was on the one hand a brilliant mathematical synthesis of the classical ideas concerned with the production and price proportions required by economic equilibrium. It was on the other hand a forerunner of modern mathematical economics, which became fully developed only some decades later, under the influence of neoclassical economics. Von Neumann generalised and employed for the first time Brouwer's fixed-point theorem in the proof of existence of competitive equilibrium, and used an explicit and full duality approach and a linear activity description of technological choice.

Von Neumann's model addressed very deep economic issues and it is no surprise that it can be fitted into most economic schools, into the – at times – Procrustean bed of neoclassical, Marxian or neo-Ricardian theoretical frameworks. And after a few years of a surprisingly chilly reception of his model, it has often been done so. (It suffices to refer here to the famous dispute between Solow and Kaldor at the 1958 Corfu conference, in which Kaldor strongly refuted the claim that the model of von Neumann was the "neo-classical school in a new disguise" (Lutz and Hague, 1961, pp. 296–297).

If not for other reasons, but because of the aforementioned merits of the model, one may understand R. Weintraub's enthusiasm when he went as far as stating "von Neumann's paper is, in my view, the single most important article in mathematical economics" (Weintraub, 1983, p. 13). Weintraub's judgement is however not shared universally by economists. At a 1974 conference in Warsaw Koopmans, while praising the many novel methodological aspects of the model, added "(t)he tremendous influence of von Neumann's paper demonstrates that contributions of great importance can be made in *a paper that is not very good economics*" (Koopmans, 1974, p. 3, emphasis added). Samuelson, who also seemed to share Koopmans' dry verdict, in his 1989 paper tried to downgrade the methodological importance of von Neumann's model as well. Nevertheless, he had to acknowledge that: "He darted briefly into our domain and it has never been the same since" (Samuelson, 1989, p. 121).

Equations and a Generalisation of Brouwer's Fixed-point Theorem). The title of the English translation became *"A Model of General Equilibrium"* and was published only in 1945.

The appearance of von Neumann's model coincided with two important developments, which left lasting effects on the development of mathematical economics and explains partly its controversial reception, too. One was the rise of quantitative economics[2] as an independent sub-discipline in the early 1930s. Another and strongly related development was the gradual expansion of the axiomatic, *a priori* modelling approach that resulted in a shift from *ex ante* to *ex post* modelling, to "a philosophy of model-building which was borrowed from Hilbert's metamathematics, to which von Neumann contributed substantially" (Punzo, 1989, p. 30). This change has gradually reached economics, too, as Weintraub (1983; 1985) described very vividly. The number of mathematically trained and oriented economists has been steadily growing. They have brought into economics a radically changed perception of the subject matter and the methodology of mathematics ("Bourbakism came to mathematical economics", cf. Weintraub and Mirowski, 1994).

The adoption of the formal axiomatic approach and mathematical reasoning did accelerate the development of mathematical economics, but this progress incurred significant costs, too. The focus of research had swiftly shifted from the applied (concrete) to the pure (abstract), to "implicit theorising" (Leontief). The requirements of logical consistency and mathematical elegance gained power over empirical relevance. Mathematics, "because of the lack of sufficiently secure experimental base" (Debreu, 1991), became increasingly a tool of logical calculus, instead of providing means for making quantitative empirical predictions. Beyond the traditional ideological and methodological schisms, the economics profession became further divided by language (verbal vs. mathematical) and methodology (analytical-formalist vs. historical-social) as well.

Marshall, one of the founders of modern economics, was among the first who warned against the extensive and unjustified use of mathematics in economics, because it "might lead us astray in pursuit of intellectual toys, imaginary problems" (Pigou, 1925, p. 84, quoted by Ekelund and Hébert, 1997). Von Neumann was also very much aware of the dangers involved, not only in sciences dealing with real phenomena, such as economics, but for the development of mathematics itself. "As a mathematical discipline travels far from its empirical source, or still more … if it is indirectly inspired by ideas coming from 'reality', … it becomes

2 It has initially appeared under the name of econometrics, but later it became divided into three somewhat independent sub-disciplines: mathematical economics, operations research and econometrics. It is usually connected to the foundation of the Econometric Society (1930), and the journal of *Econometrica* (1933), although its roots can be traced back at least as far as to the book of Cournot (1838).

more and more purely astheticizing, more and more purely l'art pour l'art" (Neumann, 1947, p. 234).

Although the model of von Neumann was in many ways a prototype of the *a priori* (*ex ante*) models in economics, he often cautioned against the misuse of such models. Morgenstern (1976) recalls that von Neumann has repeatedly criticised economists for not using more appropriate mathematics and emphasised the need for more comprehensive mathematical tools than those borrowed from classical physics. It is perhaps not by accident that von Neumann did not continue his research on the abstract models of general equilibrium. Indeed, he did not just advocate the need for a new methodology, but set an excellent example for others to follow by their work written together with Morgenstern on game theory, initiating a totally new discipline almost from scratch.[3]

The paper is organised as follows. *Section 2* reconstructs the way von Neumann set up his model. *Section 3* contains notes on some salient aspects of the model and critically reviews some important attempts to generalise it. Some issues related to consumption, decomposability and uniqueness will be especially scrutinised. Although it is difficult to add much new to the vast literature, the author would like the reader to find some fresh ideas and new insights, too, in this appraisal nevertheless. The *remaining sections* are devoted to the most prominent models of general equilibrium that appeared before or roughly at the same time as von Neumann's paper. The static and stationary models of Walras, Cassel, Schlesinger and Wald, and Leontief will be revisited and compared with von Neumann's. It will be demonstrated that none of them had any noticeable influence on von Neumann's model, which is genuinely distinct, ideologically free and methodologically very fresh and forward-looking. And that is all true despite the fact that the model can be viewed as a brilliant mathematical metaphor of some deep-rooted visions, pertaining to the core issues of commodity production.

2. THE VON NEUMANN MODEL OF ECONOMIC EQUILIBRIUM

The *table* below illustrates the key components and the basic accounting framework of the model. The rows ($i = 1, 2,\ldots, n$) refer to economic goods, the columns ($j = 1, 2,\ldots, m$) to economic processes. The table contains their lists and the amounts of the goods produced (Y_{ij}) and used (X_{ij}) in the various processes in some period of time. The last column contains the unit prices of the goods and the

[3] In this paper we shall not address issues related to von Neumann's contribution to modern game theory. A separate paper by F. Forgó (2003), also included in this volume, discusses that topic in detail.

last row the levels of the processes (activities). The latter refers to the intensities of operation of the processes.

The basic accounting framework (goods, activities, inputs and outputs)
in the von Neumann model

	Process 1	Process 2	…	Process j	…	Process m	Prices
Good 1	$Y_{11} \vee X_{11}$	$Y_{12} \vee X_{12}$	…	$Y_{1j} \vee X_{1j}$	…	$Y_{1m} \vee X_{1m}$	p_1
Good 2	$Y_{21} \vee X_{21}$	$Y_{22} \vee X_{22}$	…	$Y_{2j} \vee X_{2j}$	…	$Y_{2m} \vee X_{2m}$	p_2
⋮	⋮	⋮		⋮		⋮	⋮
Good i	$Y_{i1} \vee X_{i1}$	$Y_{i2} \vee X_{i2}$	…	$Y_{ij} \vee X_{ij}$	…	$Y_{im} \vee X_{im}$	p_i
⋮	⋮	⋮		⋮		⋮	⋮
Good n	$Y_{n1} \vee X_{n1}$	$Y_{n2} \vee X_{n2}$	…	$Y_{nj} \vee X_{nj}$	…	$Y_{nm} \vee X_{nm}$	p_n
Activity levels	x_1	x_2	…	x_j	…	x_m	

Von Neumann considers an economy, in which production takes place in uniform, discrete periods of time, with exchange only at the turn of such intervals. By this assumption, the output of a given period can only be used in the next period. Because of the assumption of uniform production periods, he has to assume that "processes of longer duration (have) to be broken down into single processes of unit duration introducing if necessary intermediate products as additional goods" (1945, p. 2). He also postulated that "capital goods are to be inserted on both sides of (the processes); wear and tear of capital goods is to be described by introducing different stages of wear as different goods, using a separate (process) for each of those" *(ibid.)*. In this way, he actually turned fixed capital into circulating capital and assumed that capital circulation took uniformly one time period. He thus did not have to face the problems caused by the proper measurement of capital tied up in production.

Von Neumann assumes that X_{ij}, the amount of good i consumed in activity j, contains not only the direct material inputs of production but also the consumption of labour (and their households) engaged in it. For the sake of simplicity he also presumes that the consumption of the households consists only of necessities of life, and "all income in excess of necessities of life will be reinvested" *(ibid.)*, moreover, consumption patterns do not change over time and there is no technical progress. And a final assumption: "the natural factors of production, including labour, can be expanded in unlimited quantities" *(ibid.)*. The rate of growth depends thus on "man-made" factors of production only, i.e., on the rate of accumulation of capital goods.

With these assumptions von Neumann defines an abstract, quasi-stationary economy, in which there is no reason for the proportions of the production and prices to change, once a state of equilibrium has been reached. The equilibrium of a quasi-stationary economy is a steady state, in which every physical quantity (the activity levels, the production and the use of various goods) changes by the same constant rate (λ). They increase, stagnate or decrease depending on the sign of λ.

