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VON NEUMANN, THE CLASSICAL ECONOMISTS AND ARROW–DEBREU: SOME NOTES*

H. D. KURZ – N. SALVADORI

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The paper compares the von Neumann model with the approach to the theory of value and distribution of the classical economists on the one hand and that of the neoclassical economists on the other. It is shown that all salient features of the von Neumann model are classical in spirit: its long-period method; its analytical structure revolving around the concept of the surplus product, the latter's appropriation in the form of interest and its use for the purpose of accumulation; and its constitutive aspects. While the von Neumann model focuses attention exclusively on the problem of reproducibility, in neoclassical theory the problem of scarcity assumes centre stage of the analysis. The paper concludes with a remark on a proposition by Arrow and Debreu in their seminal 1954 paper. They contended that their assumption, according to which each and every agent is possessed of an endowment of goods such that he or she can survive under any circumstances whatsoever plays the same role as von Neumann's assumption that each and every commodity enters or exits each and every process of production. It is argued that this proposition cannot be sustained.

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JEL classification index: B12, B31, C61, D50, O41

1. INTRODUCTION

John von Neumann gave his paper on equi-proportionate growth for the first time in the Winter of 1932 at the Mathematical Seminar of Princeton University where he had been offered a professorship in 1931. In 1936 he gave the paper in Karl Menger's Mathematical Colloquium at the University of Vienna. The paper was

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Corresponding author: H. D. Kurz, Department of Economics, University of Graz, RESOWI-Zentrum, F4 A-8010 Graz, Austria. E-mail: heinz.kurz@uni-graz.at

published for the first time in German in the proceedings of the colloquium, *Ergebnisse eines mathematischen Kolloquiums* (von Neumann, 1937).[1] In 1945, upon the initiative of Nicholas Kaldor who was a friend of von Neumann's, an English translation of the paper was published in the *Review of Economic Studies*, which was then edited by Kaldor, together with a commentary by the Oxford economist David Champernowne (von Neumann, 1945; Champernowne, 1945).

Von Neumann's paper is rightly famous in economics, although it has repeatedly been maintained that from an economic as opposed to a mathematical point of view the paper is not all that interesting. One author even contended that the paper contains "not very good economics" (Koopmans, 1974). We do not agree with this judgement, which assesses the model against the background of a particular point of view, neoclassical analysis, of which von Neumann was critical (see Section 3 below). From another point of view, that of classical analysis, the von Neumann model turns out to be a very important contribution to the theory of "normal" prices, or "prices of production", and income distribution in the tradition of such authors as William Petty, the Physiocrats, the English Classical economists, especially David Ricardo and Robert Torrens, Vladimir K. Dmitriev, Ladislaus von Bortkiewicz and Georg von Charasoff, a tradition which culminated in the work of Piero Sraffa (1960). In this paper we make an attempt to clarify the issues at hand. We do this partly with recourse to previous contributions by us to an interpretation of the von Neumann model (Kurz and Salvadori, 1993; 1995, ch. 13; 2001). However, we add further flesh to our argument and present it more succinctly and, hopefully, also more convincingly.[2]

The composition of the paper is the following. *Section 2* provides a discussion of the analytical structure of the classical approach to the theory of value and distribution and its constitutive substantive elements. The emphasis will be on the classical economists' concern with the properties of the economic system as a whole, conceived of as a system of production, income distribution, and given needs and wants. These properties refer first and foremost to the system of prices supporting a given system of production, and the way in which its surplus product is distributed amongst different groups of society (workers, capitalists and landlords) in conditions of free competition. The properties concern necessary relationships between what are the unknowns or dependent variables of the problem under consideration imposed by the data or independent variables, which determine the former magnitudes. *Section 3* expounds briefly the salient features of the

1 The *Ergebnisse* which were terminated shortly afterwards due to the "Anschluß" of Austria to Nazi Germany have recently been reprinted (Dierker and Sigmund, 1998).