In view of the rather stringent assumptions, von Neumann carefully avoided treating his model as a complete description of the working of a real economy, unlike some modern followers of general equilibrium theory. Quite to the contrary, he made it clear that his model was a very abstract metaphor of a real economy, with the help of which one can shed light on some specific features of modern commodity production systems. He set out to analyse, first and foremost, the mutual dependence ("remarkable dual symmetry") of the rules guiding the selection of efficient (optimal) technologies on the one hand, and the determination of equilibrium prices (which make their use profitable) on the other, resulting from the circular nature of reproduction:

> "In order to be able to discuss (the mentioned properties of the economic system) quite freely we shall idealise other elements of the situation … Most of these idealisations are irrelevant, but this question will not be discussed here" (Neumann, 1945, p. 1).

He was aware of it that economics is as yet not a well-developed scientific discipline and therefore the use of stringent abstractions is unavoidable. It is worth quoting him at length on this subject:

> "It is frequently said that economics is not penetrable by rigorous scientific analysis, because one can not experiment freely. … Experimentation is a convenient tool, but large bodies of science have been developed without it. … What seems to be essentially difficult in economics is the definition of categories. … it is always the conceptual area that the lack of exactness lies. … Now, all science started like this, and economics, as a science, is only a few hundred years old. The natural sciences were more than a millennium old when the first really important progress was made. … methods in economic science are not worse than they were in other fields. But we will still require a great deal of research to develop the essential concepts – the really usable ideas." (Neumann, 1955, in Bródy and Vámos, 1995, p. 639)

Based on the above assumptions let us now formulate the conditions of balanced supply and demand, using equations as his predecessors who followed the classical *ex post* modelling approach:

$$Y_{i1} + Y_{i2} + \ldots + Y_{im} = (1+\lambda)\cdot(X_{i1} + X_{i2} + \ldots + X_{im}), \qquad i = 1, 2, \ldots, n. \qquad (1)$$

The equilibrium prices (p_i, $i = 1, 2, \ldots, n$) of such an economy must yield the same (π) rate of return (interest, as von Neumann called it) on capital in every activity used. The prices (as a matter of fact, only price ratios) can thus be defined by the following set of equations:

$$p_1 \cdot Y_{1j} + p_2 \cdot Y_{2j} + \ldots + p_n \cdot Y_{nj} = (1+\pi) \cdot (p_1 \cdot X_{1j} + p_2 \cdot X_{2j} + \ldots + p_n \cdot X_{nj}), \quad j = 1, 2, \ldots, m. \quad (2)$$

By dividing each equation by the level of the corresponding activity (x_j) and assuming constant output ($b_{ij} = Y_{ij}/x_j$) and input ($a_{ij} = X_{ij}/x_j$) coefficients (constant returns to scale), the equilibrium conditions can be rewritten into the following symmetric, dual forms:

$$b_{i1} \cdot x_1 + b_{i2} \cdot x_2 + \ldots + b_{im} \cdot x_m = (1+\lambda) \cdot (a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \ldots + a_{im} \cdot x_m), \quad i = 1, 2, \ldots, n, \quad (3)$$

$$p_1 \cdot b_{1j} + p_2 \cdot b_{2j} + \ldots + p_n \cdot b_{nj} = (1+\pi) \cdot (p_1 \cdot a_{1j} + p_2 \cdot a_{2j} + \ldots + p_n \cdot a_{nj}), \quad j = 1, 2, \ldots, m. \quad (4)$$

As far as the economic content is concerned, there is nothing novel in the above description of the conditions of equilibrium of an economy, its origins can clearly be traced back to the classical economists. Champernowne (1945) first asserted the classical origin of the model in his paper accompanying the English publication of von Neumann's paper. A typical example of a similar quasi-stationary economic model is Marx's scheme of simple and extended reproduction, which in turn was inspired by the work of Quesnay. Sraffa (1960) also used a similar model to analyse some features of long-term equilibrium prices. He defined a "self-reproducing standard system", in which production expands at an equi-proportional rate of growth, but that was not meant to be a model of actual production. (See Kurz and Salvadori, 2001 for a comparison of the models of Sraffa and von Neumann, and also for further references.)

This fact may explain the puzzling remark of von Neumann: "(i)t is obvious to what kind of theoretical models the above assumptions correspond" (Neumann, 1945, p. 2).[4] What was novel in von Neumann's approach was the precise mathematical reformulation of the classical concept for the case of joint production and choice of techniques, and the proof of existence of an equilibrium solution. Namely, unlike in other models of general equilibrium of his time, von Neumann took joint production (activities may produce several goods together) and technological choice (the same commodity may be produced by several activities) explicitly into account in his model.

The above circumstances imply that the system of equations (3) and (4) will be irregular (they will as a rule have much more variables than equations) and therefore the traditional method of counting equations cannot be used. One cannot as-

4 Von Neumann did not reveal the origins of his model. On the alternative interpretations of the von Neumann model see Kurz and Salvadori (1995, esp. pp. 407–414). I shall also come back to this issue at the end of this paper.

sume, in general, that there exists a structure of production (**x**) for which goods are produced (**Bx**) and used (**Ax**) in the same proportions. Nor can one expect, in general, to find a price system for which the ratios of revenues (**pB**) and costs (**pA**) will be the same for each process. The system of equations (3) and (4) may thus not have solutions at all. But even if it does, some variables may assume negative values, which would normally violate its economic content. To be more precise, one cannot accept negative values for the activity levels or the prices if one assumes the irreversibility of the activities and free disposal. Indeed, von Neumann did implicitly adopt these assumptions by restricting the values of the activity levels and prices to be non-negative.

In order to solve the problem, von Neumann relaxed the equilibrium conditions. On the one hand, he introduced the possibility of excess supply of some goods and on the other hand, extra costs for some processes. In order to stay in line with the rule of supply and demand, von Neumann had to complement the above assumptions with two rules. Namely, any commodity in excess supply will be a free good and hence its price zero (the Rule of Free Goods), and the activities that do not yield the maximum rate of return will not be used in equilibrium (the Rule of Idle Activities).

Therefore, equilibrium conditions should be formulated as a complementarity problem and use the following system instead of equations (3) and (4):

$$\textstyle\sum_j b_{ij}\cdot x_j \geq (1+\lambda)\cdot\sum_j a_{ij}\cdot x_j, \qquad i = 1, 2,\ldots, n, \tag{5a}$$

$$p_i\cdot\textstyle\sum_j b_{ij}\cdot x_j = (1+\lambda)\cdot p_i\cdot\sum_j a_{ij}\cdot x_j, \qquad i = 1, 2,\ldots, n, \tag{5b}$$

$$\textstyle\sum_i p_i\cdot b_{ij} \leq (1+\pi)\cdot\sum_i p_i\cdot a_{ij}, \qquad j = 1, 2,\ldots, m, \tag{6a}$$

$$x_j\cdot\textstyle\sum_i p_i\cdot b_{ij} = (1+\pi)\cdot x_j\cdot\sum_i p_i\cdot a_{ij}, \qquad j = 1, 2,\ldots, m. \tag{6b}$$

Von Neumann was not the first to use complementary slackness conditions, which has become a standard tool in equilibrium models. In a different context Zeuthen (1933) and Schlesinger (1935) also suggested the use of the Rule of Free Goods in order to avoid the negative prices in the Cassel model. But von Neumann was the first to formulate duality and complementary slackness conditions in a symmetric, full-fledged manner.

As can be seen, the equilibrium conditions determine only the relative sizes (proportions) of the variables x_j and p_i. If some values of x_j and p_i satisfy the above system, then $s\cdot x_j$ and $v\cdot p_i$ will also satisfy it, as long as s and v are positive scalars. The trivial solutions (no production at all or nothing but free goods) have no economic relevance, so they can be ruled out at the outset. One can thus set the activity and price levels in any meaningful way. Von Neumann did it by setting their sums equal to one ($\sum_j x_j = \sum_i p_i = 1$), that is, restricted their domain to the so-called standard (unit) simplex.

Observe that by multiplying both sides of inequalities (5a) and (6a) with the corresponding (complementing) p_i and x_j variables, respectively and taking their sum, one can derive the following series of inequalities:

$$(1+\lambda)\cdot\sum_{ij} p_i\cdot a_{ij}\cdot x_j \leq \sum_{ij} p_i\cdot b_{ij}\cdot x_j \leq (1+\pi)\cdot\sum_{ij} p_i\cdot a_{ij}\cdot x_j. \qquad (7)$$

There are two important conclusions that follow from the above inequalities. First, if the value of total output ($\sum_{ij} p_i\cdot b_{ij}\cdot x_j$) is positive, as required from any meaningful solution, then $\lambda = \pi$, i.e., the equilibrium rate of growth and interest (return on capital) will be equal. Second, if $\lambda = \pi$, then the complementary slackness conditions will be automatically met by the solutions of (5a) and (6a).

Von Neumann postulated that a_{ij}, $b_{ij} \geq 0$ and $a_{ij} + b_{ij} > 0$ for all i and j, that is, each commodity takes part in every activity either as input and/or output. This assumption guarantees that the value of total output will be positive in the case of any feasible (primal) solution and therefore the equilibrium rates of growth and interest will be equal and uniquely determined by the coefficients of the model. The same conditions imply also, as indicated above, the fulfilment of the complementary slackness conditions. One can thus leave equation (5b) and (6b) out of the final form of the model and simplify it further by introducing a common factor of growth and interest ($\alpha = 1 + \lambda = 1 + \pi$).

$$\mathbf{x}, \mathbf{p} \geq \mathbf{0}, \alpha > 0, \qquad (8a)$$

$$\mathbf{1x} = \mathbf{p1} = 1, \qquad (8b)$$

$$\mathbf{Bx} \geq \alpha\mathbf{Ax}, \qquad (8c)$$

$$\mathbf{pB} \leq \alpha\mathbf{pA}, \qquad (8d)$$

where (for shorthand we switched to matrix-vector notation) $\mathbf{x} = (x_j)$, $\mathbf{p} = (p_i)$, $\mathbf{B} = (b_{ij})$, $\mathbf{A} = (a_{ij})$ and $\mathbf{1}$ is a (summation) vector, the elements of which are all equal to 1. Von Neumann provided a rigorous proof showing the existence of a solution of the above system.

3. SOME NOTES ON THE PROPERTIES AND GENERALISATIONS OF THE VON NEUMANN MODEL

3.1. The uniqueness of the equilibrium in terms of the rate of growth and interest was crucial for von Neumann for at least two reasons. First, it made the duality of the two (quantity and value) sides of the model complete. The common equilibrium rate is the highest possible uniform rate of growth on the one hand, and the smallest possible equilibrium rate of interest on the other. In other words

$$\lambda^* = \alpha - 1 = \max \{ \lambda : \exists\ \mathbf{x} \geq \mathbf{0}, \mathbf{1x} = 1, \mathbf{Bx} \geq (1 + \lambda)\mathbf{Ax} \}, \tag{9a}$$

$$\pi^* = \alpha - 1 = \min \{ \pi : \exists\ \mathbf{p} \geq \mathbf{0}, \mathbf{p1} = 1, \mathbf{pB} \leq (1 + \pi)\mathbf{pA} \}. \tag{9b}$$

Second, this equality established the crucial mathematical link between the existence of equilibrium in the model of balanced growth and that of the two-person zero-sum game. For, it seems evident that it was the former model that provided directly the mathematical inspiration for his growth model. Von Neumann used almost identical mathematical forms in both and the same technique in proving the existence of solution, namely the minimax (saddle point) approach, for which he generalised Brouwer's fixed-point theorem.[5]

3.2. The equilibrium conditions in the growth model are, as von Neumann pointed out, the necessary conditions for a minimax solution (saddle point) of the following function:

$$F(\mathbf{x}, \mathbf{p}) = F(x_1, x_2 \ldots x_m ; p_1, p_2 ,\ldots, p_n) = \textstyle\sum_{ij} p_i{\cdot}b_{ij}{\cdot}x_j \,/\, \sum_{ij} p_i{\cdot}a_{ij}{\cdot}x_j, \tag{10}$$

where the denominator, the total value of the inputs, is assumed to be positive. Function F can be called the *profit function*, since its value at ($\mathbf{x}$, $\mathbf{p}$) determines the profit factor.

It is easy to show that (α^*, $\mathbf{x}^*$, $\mathbf{p}^*$) is an equilibrium solution of the von Neumann model if and only if $F(\mathbf{x}^*, \mathbf{p})$ reaches its minimum in $\mathbf{p}$ at $\mathbf{p}^*$, and $F(\mathbf{x}, \mathbf{p}^*)$ reaches its maximum in $\mathbf{x}$ at $\mathbf{x}^*$, where the value of $F(\mathbf{x}^*, \mathbf{p}^*)$ is equal to α^*, that is,

$$F(\mathbf{x}, \mathbf{p}^*) \leq F(\mathbf{x}^*, \mathbf{p}^*) \leq F(\mathbf{x}^*, \mathbf{p}). \tag{11}$$

[5] Kakutani (1941) provided later a more general theorem with much shorter proof that became the standard reference in the existence proofs of general equilibrium. As a matter of interest, Kakutani did not know von Neumann's theorem when he prepared the first draft of his paper. He consulted nevertheless often with von Neumann as he was finalising his paper for publication at Princeton (cf. Weintraub, 1983).

3.3. Von Neumann has called attention to an interesting formal analogy that exists between economic phenomena and thermodynamics. Namely that the role of the profit function "appears to be similar to that of thermodynamic potentials", and he conjectured that "the similarity will persist in its full phenomenological generality (independently from our restrictive idealisations)" (1945, p. 1).

Thermodynamics and physics, in general, have significantly influenced the development of the methodology of economics. The founders of modern neoclassical analysis, Hicks (1939) and Samuelson (1947) borrowed, for example, the basic tools of their mathematical analysis from classical thermodynamics. Georgescu-Roegen (1971) devoted a whole book to illustrate similarities between economics and thermodynamics. More recently Bródy (1989) has revisited von Neumann's conjecture and offered a potential explanation of its deeper meaning (see also Bródy, Martinás and Sajó, 1985; Gilányi and Martinás, 2000).

3.4. The uniqueness of the equilibrium rate of growth is of special interest because it means that it is the *maximal* rate of expansion allowed by the input-output coefficients. The unique path of steady-state growth, called *the von Neumann-path*, exhibits an interesting property that was first pointed out by Dorfman, Samuelson and Solow (1958), and called aptly a *turnpike* (express highway) property. Several turnpike theorems have followed. They prove in essence that optimal growth paths, even if they start and terminate outside of the von Neumann-path, will run near to or on the von Neumann-path most of the time, provided that the time horizon is long enough (for references see the review article of Koopmans, 1964).

3.5. The assumption $a_{ij} + b_{ij} > 0$ guaranteed for von Neumann the existence and uniqueness of the common equilibrium rate of growth and interest. It is however a rather strong assumption that cannot be defended on economic grounds. Kemeny, Morgenstern and Thompson (1956) replaced it by much weaker postulates. At the same time they simplified the existence proof without invoking a fixed-point theorem. The KMT conditions are as follows:

$$\sum_j b_{ij} > 0 \text{ for all } i \text{ and } \sum_i a_{ij} > 0 \text{ for all } j, \tag{12}$$

which state that each good is reproducible and that each process requires (directly) at least one product as input (either in production or in consumption).

Both postulates are quite natural assumptions. Von Neumann himself, as a matter of fact, adopted the first one, when he excluded the natural factors of production from his model. One needs the second one for exactly the same reason, to ensure that the rates of growth and interest remain finite despite the absence of exogenous resource constraints. (Incidentally, allowing for not only direct but indirect requirements, too, one can further relax this assumption.) The above assumptions, however, do not guarantee that the total value of production will be positive (and the rates of growth and interest equal). As a consequence of this modification, the positivity of the value of total output ($\sum_{ij} p_i \cdot b_{ij} \cdot x_j > 0$) had to be added to the equilibrium conditions, as a special requirement, in order to make the solutions economically meaningful and also to secure the equality of the two factors. We shall refer to the resulting variant of the von Neumann model as the *KMT model*.

The same authors have also shown that under the revised conditions the equilibrium factor of growth and interest is no longer necessarily unique, the number of possible values of factors is finite and cannot exceed the minimum of the number of activities and goods. Multiple solutions can exist if the model-economy is *decomposable*, meaning that some groups of activities can be operated without using goods that can be produced only by activities not belonging to that group.

3.6. The above uniqueness, as pointed out earlier, was crucial for von Neumann. He made that point clear as he commented on his assumption that $a_{ij} + b_{ij} > 0$: "it must be imposed in order to assure uniqueness of α, β ($1+\lambda$ and $1+\pi$ in our notation) as otherwise W (the system) might break up into disconnected parts" (1945, p. 3). This quotation reveals also that he knew exactly that with his assumption he actually ensured the indecomposability of the economy. Gale (1960) therefore suggested relaxing von Neumann's original assumption by simply postulating that all goods must be produced in any solution that fulfils the balance conditions given by (5a). This is however the consequence rather than the proper definition of the indecomposability of an economic system given by constant input and output coefficients. Móczár (1995) provided later a proper structural characterisation of (in)decomposability, in terms of the input and output coefficient matrices of von Neumann.

A wide range of literature has been devoted to the consequences of decomposability in models of the von Neumann type. The mathematical properties of multiple solutions have been fully explored by several authors. The comprehensive characterisations given by Morishima (1971) and Bromek (1974a) deserve special attention. From a mathematical point of view the investigations are interesting, but the economic relevance of multiple equilibrium solutions, in terms of the rate of growth and interest, is in our opinion, to say the least, doubtful (we shall come back to this issue later).

3.7. The only point that brought von Neumann closer to the neoclassical rather than the classical terminology was his usage of the term of interest instead of profit. He viewed profit, like most neoclassical economists, as an excess income above normal costs (including interest on capital), which the classical economists called "extra-profit". This can be distilled from the passage in which von Neumann comments on the conditions of equilibrium: "in equilibrium no profit can be made on any process … else prices or the rate of interest would rise – it is clear how this abstraction is to be understood" (*ibid.,* p. 3). This puzzling remark, typical of von Neumann, suggests that he must have assumed that in an economy "without monetary complications" the equilibrium rate of interest would be equal to the uniform rate of surplus or "profit", using the latter term in its classical meaning.

Following von Neumann's instruction, consumption can be explicitly introduced in the model[6] by decomposing the inputs $\mathbf{A} = (a_{ij})$ into uses in production $\mathbf{R} = (r_{ij})$ and in consumption $\mathbf{C} = (c_{ij})$, where $a_{ij} = r_{ij} + c_{ij}$. The cost of necessary consumption ($w_j = \sum_i p_i \cdot c_{ij}$) can be interpreted as the unit wage cost because the cost of consumption enters the definition of prices in its place (material cost + cost of consumption + plus interest). What else could it represent anyway? Classical economists have also frequently used this simplifying solution, assuming that workers spend all their income on consumption. With this in mind, one can rightly replace interest with the classical notion of *profit*, whereby the equilibrium prices become the well-known *prices of production*, used by classical economists.