2 The interpretation of the von Neumann model first suggested in our 1993 paper was favourably received in the literature; see, for example, Leonard (1995).

von Neumann model. It is shown that both its long-period method, its analytical structure and its constitutive elements are classical in spirit. The emphasis is on the capacity of the economic system to generate a physical surplus and the distribution of the surplus in terms of a uniform rate of return on capital. Interestingly, in von Neumann's paper the problem of scarcity plays no role whatsoever. *Section 4* turns to the Arrow–Debreu model and specifies its alternative neoclassical, short-period features. Here the problem of scarcity assumes centre stage of the analysis. A main assumption of the model are given initial endowments of the economy of all sorts of physically and temporally specified factors of production: capital goods' services, labour services and the services of natural resources, including land. *Section 5* compares the von Neumann model and neoclassical models; the emphasis will be on the Arrow–Debreu model and especially on one important assumption in each of the two models. These concern von Neumann's premiss that each and every commodity enters and/or exits each and every process of production and the premiss of Arrow–Debreu that each agent possesses an initial endowment that allows him to survive. Interestingly, some interpreters have maintained that the two assumptions are equivalent. It will be shown that they are not, and that they highlight the fact that the two models belong conceptually to two different traditions of economic thought. *Section 6* contains some concluding remarks.

2. CLASSICAL THEORY[3]

The economy experienced by the classical economists from William Petty to David Ricardo typically generated an annual *social surplus* which was distributed amongst the propertied classes in the form of rents or profits, and was used for the purposes of consumption and capital accumulation. The surplus refers to those quantities of the different commodities that were left over after the necessary means of production had been used up and the means of subsistence in the support of workers have been deducted from the gross outputs produced during a year. In this conceptualisation the necessary real wages of labour were considered no less indispensable as inputs, and thus agents of production, than raw materials, tools, or machines. What became known as the "surplus interpretation" of the classical economists focuses attention on the mature classical economists' approach to how

[3] Our interpretation of the classical economists draws heavily on Piero Sraffa's findings both in his published work on the matter and in his unpublished papers kept at the Wren Library of Trinity College, Cambridge. Sraffa deserves the credit for having rediscovered the distinct classical approach to the theory of value and distribution and for having reformulated it in a logically coherent and substantively encompassing way.

the surplus is distributed and which system of exchange values of the different commodities can be expected to emerge as the result of the gravitation of "market" or "actual" prices to their "natural" or "ordinary" levels, or "prices of production". In conditions of *free competition* – that is, in the absence of significant barriers to entry and exit from all markets – prices can be taken to oscillate around levels characterised by a *uniform rate of profits* (or *interest*) on the value of the capital advanced at the beginning of the uniform production period and a uniform rate of rent for each of the different qualities of land.

The determination of the general rate of profits, the rents of land, and the corresponding system of relative prices constitutes the analytical centrepiece of classical political economy. It was designed to lay the foundation of all other economic analysis, including the investigation of capital accumulation and technical progress; of development and growth; of social transformation and structural change; and of taxation and public debt. The pivotal role of the theory of value and distribution can be inferred from the fact that the latter is typically developed right at the beginning of major classical works: think of Adam Smith's *Wealth of Nations* (WN, I, vi–xi), or of David Ricardo's *Principles* (*Works*, I, chs I–VI).

The classical economists were concerned with the laws governing the emerging capitalist economy, characterised by the stratification of society into three classes: workers, land owners, and the rising class of capitalists; wage labour as the dominant form of the appropriation of other people's capacity to work; an increasingly sophisticated division of labour within and among firms; the coordination of economic activity via a system of interdependent markets in which transactions were mediated through money; and significant technical, organisational, and institutional change. In short, they were concerned with an economic system that was incessantly in motion. How should one analyse such a system? The ingenious device of the classical authors for seeing through the complexities of the modern economy consisted in distinguishing between the "actual" values of the relevant variables – the distributive rates and prices – and their "natural", "normal" or "necessary" values. The former were taken to reflect all kinds of influences, many of an accidental or temporary nature, about which no general propositions were possible. The latter were instead seen to express the persistent, non-accidental and non-temporary factors governing the economic system as a whole and guaranteeing its reproduction and possible expansion. Only these could be systematically studied. Accordingly, the attention focused on the determination of the normal rate of profits and normal prices.

The classical economists proceeded essentially in two steps. In the first they isolated the kinds of factors that were seen to determine income distribution and the system of relative prices supporting that distribution in specified conditions,

that is, *in a given place and time*. The theory of value and distribution was designed to identify *in abstracto* the dominant factors at work and to analyse their interaction. In a second step, the classical authors then turned to an investigation of the causes that, *over time*, affected the factors at work from within the economic system, i.e. *endogenously*. This involved the classical analysis of capital accumulation, technical and organisational change, economic growth and socio-economic development.