3.8. Morishima (1964) linked consumption to the amount of labour used. He defined the consumption coefficients as $c_{ij} = c_i \cdot m_j$, where c_i is the amount of good i required for the reproduction of one hour labour and m_j is the number of hours employed in activity j, operated at unit level of intensity. With matrix notation: $\mathbf{C} = \mathbf{c} \circ \mathbf{m}$, where symbol $\mathbf{a} \circ \mathbf{b}$ denotes the outer product of vectors $\mathbf{a}$ and $\mathbf{b}$. The hourly wage rate is thus $w = \mathbf{pc} = \sum_i p_i \cdot c_i$, the total number of hours employed in the economy as a whole is $L = \mathbf{mx} = \sum_j m_j \cdot x_j$ and the total wage bill $w \cdot L = \mathbf{(pc)(mx)} = \mathbf{p(c \circ m)x} = \sum_{ij} p_i \cdot c_{ij} \cdot x_j$. The equilibrium conditions in Morishima's explication (concretion) of the von Neumann model take thus the following form:

$$\mathbf{x}, \mathbf{p} \geq \mathbf{0}, \alpha > 0, \tag{13a}$$

$$\mathbf{1x} = \mathbf{p1} = 1, \tag{13b}$$

$$\mathbf{Bx} \geq \alpha(\mathbf{R} + \mathbf{c} \circ \mathbf{m})\mathbf{x}, \tag{13c}$$

$$\mathbf{pB} \leq \alpha \mathbf{p}(\mathbf{R} + \mathbf{c} \circ \mathbf{m}), \tag{13d}$$

$$\mathbf{pBx} > 0. \tag{13e}$$

6 See the review article of Bauer (1974) on the various possibilities offered in the literature.

Such a generalisation of the von Neumann model (by making its specification more concrete) is perfectly legitimate, as long as one assumes that *labour power is homogeneous* and *indispensable*. This latter assumption means that reproduction cannot take place at any positive level without using labour power. It can be postulated as follows: $\sum_j m_j{\cdot}x_j > 0$, for all $x_j \geq 0$ such that there exists $\alpha > 0$, at which conditions (5a) are fulfilled in such a way that at least one good is produced.

It has been shown that for certain magnitudes of exogenously given consumption coefficients the KMT model may not have such a solution, in which the price of some "necessities of life", that is, the wage rate is positive. It has also been shown that in such cases some or all consumption coefficients could be increased without decreasing the rate of interest or growth. What is perhaps even more important, Bromek (1974b) has shown that the common equilibrium rate of growth and interest must be the highest possible rate of expansion, whenever a positive wage can be associated with it.

3.9. This last observation is crucial and must hold for more general models, too, if wages are defined as the cost of a given basket of wage goods. In such an interpretation and the corresponding extensions of the von Neumann model, one can thus re-establish von Neumann's assertion that the equilibrium rates of growth and interest are equal and uniquely determined. One can in fact do it using a requirement that is somewhat weaker and more plausible than indecomposability. One simply augments the definition of equilibrium with requiring the positivity of the total value of consumption $\sum_{ij} p_i{\cdot}c_{ij}{\cdot}x_j > 0$ instead of the total value of production $\sum_{ij} p_i{\cdot}b_{ij}{\cdot}x_j > 0$ as Kemeny, Morgenstern and Thompson did. Note that this latter constraint of ours implies the one in KMT, which is not necessarily the case the other way around, as noted above.

Any economist should *ab ovo* exclude solutions in which the value of consumption is nil. In a model of long-term economic equilibrium, in which the consumption coefficients are given exogenously, in fact arbitrarily, one cannot justify such a situation on sound economic grounds. Such a situation could not be sustained for any period of time. They are mathematical artefacts, totally irrelevant from the point of view of an economist. Von Neumann was thus completely right in our view to postulate that the common equilibrium rate of growth and interest is unique and maximal, although he failed to provide a convincing argument for that. There is, however, one problem with that assumption. Namely, there is no easy way to guarantee that the equilibrium solution will be a saddle-point. This may disappoint some followers of von Neumann.

3.10. Let us now introduce a scalar *variable* to measure the *level* of (necessary) consumption and denote it by γ. The simplest way to let the level of consumption vary is to treat its structure constant. In such a case the c_{ij} (per activity level) consumption coefficients can be determined as $c_{ij} = \gamma \cdot s_{ij}$, where the s_{ij} coefficients are appropriately chosen structural constants, and γ is variable, the level of which is set either exogenously or endogenously.

One could of course introduce more elaborate demand functions, too. That would only make the model and the analysis technically more complicated, but hardly change the results. One should be able to demonstrate a real trade-off between γ and α, i.e., to show that one is a strictly decreasing function of the other. Any meaningful demand system should fulfil this requirement. It follows from the set-up of the model that γ represents both the level of consumption and that of the real wage, in the same way indeed as α stands for both the factor of growth and profit. The trade-off curve defined by the values of γ and α gives the (optimal) *consumption-investment* as well as the *wage-profit frontier,* as they are called in the neoclassical theory of growth.

The classical concept of wage-profit correspondence was reintroduced by Hicks (1939), as the factor-price frontier. Bruno (1969) proved and called attention to the duality (coincidence) of the wage-profit and the consumption-investment frontier in the neoclassical model of optimal growth. This was not the case in the KMT model because of the possibility of multiple equilibrium solutions. Morishima (1971) therefore reinterpreted their concepts and their duality for the von Neumann model. He redefined them as the *maximal* consumption-investment (primal) and the *minimal* wage-profit (dual) frontier, using definition (9a) and (9b) with a varying level of consumption. The two frontiers will as a rule differ in decomposable economies. Morishima has in fact defined several (subordinate) frontiers, each corresponding to one of the equilibrium rates of growth (profit) feasible at the given level of consumption (real wage).

Bromek (1974b) on the other hand has proposed to re-establish the coincidence (the strict duality) of the two frontiers, by requiring the wage level to be positive in equilibrium, as we did before. Burmeister and Kuga (1970) suggested an almost identical solution in a somewhat more general intertemporal model allowing for joint production. Bromek has shown that γ is a strictly decreasing function of α, but it may be discontinuous at some (singular) points, if the system is decomposable. The domain of the inverse function, the consumption-investment frontier, will thus be disconnected at the corresponding values.

3.11. Let us confine the range of equilibrium solutions in Morishima's explication of the von Neumann model by requiring the value of consumption (the wage rate) to be positive ($\mathbf{pc} > 0$). We must assume of course, as indicated above, that labour is indispensable (which guarantees that $\mathbf{mx} > 0$ in any feasible solution). From the analysis of Morishima and Bromek it follows that the conditions of the equilibrium can be rewritten as follows:

$$\mathbf{x}, \mathbf{p} \geq \mathbf{0}, \alpha > 0, \tag{14a}$$

$$\mathbf{mx} = \mathbf{ps} = 1, \tag{14b}$$

$$\mathbf{Bx} \geq \alpha(\mathbf{Rx} + \gamma\mathbf{s}), \tag{14c}$$

$$\mathbf{pB} \leq \alpha(\mathbf{pR} + \gamma\mathbf{m}), \tag{14d}$$

where γ is here the level of per hour consumption and the $\mathbf{s} = (s_i)$ coefficients are the amounts consumed at unit level of γ. The level of γ is assumed to be given exogenously here, as in the case of the original von Neumann model.

All we have done was setting the levels of the variables differently, using the equations given in (14b). They together imply that the total value of consumption will be positive, as required for meaningful solutions, so there is no need to add them. Notice that for the same reason

$$(\mathbf{R} + \gamma\mathbf{s}\circ\mathbf{m})\mathbf{x} = \mathbf{Rx} + \gamma\mathbf{s}, \text{ and } \mathbf{p}(\mathbf{R} + \gamma\mathbf{s}\circ\mathbf{m}) = \mathbf{pR} + \gamma\mathbf{m}, \tag{15}$$

and this is why we could replace inequalities (8c) and (8d), homogeneous in variables **x** and **p,** respectively, by their inhomogeneous counterparts.

System (14) provides a complete characterisation of equilibrium in the extended von Neumann model, with homogeneous labour and a parametrically changing level of consumption. It is equivalent to system (13), the conditions of Morishima's model, except for the last one, which was replaced by $\mathbf{pCx} > 0$ or $\mathbf{p}(\mathbf{s}\circ\mathbf{m})\mathbf{x} > 0$ in our case. This characterisation of the equilibrium is interesting for several reasons. First, it introduces properly the real wage ($w = \gamma$) into the model by choosing the unit consumption bundle as numeraire ($\mathbf{ps} = 1$). Second, this generalised form proved to be useful in comparing the analysis of von Neumann and Sraffa.[7] But most of all, it allows one to change the exogenous-endogenous role of the growth (profit) factor and the level of consumption (real wage). One could do

7 The relation of the von Neumann model to Sraffa's model of prices in the case of joint production became a subject of special interest in the literature (see Kurz and Salvadori, 1995). For the lack of space we cannot discuss this issue here in detail but mention a few crucial points. Sraffa assumed that, at any feasible rate of profit, there existed "square" techniques (the same number of processes as goods) that determined equilibrium prices. Bidard (1986) has shown that, except for single odd cases, there exists square solution to a von Neumann model, thus the crucial formal link between the two analyses can be established. This gives also an important link to joint production versions of the Leontief-type models. For this latter theme see Konijn and Steenge (1995). Needless to say, there remain many important differences between the above-mentioned models and approaches with regards to the purpose and economic context of the analysis.

that both for technical advantages and in order to change the underlying economic hypothesis.