It is another characteristic feature of the classical approach to profits, rents and relative prices that these are explained essentially in terms of magnitudes that can, in principle, be observed, measured or calculated. The *objectivist* orientation of classical economics has received its perhaps strongest expression in a famous proclamation by William Petty, who was arguably its founding father. Keen to assume what he called the '"physician's" outlook, Petty in his *Political Arithmetic* (1690) stressed:

> "The Method I take to do this, is not yet very usual; for instead of using only comparative and superlative Words, and intellectual Arguments, I have taken the course (as a Specimen of the Political Arithmetic I have long aimed at) to express my self in Terms of *Number*, *Weight* or *Measure*; to use only Arguments of Sense, and to consider only such Causes, as have visible foundations in Nature; leaving those that depend upon the mutable Minds, Opinions, Appetites and Passions of particular Men, to the Consideration of others ..." (Petty, [1899] 1986, p. 244; emphases in the original).

This brings us to the substantive part of the problem under consideration. *Five aspects* are especially worth noting in the context of the present comparative study.

(1) *Production as a circular flow*

First, the concept of production the classical economists deemed appropriate to the kind of interdependent economy under investigation was that of a *circular flow*. This idea can be traced back to William Petty and Richard Cantillon and was most effectively expressed by François Quesnay in the *Tableau économique*, which foreshadowed Marx's schemes of reproduction and Leontief's input-output analyses. Actually the classical economists, at least in some of their analyses, advocated the radical view that commodities are in the last instance produced by commodities. Thus, James Mill in his *Elements of Political Economy* insisted that the agents of production are the commodities themselves (Mill, 1826, p. 165), that is, (a) the food of the labourer; (b) the tools and the machinery with which he works; and (c) the raw materials that he works upon.

(2) *Physical real cost*

This view led naturally to the concept of *physical real cost*. Man cannot create matter, man can only change its form, decompose and recompose it, and move it. Production involves destruction, and the real cost of a commodity consists in the commodities destroyed in the course of its production.

(3) *Labour*

In the writings of the classical economists the size of the work force is seen essentially as generated by the accumulation process itself. In other words, labour power is treated as a kind of producible commodity. It differs from other commodities in that it is not produced in a capitalistic way by a special industry on a par with other industries, but is the result of the interplay between the generative behaviour of the working population and socio-economic conditions. In the simplest conceptualisation possible, labour power is seen to be in infinitely elastic supply at a given real wage rate. Increasing the amount of real wages involves a proportional increase of the work force. In this view the rate of growth of labour supply adjusts to any given rate of growth of labour demand without necessitating a variation in the real wage rate.[4] Labour can thus put no limit to growth because it is "generated" within the growth process itself. The only limit to growth can come from other, non-accumulable factors of production.

(4) *Natural resources, especially land*

The classical economists were first and foremost concerned with commodities that can be produced and *reproduced*. This case is best studied by setting aside the only candidate for bringing the concept of *scarcity* into the classical frame of thought, that is, the limited availability of natural resources. Setting aside scarce natural resources and taking the set of methods of production from which cost-minimising producers can choose as given (i.e. setting aside technical improvements), the economy could grow forever. Interestingly, the champion of extensive and intensive diminishing returns and the corresponding concepts of extensive and intensive rent in classical political economy, David Ricardo himself, contemplated this case. In a letter to Malthus dated 18 December 1814 he wrote:

[4] In the more sophisticated conceptualisations underlying the arguments of Smith and Ricardo, higher rates of growth of labour supply presuppose higher levels of the real wage rate. However, the basic logic remains the same: in normal conditions the pace at which capital accumulates regulates the pace at which labour grows (Kurz and Salvadori, 2003).

> "Accumulation of capital has a tendency to lower profits. Why? Because every accumulation is attended with increased difficulty in obtaining food, unless it is accompanied with improvements in agriculture; in which case it has no tendency to diminish profits. *If there were no increased difficulty, profits would never fall, because there are no other limits to the profitable production of manufactures but the rise of wages. If with every accumulation of capital we could tack a piece of fresh fertile land to our Island, profits would never fall*" (Ricardo, VI, p. 162; emphasis added).

Similarly, in his letter to Malthus of 17 October 1815 he stated that

> "... profits do not *necessarily* fall with the increase of the quantity of capital because the demand for capital is infinite and is governed by the same law as population itself. They are both checked by the rise in the price of food, and the consequent increase in the value of labour. *If there were no such rise, what could prevent population and capital from increasing without limit*?" (Ricardo, VI, p. 301; first emphasis in the original).