In the original model of von Neumann the level of consumption was determined exogenously. He did not refer to any mechanism that establishes its level and structure. He might have borrowed the concept of necessary consumption from classical economics. Once the level of necessary consumption is given, this determines the rate of profit and growth. But one could easily turn the causal relationship around: take the rate of growth as given exogenously and let the level of consumption become endogenous instead. One can, for instance, assume a steadily growing population and constant per capita consumption needs. That will determine the necessary rate of growth, like in some neoclassical models of economic growth. (Cassel, as we shall see shortly, assumed that all natural factors grew at the same constant rate.)

But, as long as the equilibrium rate of growth and profit is unique, there is a one-to-one correspondence between α and γ, that is, between the feasible values of the growth factor and the level of consumption. From a purely mathematical point of view therefore it does not matter which is determined first. Once one of them is given, the level of the other is determined, too. In the analysis of the consumption-investment or the wage-profit function it proves to be useful to change the mathematical role of the two potential variables. For, as was pointed out, certain levels of consumption may not be associated with an equilibrium rate of growth. But this is not the case the other way around.

It is easy to show (Bromek, 1974b or Zalai, 2002) that the domain of the feasible equilibrium factors of growth (profit) is a connected set, $H = (0; \Lambda)$, which can be determined as

$$H = \{\alpha > 0\colon \exists\, \mathbf{x} \geq \mathbf{0},\, [(1/\alpha)\mathbf{B} - \mathbf{R}]\mathbf{x} \geq \mathbf{s}\}, \tag{16}$$

where Λ is, under natural assumptions (i.e., constant capital is indispensable) a finite number and H may be closed from above, thus, $H = (0; \Lambda]$.

It can also be shown that treating α as parameter (exogenous variable) and γ as (endogenous) variable, the equilibrium of the generalised von Neumann model, system (14), can be found by solving the following (primal-dual) pair of parametric (α) linear programming problems:

Primal problem	Dual problem	
$y \geq 0,\ \mathbf{x} \geq \mathbf{0}$	$w \geq 0,\ \mathbf{p} \geq \mathbf{0}$	
$[(1/\alpha)\mathbf{B} - \mathbf{R}]\mathbf{x} \geq y\mathbf{s}$	$\mathbf{p}[(1/\alpha)\mathbf{B} - \mathbf{R}] \leq w\mathbf{m}$	(17)
$\mathbf{mx} \leq 1$	$\mathbf{ps} \geq 1$	
$y \to \max!$	$w \to \min!$	

One can easily see the mathematical equivalence of the solutions of the two systems, by taking $\gamma = y = w$. This reformulation of the problem allows one to gain further interesting results by using the theorems and techniques of linear programming, with which economists are usually more familiar than with topology or convex analysis.

3.12. It is apparent that the equality of the equilibrium rate of growth and interest (profit) in the von Neumann model is a direct consequence of his assumption that all income in excess of necessary consumption is reinvested. It is also easy to show that the rates of growth and profit will as a rule differ if one introduces luxury consumption as well into the model. Let us denote the coefficients of luxury consumption by f_{ij}. If we add them to the model, the coefficients determining total use ($d_{ij} = a_{ij} + f_{ij}$) and those defining the cost of production (a_{ij}) will no longer be the same. The basic inequalities of equilibrium in this extended model will be as follows:

$$\sum_j b_{ij} \cdot x_j \geq (1+\lambda) \cdot \sum_j (a_{ij} + f_{ij}) \cdot x_j , \tag{5c}$$

$$\sum_i p_i \cdot b_{ij} \leq (1+\pi) \cdot \sum_i p_i \cdot a_{ij} . \tag{6a}$$

Clearly, the introduction of luxury consumption does not affect the definition of the profit rate and the prices. It can also immediately be seen that the above generalisation of the von Neumann model abolishes its elegant (symmetric) duality.[8] In the above extension of the von Neumann model the rate of profit sets an upper limit for the potential growth rate ($\lambda \leq \pi$). Under some normal conditions, the growth rate will decrease as the level (some or all coefficients) of luxury consumption increases. It can in fact increase to a level at which there will be no growth at all ($\lambda = 0$, the case of simple reproduction), whereas the rate of profit remains positive. This is again in perfect harmony with the classical analysis of the capitalist mode of reproduction.

4. THE WALRAS–CASSEL MODEL AND ITS SCHLESINGER–WALD VARIANT[9]

Let us turn our attention now to other salient models of general economic equilibrium that appeared prior to or concurrently with von Neumann's model. The concept of economic equilibrium goes back at least to the classical economists who

[8] See Morishima (1964) and Łoś and Łoś (1974) for more on this or similar asymmetric extensions of the von Neumann model.

[9] This section draws heavily on Punzo's (1989) penetrating analysis of the related issues.

found a convincing analogy between the laws of nature and competitive markets (the famous "invisible hand" of Adam Smith). Both sets of laws seemed to be capable of securing long-term harmony, stability and efficiency. This ideal state is behind the concept of general economic equilibrium, and not just the law of supply and demand. The classical economists did not put that vision or parable into formal models. Even Cournot (1838), whom many consider as the first proper mathematical economist, saw the formulation of such model easy, but completely useless for any practical purposes (because of lack of data, computational methods and facilities).

Walras (1874) was the first who put aside such practical concerns and, in the spirit of pure science, "invented" the first models of general equilibrium. In this sense, he is rightly considered to be the father of general equilibrium models. In the various editions of his famous book Walras presented a series of general equilibrium models of decreasing level of abstraction (pure exchange economy, production economy without and with capital goods). It is interesting to note that Marx, a contemporary to Walras, was the other economist who, in the words of the Nobel-laureate Arrow (1974), got much closer to the modern theory and models of general equilibrium than any of his predecessors. He meant Marx's two-sector schemes of reproduction and equilibrium prices.

Despite the priority of Walras, we shall start our review with a model of Cassel, a Swedish economist. His formal (static) model can be seen as a simplified version of that of Walras (it is often referred to as the Walras–Cassel model). Cassel's influence on later developments became more crucial than that of Walras. His model had attracted the attention of one of the famous Viennese circles of scholars in the 1920s. They were interested in the formal analysis of some concepts of the Austrian neoclassical school of economics. The problem at the focus of their attention was the so-called *principle of imputation* (Zurechnung).

Carl Menger (1871) divided the economic goods into the groups of final products and factors of production (of various orders). The equilibrium prices of the final products, in his view, are directly established by the consumers' preferences, which in turn determine, via imputation, the prices of the factors of production (distributing the revenue among their suppliers). Once we know the prices (p_i) and the input requirements (d_{ki}) of the products, the prices of the factors of production (w_k) can be found, as Wieser (1893) claimed it, by solving the following system of equations:

$$w_1 \cdot d_{1i} + w_2 \cdot d_{2i} + \ldots + w_m \cdot d_{mi} = p_i, \qquad i = 1, 2, \ldots, n \tag{18}$$

where n is the number of outputs, m is the number of inputs.

For a mathematician it was clear that the above price formation rule is far from being a trivial mathematical problem. What ensures the *regularity* ($n = m$) of the system of equations? Even if it were a regular system, what would guarantee that it has a non-negative solution? The examination of this problem made the Viennese scholars interested in the model of Cassel, who complemented the price imputation formula (18) with equations:

$$d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \ldots + d_{kn} \cdot y_n = s_k, \qquad k = 1, 2, \ldots, m \tag{19}$$

that require the equality of demand and supply of the production factors. y_i is here the production of final good i, determined by a $y_i(p_1, p_2, \ldots, p_n)$ demand function, and s_k is the supply of the kth factor of production, exogenously given (fixed) in the model. (Cassel fixed the level of prices by setting the level of total income to 1, therefore income did not appear in his demand functions as a variable. This is sometimes not well understood.)

The conditions of equilibrium of the Walras–Cassel model constitute thus a *regular* system of equations. This made it possible for Cassel to rely on the classical method of counting equations. Moreover, the model can be reduced to a simple form as follows. Substitute first the expression (given in terms of factor prices) on the left hand side of (18) for the product prices in the $y_i(\cdot)$ demand functions, and next replace the production level variables in (19) with the resulting demand expression. At the end one arrives at a reduced regular form,

$$d_k(w_1, w_2, \ldots, w_m) = s_k, \qquad k = 1, 2, \ldots, m, \tag{20}$$

where the kth equation expresses the equality of *supply* (s_k) and *demand* (d_k) on the market of the kth production factor.

It should be emphasised that Cassel formulated his model according to the *ex post* modelling tradition of classical physics. The values of the variables of his model were assumed to be directly observable. Moreover, their observed (naturally positive) longer-term average values were assumed to be equilibrium values, which thus had to satisfy the above conditions. The solvability of the equation system, i.e., the existence of equilibrium, was therefore not a question of mathematical feasibility for Cassel (or for Walras for this matter) but an observed (observable) fact of life.

Also, because the model was a timeless (not static, as is often mistakenly contended) expression of a longer period equilibrium position, by assumption only *scarce* production factors, i.e., factors with positive prices appeared in his model. The parameters of the system were supposed to be estimated, calibrated in a way that would guarantee the existence of a solution with positive (equilibrium) prices.

And if one intends to use the calibrated model for comparative static analysis, he should also ensure that the solution is locally unique and robust. Robust in the sense that small perturbation of the parameters will result in a system which will also have a positive solution. (This condition was emphasised and called local stability by Hicks.)

The Viennese scholars thought that "Cassel's clever idea" (Punzo) gave the proper solution to the price imputation problem investigated by them. But Schlesinger (1935), following the Austrian approach, used *inverse*, $p_i(y_1, y_2,\dots, y_n)$ demand functions, in which prices are determined by the quantity of demand, and not the other way around. This seemingly innocent formal change resulted in significant consequences. Schlesinger's variant of the Walras–Cassel model can no longer be reduced by Cassel's method. One can only reduce the model to the following form:

$$w_1 \cdot d_{1i} + w_2 \cdot d_{2i} + \dots + w_m \cdot d_{mi} = p_i(y_1, y_2,\dots, y_n), \qquad i = 1, 2,\dots, n, \tag{18a}$$

$$d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \dots + d_{kn} \cdot y_n = s_k, \qquad k = 1, 2,\dots, m. \tag{19}$$

Although the resulting system is regular, it still does not solve the mathematical problem of factor price imputation. Consider the equation subsystem (19) that determines the production possibility set. Suppose that $(y_1, y_2,\dots, y_n)$ satisfies these constraints, and substitute them into the demand functions. The equation system (18a) is as a rule not regular, it could be under- or overdetermined, in the same way as in the original price imputation problem. So they were back to the square, Cassel's clever idea did not help the Viennese scholars at all. What could have appeared as a negative factor price in Cassel's model appeared in the form of irregularity in their version of the model (cf. Punzo, 1989).

In order to overcome the mathematical problem, Schlesinger (1935) in the end proposed to use inequalities and complementary slackness conditions instead of equations:

$$d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \dots + d_{kn} \cdot y_n \leq s_k, \qquad k = 1, 2,\dots, m, \tag{19a}$$

requiring the price to be zero whenever a factor is oversupplied:

$$w_k \cdot (d_{k1} \cdot y_1 + d_{k2} \cdot y_2 + \dots + d_{kn} \cdot y_n) = w_k \cdot s_k, \qquad k = 1, 2,\dots, m. \tag{19c}$$

It was with regard to the model given by conditions (18a), (19a) and (19c) that Wald (1935) proved, roughly at the same time, but apparently fully independently from von Neumann, the existence of general equilibrium.[10] As one can clearly see,

[10] Von Neumann's proof was the first by all accounts; it must have been ready sometime in the period of 1928–1932. It was, however, not published and known in Vienna, despite his close

both their models and the mathematical techniques used in their proofs were completely different.

Punzo (1989) was completely right to point out that Schlesinger and Wald, by the above seemingly innocent change, have actually transformed Cassel's *ex post* model into an *ex ante* model. The constituents of such a model are no longer observed (or potentially observable) economic magnitudes but those of an abstract metamathematical structure. The input coefficients are no longer observed long-term averages, assumed to change relatively slowly with prices, but fixed technological coefficients. In such an abstract model it is not possible to assume *ab ovo* that a state characterised by the conditions of equilibrium exists at all, and if it does, determine in advance which factors will be scarce. Everything depends on the model parameters, which could take any value, except for the required sign.

From this point of view, the von Neumann model and Schlesinger–Wald variant of Cassel's model are similar. Apart from that it is only the use of the complementary slackness conditions that establishes similarity between them. Observe however that Schlesinger and Wald used these conditions only partially, only for the factors of production, but not for the products or the activities. This was made possible by a stringent assumption adopted by Wald for the demand functions. Namely, he postulated that the price of any given product would go to infinity, as its available amount approached to zero. This implies that every product is needed for consumption and will be produced (i.e., every activity used in equilibrium).

5. CASSEL'S MODEL OF STATIONARY GROWTH

One of the apparent differences between the discussed model of Cassel and that of von Neumann is the timeless nature of the former. Cassel has only verbally outlined a multi-period extension of his model. It is very easy to put it into formal terms, which is probably the reason why Cassel did not do it himself. In outlining his multi-period vision of the conditions of equilibrium, Cassel assumed, too, that production took place in uniform, discrete periods of time and exchange only at the end of each period. He continued to portray an economy, in which the products served only for the purpose of final consumption and were produced from primary factors alone. As a result, it is only the availability of the primary (natural) factors (including labour) that can limit the level of production and its growth in time. Cassel postulated a stationary economy in which the various primary factors as

contact to K. Menger. It was Menger who had informed von Neumann about the work of Schlesinger and Wald that prompted him to submit his paper for publication in a volume edited, as a matter of fact, by Wald.

well as the production and consumption of various final goods grew at the same (ρ) rate over time.

From these assumptions it follows that the equilibrium conditions (19), which state the equality of demand and supply of the input factors, remain the same. (Both sides of the equations must be multiplied by the same growth factor from one period to another.) Because of the assumed one-period lag between purchasing the factors of production and selling the final goods produced, the revenue received in equilibrium must cover interest as well. This means that we have to multiply the cost side of equations (18) by the factor of interest. Assuming that all income is spent in each period, the expenditure of a given period will be equal to the revenue yielded by the production of the previous period. Therefore, the factor of interest must be the same as the factor of growth. All these assumptions result in a system of equations that is different from the condition of the timeless model only because of the inclusion of a factor of interest, $(1+\rho)$ given exogenously by the factor of growth:

$$(1+\rho)\cdot(w_1\cdot d_{1i} + w_2\cdot d_{2i} +\ldots+ w_m\cdot d_{mi}) = p_i, \qquad i = 1, 2,\ldots, n, \tag{21}$$

$$d_{k1}\cdot y_1 + d_{k2}\cdot y_2 +\ldots+ d_{kn}\cdot y_n = s_k, \qquad k = 1, 2,\ldots, m, \tag{22}$$

where $y_i = y_i(p_1, p_2,\ldots, p_n)$.

6. CAPITAL GOODS IN THE WALRASIAN MODEL OF GENERAL EQUILIBRIUM

The models of Cassel and von Neumann are thus very different from each other, they are rather complements than variants of the same model. Capital shows up in Cassel's model merely in the form of advanced cost, and physical capital goods and their accumulation are completely missing from it. This is a typical neoclassical feature of the model. In this respect Walras was still fairly classical, because he had introduced capital goods into one of his general equilibrium models. We shall have a look at that model now.

In fact we present a slightly more general version of the original model given by Walras. Unlike in his model, final (consumption) goods and capital goods in our model are physically not necessarily distinct commodities. The goods that appear in our model will be classified into three groups:

– *final products* (goods currently produced but not used as material inputs in production, including pure capital goods produced in the given period too),

- capital stocks (accumulated final products, assumed to be physically the same as their currently produced counterparts) and
- primary (non-producible) factors.

Let us now decompose final use (y_i) into consumption (v_i) and accumulation (z_i). Let k_i denote the already accumulated stock of product i (the supply of capital goods) and s_k the supply of the kth primary factor of production. Let us denote by k_{ij} the input coefficients of the capital goods and by d_{kj} the input coefficients of the primary factors, as before. We shall introduce q_i to denote the price (or cost) of the ith capital good defined by Walras as the sum of the cost of amortisation and the net rate of return on capital (the third element, the cost of risk insurance, will be disregarded here).

The necessary conditions of general equilibrium in a model based on the above assumptions can be formulated in the spirit of Walras as follows:

$$q_i = (r_i + \pi_i)\cdot p_i, \qquad i = 1, 2,\ldots, n, \tag{23}$$

$$v_i(\mathbf{p}, \mathbf{q}, \mathbf{w}) + z_i = y_i, \qquad i = 1, 2,\ldots, n, \tag{24}$$

$$d_{k1}\cdot y_1 + d_{k2}\cdot y_2 +\ldots+ d_{kn}\cdot y_n = s_k(\mathbf{p}, \mathbf{q}, \mathbf{w}), \qquad k = 1, 2,\ldots, m, \tag{25}$$

$$k_{i1}\cdot y_1 + k_{i2}\cdot y_2 +\ldots+ k_{in}\cdot y_n = k_i, \qquad i = 1, 2,\ldots, n, \tag{26}$$

$$q_1\cdot k_{1i} + q_2\cdot k_{2i} +\ldots+ q_n\cdot k_{ni} + w_1\cdot d_{1i} + w_2\cdot d_{2i} +\ldots+ w_m\cdot d_{mi} = p_i, \quad i = 1, 2,\ldots, n, \tag{27}$$

where π_i is the net rate of return on good i used in the form of capital, r_i is the rate of amortisation, $v_i(\mathbf{p}, \mathbf{q}, \mathbf{w})$ is the consumers' demand function for good i, and $s_k(\mathbf{p}, \mathbf{q}, \mathbf{w})$ is the supply function of the kth primary factor of production, and $\mathbf{p} = (p_i)$, $\mathbf{q} = (q_i)$ and $\mathbf{w} = (w_k)$.

The number of the variables (y_i, z_i, p_i, q_i, π_i, w_k) in the (23)–(27) systems of equations is $(5n + m)$, whereas the number of the equations is $(4n + m)$. The system is thus underdetermined as yet and there remain n degrees of freedom. At the same time, the demand for goods for the purpose of accumulation (or, which is almost the same, the supply of capital stocks) has not yet been specified. Neither has been postulated the equality of the net rates of return on capital, which must hold in a long-run equilibrium. The remaining degrees of freedom can, however, be removed by adding only one of the above two missing sets of specification.

Walras was perfectly aware of the fact that the equality of the rates of return requires the harmonisation of the accumulation of capital stocks with their demand. If he chose to close the model by prescribing the equality of the rates of return, i.e., substituting

$$q_i = (r_i + \pi)\cdot p_i, \qquad i = 1, 2,\ldots, n. \tag{28}$$

for equations (23) and getting rid of the variables π_i, nothing would ensure the above harmony. For, the values of variables z_i were determined as residuals by equations (25)–(28) (they do not provide any feedback between z_i and the other variables). If, on the other hand, he specified the investment demand in one way or another, then nothing would guarantee the uniformity of the rates of return.