If land of the best quality were abundant (and its ownership sufficiently dispersed), its use would be a *free good*: there could be no rent paid to the landowner (Ricardo, I, p. 69). From an economic point of view land can therefore be ignored like air or sunlight.

(5) *Fixed capital and joint production*

In the classical economists we encounter both a discussion of the role of fixed capital in production and distribution and of joint production. Here the following aspects deserve mentioning. As regards fixed capital, it was Robert Torrens who first suggested to treat what is left of fixed capital at the end of the production period as a kind of joint product. Thereafter the method was generally adopted by Ricardo, Malthus, James Mill and Marx (Sraffa, 1960, pp. 94–95). The fact that joint production proper was ubiquitous did, of course, not escape the attention of the classical authors: cases of wool and mutton, corn and straw, etc., could hardly be overlooked. Adam Smith pointed out that with joint production the proportion in which the products can be produced need not coincide with those in which they are wanted. Hence some products may be overproduced, with the consequence that "the greater part of them would be thrown away as things of no value" (Smith, 1976, I,xi,c.4). Here we encounter, possibly for the first time in the history of economic thought, the application of the *Rule of Free Goods* to products. (As we have seen under (4) that rule was also known to the classical economists as regards natural resources that are in excess supply.)

3. THE VON NEUMANN MODEL

Nicholas Kaldor who knew von Neumann from Budapest, their home town, recalls that "One day he expressed an interest in economics and he asked me whether I could suggest a short book which gives a formal mathematical exposition of prevailing economic theory" (Kaldor, 1989, p. viii). Kaldor suggested Knut Wicksell's *Über Wert, Kapital und Rente nach den neueren nationalökonomischen Theorien* (cf. Wicksell, 1893; 1954). "He read it in a very short time and expressed some scepticism of the 'marginalist' approach on the grounds that it gives too much emphasis to substitutability and too little to *the forces which make for mutually conditioned expansion*." According to Kaldor, von Neumann subsequently had a look at the original Walrasian equations (cf. Walras, [1874] 1954): "He told me afterwards that they provide no genuine solution, since the equations can result in negative prices (or quantities) just as well as positive ones" (Kaldor, 1989, p. viii; emphasis added). Thus, while the works of Wicksell and Walras, two foremost marginalist authors, appear to have been a source of inspiration to von Neumann, according to Kaldor's recollection he was not only dissatisfied with the fact that no proper proof of equilibrium was provided but also with the economic substance of the argument put forward.

There is no evidence that John von Neumann was familiar with the writings of the classical economists or those working in that tradition, such as the eminent statistician and economic theorist Ladislaus von Bortkiewicz who was von Neumann's colleague at the University of Berlin during the latter's staying there from 1927 to 1929 as a *Privatdozent*. Von Neumann may have met von Bortkiewicz and talked to him, but we are not aware of any document confirming this. The interesting thing to note here is that despite his reported acquaintance with major contributions of two major marginalist authors, von Neumann would develop a model which determines relative prices, the rate of interest, relative quantities and the rate of growth in a manner which is decidedly different from the marginalist one, passing over both consumer preferences and initial endowments of factors. Instead he would put forward a model which in crucial respects exhibits a close family resemblance with the theory of the classical economists, relying exclusively on magnitudes that can be observed and measured.

Before we demonstrate this, one observation is apposite. As we have argued in some detail in another paper (cf. Kurz and Salvadori, 1993), the proximate starting point of von Neumann's paper was in all probability a paper by Robert Remak, a colleague of John von Neumann at the Berlin Institute of Mathematics. Remak's paper of 1929 was entitled *Kann die Volkswirtschaftslehre eine exakte Wissenschaft werden? (Can Economics Become an Exact Science?)*. Interestingly, similar to von Neumann, Remak was also critical of the received marginalist approach

in terms of demand and supply functions and maintained that the behaviour of agents which is at the back of such functions "can be neither experimentally nor theoretically ascertained quantitatively" (Remak, 1929, p. 712). He therefore suggested the concept of "superposed prices" whose determination presupposes nothing but the knowledge of the socio-technical relations of production, that is, the methods of production in use and the needs and wants of producers. The system investigated by Remak is a system of single production (joint production is not considered), and thus is only circulating capital, which produces just enough of each commodity (e.g. corn) to replace what has been productively consumed of it during the year in terms of means of production (e.g., seed corn) and means of sustenance of producers (e.g., corn as a consumption good). We may characterise Remak's model as a simple classical model of production without a surplus, or, using Sraffa's concept, of "production for subsistence" (Sraffa, 1960, ch. I). Remak gave his paper at a meeting of the Berlin Mathematical Society and his ideas were discussed at the Institute of Mathematics in Berlin. There is also reason to presume that von Neumann was familiar with Remak's contribution and was critical of it (Kurz and Salvadori, 1993, pp. 148–149). A comparison of von Neumann's model with that of Remak shows that the former is in every respect more general than the latter, and that whenever Remak drops an idea or poses a question that is beyond the scope of his own model, von Neumann offers a conceptualisation and provides an answer.[5]

In his paper von Neumann assumed that there are *n* goods which can be produced by *m* constant returns to scale production processes. The problem was to establish which processes will actually be used and which not, being "unprofitable". Von Neumann (1945) based his analysis on the following assumptions:

(1) "Goods are produced not only from 'natural factors of production', but *in the first place from each other*. These processes of production may be *circular*" (*ibid.,* p. 1; emphases added). In fact, in von Neumann's paper natural factors of production are either set aside (land) or hidden (labour); see the following two paragraphs. This is reminiscent of the circular flow perspective of the classical economists and especially of James Mill's dictum: "the agents of production are the commodities themselves".
(2) "Consumption of goods takes place only through the processes of production which include necessities of life consumed by workers and employees" (*ibid.,* p. 2). Von Neumann thus takes the real wage rate, consisting of the "necessities of life", to be given and paid at the beginning of the uniform period of production, and therefore like the classical economists reckons real wages among

[5] There is evidence that Remak was in contact with von Bortkiewicz; see on this Wittmann (1967).

the capital to be advanced *(ibid.).* In von Neumann's view, as in that of the classical economists, costs are thus essentially *physical real costs*: means of production and means of subsistence.

(3) As regards labour and land he assumed that they cannot limit the growth of the economic system: "the natural factors of production, including labour, can be expanded in unlimited quantities" *(ibid.)*. His assumptions are equivalent to those discussed under (3) and (4) in *Section 2*, where labour is in infinitely elastic supply at a given real wage rate and "with every accumulation of capital we [can] tack a piece of fresh fertile land to our [economy]" as Ricardo put it (Ricardo, VI, p. 162).

(4) The processes of production "can describe the special case where good G_j can be produced only jointly with certain others, viz. its permanent joint products" (Neumann, 1945, p. 2). Hence, both circulating and fixed capital can be dealt with: "wear and tear of capital goods are to be described by introducing different stages of wear as different goods, using a separate P_i [process i] for each of these" *(ibid.).* This is the joint-products method that we encountered in (5) in the previous section. It effectively reduces fixed capital to circulating.

(5) Since he allowed for universal joint production, von Neumann also had to face the problem of overproduction of some products. In this he had recourse to the Rule of Free Goods we already encountered: "if there is excess production of G_j, G_j becomes a free good and its price $[p_j] = 0$" (*ibid.,* p. 3).

(6) Finally, he assumed "that all income in excess of necessities of life will be reinvested" (*ibid.,* p. 2). According to this assumption interest receivers are "mere investing machines" (Lancaster).

Von Neumann's approach can be summarised as follows. Let **A** and **B** be the $m \times n$ input and output matrices, respectively, where **A** includes the means of subsistence in the support of workers; and let **q** be the m-dimensional vector of activity levels and **p** the n-dimensional price vector. $\alpha = 1 + g$ is the expansion factor, where g is the expansion or growth rate; $\beta = 1 + r$ is the interest factor, where r is the rate of interest (or rate of profits). The model is subject to the following axioms.

$$\mathbf{q}^T\mathbf{B} \overset{\geq}{=} \alpha\mathbf{q}^T\mathbf{A} \tag{1a}$$

$$\mathbf{B}\mathbf{p} \overset{\leq}{=} \beta\mathbf{A}\mathbf{p} \tag{1b}$$

$$\mathbf{q}^T(\mathbf{B} - \alpha\mathbf{A})\mathbf{p} = 0 \tag{1c}$$

$$\mathbf{q}^T(\mathbf{B} - \beta\mathbf{A})\mathbf{p} = 0 \tag{1d}$$

$$\mathbf{q} \geq \mathbf{0} \text{ and } \mathbf{p} \geq \mathbf{0} \tag{1e}$$

Axiom (1a) implies that α times the inputs for a given period are not larger than the outputs of the previous period. Axiom (1b) is the no extra profits condition. Axiom (1c) states the free disposal assumption. Axiom (1d) implies that processes that incur extra costs will not be operated. Finally, (1e) requires that both the intensity and the price vector are semipositive. In order to demonstrate that for any pair of non-negative matrices **A** and **B** there exist solutions for **q** and **p** and for α, $\alpha \geq 0$, and β, $\beta \geq 0$, von Neumann in addition assumes:

$$\mathbf{A} + \mathbf{B} > \mathbf{0}, \tag{1f}$$

which implies that every process requires as an input or produces as an output some positive amount of every good.