The conditions of the model ensure that the total value of investments ($\sum_i p_i{\cdot}z_i$) equals total savings, which can be determined as

$$\textstyle\sum_i w_k{\cdot}s_k(\mathbf{p}, \mathbf{q}, \mathbf{w}) + \sum_i q_i{\cdot}k_i - \sum_i p_i{\cdot}v_i(\mathbf{p}, \mathbf{q}, \mathbf{w}).^{11} \tag{28a}$$

This convinced Walras that it was possible for investments to adjust in such a way that would maintain long-run equilibrium and equalise the rates of return. He, as a matter of fact, explicitly referred to a sort of *tâtonnement* process and eventually opted for the first closure possibility (*Keynes* chose the other option later).

Introducing equations (28) for equations (23) eliminates $n - 1$ variables and the degrees of freedom will be reduced to one. Assuming, as Walras did, that the demand and supply functions are homogeneous of degree zero (depend on relative prices only), one can fix the price level and close the system thereby. We must see however that any closure of the model is only a formal (mathematical) one and leaves the model essentially open-ended.

We close this section with some further remarks. Note, first of all, that the rate of return in the model of Walras, unlike in the case of Cassel or von Neumann, has no apparent relation with the rate of growth (as noted already above). In this connection it also worth pointing out that his model, unlike the models discussed so far, contains only the *intratemporal*, but no *intertemporal* conditions of a full-fledged multi-period model. It is only implicit in the model that accumulation in a given period will increase capital stocks available in later periods. But the intertemporal constraints, describing the expansion of the capital stocks through time, which could lend a dynamic character to the model, are completely missing from it. The Walrasian model with capital goods is essentially a static representation of long-term equilibrium. This is exactly the reason why the problem of closure emerges in the model.[12]

[11] The decision on consumption and investment is thus embedded into the demand and supply functions. Their concrete form determines how much income will be saved. Also note that nothing guarantees that it will be positive at all. The rate of return on capital is interestingly independent of the consumption and investment decision, and therefore of the overall rate of growth. It is just like in the case of the von Neumann model with luxury consumption.

[12] The issue of macro-closure has been discussed more recently in the literature on applied (computable) general equilibrium models. See, for example, Dewatripont and Michel (1987). In

It should be no surprise that the Walrasian definition of the rate of return is basically the same as the one used by von Neumann. This cannot immediately be seen from the mathematical formulas. Let us therefore elaborate on this point further. Recall the pricing equation of von Neumann and place a similar equation below it that can be derived from the Walrasian model:

$$p_1 \cdot b_{1j} + p_2 \cdot b_{2j} + \ldots + p_n \cdot b_{nj} = (1+\pi) \cdot (p_1 \cdot a_{1j} + p_2 \cdot a_{2j} + \ldots + p_n \cdot a_{nj}), \tag{29}$$

$$q_j = (r_j + \pi) \cdot (q_1 \cdot k_{1j} + q_2 \cdot k_{2j} + \ldots + q_n \cdot k_{nj} + w_1 \cdot d_{1j} + w_2 \cdot d_{2j} + \ldots + w_m \cdot d_{mj}). \tag{30}$$

These forms reveal the similarities and differences, both the formal and substantial ones, hidden behind the mathematical formulas. The second factor on the right-hand sides represent the value of capital advanced in process j in both cases. The expressions on the left-hand side show the gross revenues (returns) generated by them. $(1+\pi)$ and $(r_j + \pi)$ are thus the factors of gross return in the two models, respectively.

The differences are perhaps more telling than the similarities. Let us comment on them one by one. First, it is interesting to note that the commodity prices (p_i) of the von Neumann model correspond to the cost of capital (q_i) and not to the commodity prices (p_i) in the Walrasian model. Second, the unit revenue is the renting price of the capital good j in the case of Walras, whereas in the case of von Neumann it is the value of the jointly produced goods including different vintages of fixed capital, too. These are direct consequences of the difference in the *intratemporal* (Walras) and *intertemporal* (von Neumann) nature of the model constraints. Third, capital advance in the case of Walras contains the cost of primary factors, whereas it does not appear at all in the model of von Neumann. This is contrary to the assumption used by Walras in other contexts (e.g., labour), namely that the owners have to advance the cost of the primary factors. Such cost elements should be subtracted from the revenue of the capitalist in order to get his return. If we took the cost of the primary factors into account and revised the pricing formula of von Neumann, we would arrive at the following equation:

$$p_1 \cdot b_{1j} + p_2 \cdot b_{2j} + \ldots + p_n \cdot b_{nj} = (1+\pi) \cdot (p_1 \cdot a_{1j} + p_2 \cdot a_{2j} + \ldots + p_n \cdot a_{nj}) + \\ + (w_1 \cdot d_{1j} + w_2 \cdot d_{2j} + \ldots + w_m \cdot d_{mj}). \tag{31}$$

static models, in which final use is split into consumption and investment, one can choose from among several closure options. They can, in fact, express diametrically different (Marxian, neoclassical, Keynesian) theories of distribution or saving/investment, as Taylor (1979) rightly pointed it out.

This form is of special interest, because it is the equivalent of the Ricardo–Sraffa definition of equilibrium prices, generalised for the case of joint production and several primary factors, not just labour. Finally, let us come back to the definitions of the factors of gross return, $(1+\pi)$ and $(r_j + \pi)$, in the two models. They are equivalent only if $r_j = 1$ in the case of the von Neumann model. And this is exactly the case, because of the assumption that production processes are of one-period duration and exchange takes place instantaneously.

In fact, it is one of the special advantages of the von Neumann model that it allows for representing capital goods (fixed assets) used for several periods as a series of different vintages of the same good. *Depreciation* of the capital goods (due to wear and tear) can be expressed as the difference between their equilibrium prices, which reflect the differences in their efficiency. This is a very different approach from the concept of amortisation based on replacement needs. The rate of *amortisation* can indeed be taken to be equal to 1 in the von Neumann model, since every piece of capital is treated as circular capital, as if they were completely used up in the one period.

7. THE CIRCULAR NATURE OF PRODUCTION AND LEONTIEF'S GENERAL EQUILIBRIUM MODEL

Let us continue with the observation that the models discussed above, unlike the model of von Neumann, did not take the *circularity of production* into account. Classical economists emphasised that commodities are produced essentially by means of commodities themselves, as the title of the famous book by *Sraffa* (1960) stressed that, too. It presents itself not only through investments (fixed capital goods) in multi-period models, but also in the form of intermediate consumption (materials used in production). Intermediate consumption, the use of products in production itself, accounts for a significant part of total demand in the modern economies as well as in the costs of production.

Even in a static model, such as the Walrasian one discussed above, it is easy to make up for this deficiency and introduce the intermediate use of products. Let r_{ij} denote their input coefficients and x_j the total output of the jth product. Let us introduce these changes into the system of (24)–(28), and replace y_j by x_j in equation (25) and (26). (We shall use y_j to denote total final demand, as before, but it will not appear in the definition of the equilibrium conditions.) We must therefore rewrite the equilibrium conditions of the product markets as follows:

$$r_{i1}\cdot x_1 + r_{i2}\cdot x_2 + \ldots + r_{in}\cdot x_n + v_i(p_1, p_2, \ldots, p_n) + z_i = x_i, \qquad i = 1, 2, \ldots, n, \qquad (24a)$$

and the conditions defining equilibrium prices must be modified accordingly, too:

$$p_1 \cdot r_{1i} + p_2 \cdot r_{2i} + \ldots + p_n \cdot r_{ni} + q_1 \cdot k_{1i} + q_2 \cdot k_{2i} + \ldots + q_n \cdot k_{ni} +$$

$$w_1 \cdot d_{1i} + w_2 \cdot d_{2i} + \ldots + w_m \cdot d_{mi} = p_i, \qquad i = 1, 2, \ldots, n. \qquad (27a)$$

The conditions of equilibrium will otherwise remain the same.

The modified system of equations (24a)–(27a) and (28) yields nothing but the framework of the well-known *input-output model* of the Nobel-laureate Leontief (1928; 1941). Leontief used a much simpler version of it, because he had primarily designed his model for practical use. He avoided the use of demand functions and he did not take into account primary resources (labour and capital stocks) as potential capacity constraints. In this regard his model is similar to von Neumann's. Instead, Leontief took the parameters of both final demand and *value added* (the cost of primary resources) as exogenously given in his model. As a result, Leontief's static[13] model can be reduced to the following two sets of linear equations:

$$r_{i1} \cdot x_1 + r_{i2} \cdot x_2 + \ldots + r_{in} \cdot x_n + y_i = x_i, \qquad i = 1, 2, \ldots, n, \qquad (24b)$$

$$p_1 \cdot r_{1i} + p_2 \cdot r_{2i} + \ldots + p_n \cdot r_{ni} + h_i = p_i, \qquad i = 1, 2, \ldots, n, \qquad (27b)$$

where h_i is the coefficient of value-added (which was equal to the price itself in the Walrasian model). The duality of the two systems is perfect, but – unlike in the case of the von Neumann model – there is no cross-relationship between the definition of the production and valuation equilibrium, the two sets of equations and variables fall completely apart.

Under normal conditions[14] the above two systems of equations can be re-arranged and solved for variables x_i and p_i. This can be achieved by means of the total (direct and indirect) input coefficients, the elements of $\mathbf{S} = (s_{ij}) = (\mathbf{I} - \mathbf{R})^{-1}$, the so-called Leontief-inverse. As a result we get

$$x_i = s_{i1} \cdot y_1 + s_{i2} \cdot y_2 + \ldots + s_{in} \cdot y_n, \qquad i = 1, 2, \ldots, n, \qquad (32)$$

$$p_i = h_1 \cdot s_{1i} + h_2 \cdot s_{2i} + \ldots + h_n \cdot s_{ni}, \qquad i = 1, 2, \ldots, n. \qquad (33)$$

[13] Much later Leontief and others have also developed stationary and dynamic intertemporal versions of the model, too. For the lack of space they will not be discussed here.