On the basis of these givens von Neumann demonstrates the existence of a solution. He determines (a) which processes will be operated; (b) at what rate the economic system will grow; (c) what prices will obtain; (d) what the rate of interest will be; and (e) that, of necessity, $\alpha = \beta$, i.e. the growth and the interest factor are equal.

As mentioned above, the stimulation to publish an English version of the paper came from Nicholas Kaldor, then chairman of the editorial committee of *The Review of Economic Studies*. Kaldor arranged also for the translation of the paper and was concerned with rendering the mathematically demanding paper attractive to an audience of economists. A first step in the pursuit of this goal appears to have been the adaptation of the paper's title (cf. Kaldor, 1989, p. x), a literal translation of the original German version of which would have been "On an economic system of equations and a generalisation of Brouwer's Fixed Point Theorem". The second part of the title which reflects von Neumann's assessment that the main achievement of the paper consisted in the generalisation of a mathematical theorem was dropped entirely, and the neutral term "economic system of equations" was replaced by the not so neutral term "model of general economic equilibrium".

The second step consisted in asking David Champernowne, "the most mathematically minded economist I knew, to write an explanatory paper *ad usum delphini*, for the use of the semi-numerates, to appear alongside it in the *Review of Economic Studies*" (*ibid.,* p. x).[6] In a footnote to the introduction of his paper, Champernowne thanks Nicholas Kaldor for help with economic ideas, and Sraffa and Crum for "instruction in subjects discussed in this article" (Champernowne, 1945, p. 10, n. 1). Interestingly, in Champernowne's interpretation von Neumann's model emerges as one characterised by essentially "classical" fea-

[6] It is interesting to note that in the title of Champernowne's paper (1945) the title of the English version of von Neumann's paper is referred to incompletely: the adjective "general" is left out.

tures. This interpretation is fully confirmed by our investigation of the classical economists from Adam Smith to David Ricardo and those authors working in the "classical" tradition prior to von Neumann. Indeed, there are striking similarities between the contributions of these economists and von Neumann. These concern: (a) the concept of production as a circular flow; (b) the concept of the surplus product which forms the basis of an explanation of all shares of income other than wages; (c) the notion of a uniformly expanding economy in which the rate of expansion is determined endogenously, i.e. a "quasi-stationary system"; (d) the concept of duality of the relationship between relative quantities and the rate of growth on the one hand, and that between relative prices and the rate of interest (rate of profits) on the other; (e) the way in which the choice-of-technique problem is approached and the use of inequalities in it; and (f) the way the Rule of Free Goods is applied to primary factors of production and to products, respectively. Von Neumann applied that rule in fact in the same way as the classical economists did. While he assumed "That the natural factors of production, including labour, can be expanded in unlimited quantities" (von Neumann, 1945, p. 2), this did not make him treat all these factors alike. Rather, he singled out labour as the only factor that is exempt from that rule; all other primary factors, although needed in production, "disappear" from the scene because they are taken to be non-scarce.[7] Labour is assumed to receive an exogenously given wage bundle which is independent of the degree of employment.[8]

The contention that von Neumann's approach has been anticipated in all important aspects by authors working in the "classical" tradition is, of course, not meant to play down the importance of von Neumann's contribution. After all it was he who provided a comprehensive and general formulation of what other authors were able to put forward only partially and with respect to special cases, and it was he who was able to prove the existence of a solution.

7 Assuming that natural resources are non-scarce is of course not the same thing as assuming that there are no natural resources at all. Von Neumann's model is frequently misinterpreted in the latter sense. In this context it deserves to be noted that von Neumann does not define goods in the same way as Debreu (1959, p. 32): he does not consider a particular plot of land in a particular location as a special good.

8 "At most, one could say that a Rule of Zero 'Excess' Wages is applied because labour is less than fully employed" (Steedman, 1987, p. 419). The interpretation given by Dore of von Neumann's use (or rather non-use) of the Rule of Free Goods is difficult to sustain: according to Dore (1989, p. 83) in the von Neumann model "Cassel's 'principle of scarcity' … is given an extreme binary interpretation whereby a resource has either a positive economic value if it is fully utilized, or its value is zero. Unless every single man and woman is fully employed, the social value of labour is zero; this is indeed extreme. Why did von Neumann resort to this formulation?" The answer to this question is: he did not.

4. NEOCLASSICAL THEORY

Neoclassical models, whether short- or long-period ones, are typically characterised as follows. The unknowns or dependent variables of the theory are all prices, including the prices of "factor services", wages, rents and capital rentals or the rate of interest, and all quantities produced of the different commodities and the corresponding allocation of given productive resources. Prices are typically interpreted as indexes of scarcity. In conventional neoclassical theory they are determined simultaneously and *symmetrically* in terms of demand and supply, where the latter are conceived of as functions or correspondences. The data or independent variables from which the theory starts are the following:

(a) the initial endowments of the economy of different kinds of labour, land and capital goods and the distribution of property rights among agents;
(b) consumer preferences; and
(c) the set of available methods of production to produce the various commodities.

In *long-period* versions of neoclassical theory such as those of Eugen von Böhm-Bawerk, Knut Wicksell, John Bates Clark down to Robert M. Solow's growth model and several of the so-called new growth models, the endowment of capital must be given in terms of a single magnitude, otherwise the solution is overdetermined, and such magnitude must be related in a known and increasing relation with the value of capital since the relative scarcity of the "quantity of capital" is supposed to be reflected in the level of the rate of interest. Many, but not all, authors have given it directly as a *value* magnitude. The physical composition of that "quantity" is then supposed to be a part and parcel of the equilibrium solution of the system. This composition has to fulfil the criteria that a uniform rate of return on the supply price of all capital goods obtains – a criteria which with regard to an arbitrarily given vector of quantities of capital goods would only be satisfied by a fluke, that is, as mentioned, the problem would be overdetermined.

In *short-period* versions of neoclassical theory the capital endowment of the economy is instead given in terms of a vector of physically specified amounts of different capital goods. Léon Walras in the first three editions of his *Eléments*, first published in 1874, thought that even with such a physical specification of social capital he would in competitive conditions be able to determine a uniform rate of interest across all heterogeneous capital goods. However, in the fourth edition he saw that this was not generally the case (Kurz and Salvadori, 1995, pp. 22–26 and 439–441). In this case capital goods had rather to be conceived of totally on a par with different kinds of land which yielded their proprietors a rent, provided

they were scarce. The general rate of interest (or profit) has therefore to be replaced by rentals specific to the different kinds of capital goods.

A major motive for breaking away from traditional long-period neoclassical theory and the development of temporary and intertemporal equilibrium models had been the capital theoretic problems the former faced and could not satisfactorily solve. These problems surfaced already in the late 1920s and made Erik Lindahl and later John Richard Hicks abandon the long-period method in favour of short-period ones.[9] The intertemporal equilibrium theory made rapid progress and culminated in the publication of the essay *Existence of an equilibrium for a competitive economy* by Arrow and Debreu (1954) followed by Debreu's *Theory of Value* (1959). The model became known as the "Arrow–Debreu model". The starting point of Arrow–Debreu was Walras's theory. They motivated their analysis in the following way:

> "The investigation of the existence of solutions is of interest both for descriptive and for normative economics. Descriptively, the view that the competitive model is a reasonably accurate description of reality, at least for certain purposes, presupposes that the equations describing the model are consistent with each other. Hence, one check on the empirical usefulness of the model is the prescription of the conditions under which the equations of competitive equilibrium have a solution.
>
> Perhaps as important is the relation between the existence of solutions to a competitive equilibrium and the problems of normative or welfare economics" (Arrow and Debreu, 1954, p. 265).

They emphasised that in order to study this problem, "it is first necessary to specify more carefully than is generally done the precise assumptions of a competitive economy" (*ibid.,* p. 266). In addition to assumptions (a)–(c) above these include:

(1) "There are a finite number of distinct commodities. ... For the present purposes, the same commodity at two different locations or two different points of time will be regarded as two different commodities" (*ibid.,* p. 266). This and assumption (5) below imply a *finite, arbitrarily given time horizon of the model*.