[14] They are usually referred to as the Simon–Hawkins or productivity conditions. See Zalai (1989 and 1997) for their review and some generalisations.

Let us return to the beginning of this section, where we have noted that Cassel completely, and partly Walras, too, disregarded the circularity of production. The products in their models appeared as final goods, which could be produced from primary resources alone. As a result, the prices of the products became directly determined by the cost of primary factors. We should however be fair to them. One can assume that they have omitted the intermediate use of products for the very simple reason discussed above. Namely, because they might have realised that in the absence of joint products and technological choice the equilibrium conditions could be reduced and defined in terms of final use and value added.

It is however only one thing that products may serve as intermediate (flow) inputs in production, and thus contribute to its cost (material cost plus amortisation). It is equally important that some stocks of produced goods are tied up in production for shorter or longer periods of time as real capital. The producers have to advance their cost in the form of money capital. The occasional sales of the products and services return the advanced money, increased in value, if it was profitably invested. Their annual average value is what classical economists called circulating and fixed capital, depending on the velocity of their turnover. The classical economists were preoccupied with the phenomenon of the circulation of capital, which has not received enough attention in the neoclassical mainstream.

It is difficult to represent the physical (real) form of capital, because it changes constantly. It takes the form of inventories of materials, semi-finished and finished products, including investment goods, machinery and buildings, and other constituent parts of fixed assets. They determine the value of the capital advanced, which should in theory yield the same rate of return in the long-run in every sector of production. On the other hand, in the short run, the stocks of products accumulated in the form of circulating or fixed capital constrain the production capacity in the same way as the primary factors of production. Their returns can thus be viewed as prices (rents) determined by relative scarcity.

These features of capital turnover and their impact on the determination of equilibrium prices cannot be dealt with in their full complexity in the static or stationary models of general equilibrium. The models of von Neumann, Leontief (especially his dynamic input-output model) and Sraffa provide alternative frameworks, in which the above issues can be analysed, albeit in a rather simplified manner only. These analyses nevertheless reveal that the input-output and capital coefficients set, under normal conditions, a finite upper limit for the rate of profit, which in turn is a constraint for the general rate of growth. And this is the message of von Neumann's model as well.

8. COMPARISONS AND CONCLUDING REMARKS

Instead of presenting a full discussion of the differences and similarities among the various models discussed we shall list only some of their salient features. Almost all of them will point in favour of von Neumann's model and show his invaluable contribution to many future results:

- The full mathematical duality of the production and price decisions in his model anticipated the duality (complementary slackness) theorems of linear programming.
- Von Neumann was the first to use a general linear activity analysis model to describe the technology by allowing for joint production and technological choice. With that he also paved the way for the production-set approach.
- He was also the first to recognise and formalise the dual symmetry of the conditions characterising the choice of efficient (optimal) activities and the determination of such a(n) (equilibrium, efficiency or shadow) price system that sustains the technological choice under the conditions of a competitive equilibrium.
- He used for the first time a fixed-point theorem in the proof of existence of equilibrium that became part of a standard (almost unavoidable) tool in general equilibrium analysis. (Wald used the traditional methods exploiting the specific feature of his model).
- Von Neumann was also the first to demonstrate the close relationship that exists between the concept of equilibrium in economic systems and games.
- By taking into account the circularity of production von Neumann reintroduced real capital into general equilibrium analysis. He showed that the stocks of various kinds of capital goods were endogenously determined and could potentially create bottle-necks ("scarce" production factors) that allowed for a deeper analysis of the nature of capital and profit.
- The factors of growth and profit (interest) are endogenously determined in his model by the overall productivity of the system of reproduction (and not exogenously given as in Cassel's model).
- The above factors are equal in his model only because of the assumption that surplus is fully reinvested. (In Cassel's model they coincide by assumption.)

These are important differences that distinguish the model of von Neumann from the models of his predecessors and contemporaries. As mentioned earlier, von Neumann did not reveal much of the origins of his paper, and his puzzling remark "it is obvious to what kind of theoretical models the above assumptions correspond" gave rise to an extensive series of speculations and debates (cf. Punzo, 1989; Kurz and Salvadori, 1995).

His close relationship with K. Menger and his stay in Germany between 1926 and 1932 make it likely that von Neumann found the economic interpretation of his model in the German-language literature, either directly or indirectly (through seminars and discussions). It seems also probable that he was at that time more interested in finding an economic interpretation for his minimax theorem developed originally for game theory rather than in economics as such. In the spirit of Hilbert, the more interpretation can be associated with it, the more valuable a metamathematical model is (cf. Punzo, 1989). The original German title of the paper (literally translated: "On an economic system of equations and the generalisation of Brower's fixed-point theorem") seems to confirm his prime interest in methodology.

We do not think that he would have noticed or cared much for the nuances, which distinguish the various schools of economics from each other. He addressed such profound and universal economic issues that could be placed into the theory of almost any economic school. The model could be given interpretations pertaining to different modes of commodity production. It could be a simple (small owners') commodity production system (discussed at length by Smith and Marx), a full-fledged capitalist economy (in the way studied by the classical economists) or a centrally planned, market socialist system (decentralised in the way envisaged by Barone, Lange and their followers).

At the heart of von Neumann's model lie the old ideas of just and efficient prices and the requirements of balanced, equilibrated proportions of reproduction. He has portrayed the economy in its many aspects as classical economists did before him (see Kurz and Salvadori, 2003 for more details on this relationship). The economy in his model consists of independent departments that are not only production units, but cater also for their part of the population. They produce goods mainly for others, making use of their comparative advantages. Goods are produced thus as commodities and are exchanged on the markets. The quest is for such institutions and mechanisms that harmonise the interests of the independent units, enforce efficient decisions as well as sustain a high level of growth in the longer run.

Von Neumann's model is in our view a brilliant abstract mathematical metaphor of this classical if not ancient idea. The conditions of equilibrium establish not only the short-term (period-by-period) harmony of supply and demand, but also secure the conditions of a long-term proportional growth. The "exchange values" are just in the sense that they cover the costs of replacement of the destroyed means of production, the living cost of those engaged in production, and in addition, the cost of investments that allow each unit to grow at the same rate. The accumulation in fact makes it possible for an expanding population to enjoy the same living standard in every period. What other purpose would the accumulation serve if the per capita consumption remained constant over time?

Coming back to the comparison of the models surveyed, we would like to call attention again to the significant difference in the *ex post* and *ex ante* approaches (Punzo, 1991). By the age of von Neumann and Wald the axiomatic and mathematical approach started to gain wider acceptance in economics, too. They were been brought up in the latter philosophy, whereas their forerunners, e.g., Walras and Cassel, had used the *ex post* modelling approach of classical physics. For von Neumann and Wald, unlike for the forerunners, the existence of equilibrium in a model was no longer an empirical matter, but a feature of the model that had to be proven.

Due to the spread of the axiomatic and mathematical approaches, the mathematically inclined and trained economists, whose number was steadily growing, had radically changed their perception of the subject matter and methodology of economics. In its early years the main objective set for quantitative economics was to develop the methodology and practice of empirical research (Weintraub, 1983; 1985). There was in fact an explicit aversion against abstract mathematical economic theorising. Circumstances have, however, led quantitative economic research somewhat astray, first in the USA and later worldwide, too ("Bourbakism came to mathematical economics", cf. Weintraub and Mirowski, 1994).

The focus of research shifted from the applied (concrete) to the pure (abstract). The logical consistency and mathematical elegance of general equilibrium theory became more important than its empirical relevance. "The lack of sufficiently secure experimental base" (Debreu, 1991) can only partly justify, in our view, the prolonged preoccupation with the (otherwise outstanding) abstract models of Arrow and Debreu (1954), and McKenzie (1954). As a matter of fact, until the late 1970s general equilibrium models were relegated to the role of tool of logical calculus, rather than being developed into concrete models that can be statistically estimated and used in empirical economic policy analysis.

In the late 1970s and 1980s the perception of general equilibrium models has started to change slowly with the spread of computable (CGE) models, following the path-breaking works of Johansen (1960) and Scarf (1973). In the last 25–30 years, combining different modelling traditions, an enormous number of practically useful CGE models were developed to study a variety of policy issues. They include tax policies, development plans, agricultural programmes, international trade, energy and environmental policies and so on (see, for example, the collective works of Dervis et al., 1982; Piggot and Whalley, 1985; Bergman et al., 1990). A range of mathematical formulations and model solution techniques was developed, too, and large statistical databases were compiled. The practice of model building itself became increasingly systematised, as reflected in the increasing use of powerful standard modelling packages.

Thus, in the 1930s, or even in the 1950s the empirical data bases, the mathematical algorithms and computational techniques were not yet available for empirical applications of general equilibrium models. The dim prospects of applicability of such complex models in practice as well as the abstract theorising nature of the leading mathematical economic schools turned a large segment of the economic profession against the use of axiomatic approach and mathematical language in economics. It is perhaps not by accident that neither von Neumann nor Wald continued their research on the abstract models of general equilibrium, despite the fact that they became closely associated with the circles of quantitative economics in the US. Had von Neumann perceived that research on general equilibrium had "travelled far from its empirical source" and started to show "the signs of becoming baroque", and hence the danger of "degeneration signalled up" – to quote von Neumann (1947)? We just do not know. Science in general and economics has certainly gained by his choice.

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