(2) "The commodities, or at least some of them, are produced in *production units* (e.g., firms). The number of production units will be assumed to be a finite number *n*" (*ibid.,* p. 266). Arrow and Debreu added: "the list of production units should include not only actually existing ones but those that might enter

[9] In the 1960s and 1970s the problems were at the centre of the debate during the so-called Cambridge controversies in the theory of capital and interest (for a summary account, see, for example, Kurz and Salvadori, 1995, ch. 14). It was shown that the long-period neoclassical model could not be sustained in other than extremely special conditions.

the market under suitable price conditions" (pp. 266–267). This is designed to take care of the case of factors that are not transferable in the market. To assume a given and constant number of production units implies that *the analysis is short-run*.

(3) Returns are taken to be non-increasing.

(4) "It is impossible to have any output unless there is some input" (*ibid.,* p. 268).

(5) "The number of *consumption units* is *m*" *(ibid.)*.

(6) The set of consumption vectors "includes all consumption vectors among which the individual could conceivably choose if there were no budgetary restraints. Impossible combinations of commodities, such as the supplying of several types of labour to a total amount of more than 24 hours a day or the consumption of a bundle of commodities insufficient to maintain life, are regarded as excluded from [the set]" (*ibid.,* p. 269).

The Arrow–Debreu model assumes that there exist current markets for *all* commodities, whatever their physical, temporal (within the given time horizon) or spatial specification. Hence, in the "economy" contemplated all trade for the entire time horizon takes place at the beginning of the first period. If markets were reopened at later dates, then no additional trade would take place. As Arrow and Hahn stressed, the hypothesis that there exists a complete set of markets for current goods "'telescopes' the future into the present" (1971, p. 33). Given a set of prices, each agent chooses a plan for all the elementary periods. An equilibrium for a "private ownership economy" requires that all individual plans are, from the initial date onwards, mutually consistent for all future dates and compatible with given initial endowments.

The original Arrow–Debreu model exhibits several features that are disquieting.[10] A major difficulty concerns the treatment of time.

> "The principal objection to the restriction to a finite number of goods is that it requires a finite horizon and there is no natural way to choose the final period. Moreover, since there will be terminal stocks in the final period there is no natural way to value them without contemplating future periods in which they will be used" (McKenzie, 1987, p. 507).

[10] Here we are concerned exclusively with the original Arrow–Debreu model. Suffice it to say that several of its shortcomings have been addressed by subsequent work a discussion of which, however, is beyond the scope of this paper. Yet one aspect which was brought to our attention by Nick Baigent deserves at least to be mentioned. It concerns the fact that one strand of thought developed game-theoretic general equilibrium (cf. the contributions of Shubik, Rubinstein, Wolinsky and others), and given that von Neumann was a founder of game theory, the intriguing question is close at hand how his growth paper relates to his work on game theory.

What the Arrow–Debreu model in fact assumes is that all economic activity stops at the arbitrarily given terminal instant, that is, resources existing at the end of the time horizon have *zero* value. Due to the recursive structure of the model, all economic activities decided in the initial instant are derived with regard to the final period, since it would make no sense to transfer resources from the last but one period to the last one.[11]

As regards the instant from which the economy is analysed, i.e. the "present instant", the question arises whether there has been no economic activity prior to that date. The answer is in the negative: the economy is not created "now"; it is rather assumed that, for the purpose of analysing the economy's future development, the legacy of the past is *exclusively and completely* reflected in the amounts of resources inherited and the distribution of private ownership of these resources. In particular, it is assumed that there are no commitments carrying over from the past that constrain agents' present decisions. This implies of course that the logic of the model does not extend to the past, because otherwise one would have to admit that at some dates in the past agents entered into contracts referring to dates that are still in the future.

As we have seen, earlier neoclassical authors, most notably Walras, were concerned with the long- and short-run equilibrium relationships between the prices of durable capital goods and the prices of their services, that is, the rates of return on different kinds of capital goods, and whether the short-run relationships gravitate towards some long-run relationship characterised by a uniform rate of return throughout the economy. To this effect Walras proposed an explicit *tâtonnement* procedure which he conjectured converged to long-period equilibrium. These concerns are not present in the Arrow–Debreu model. It is not even considered how the economy is supposed to get into equilibrium. The notion of equilibrium is simply one of simultaneous clearing of all markets; there is no discussion of any adjustment process when defining equilibrium. As Arrow and Debreu stressed: "Neither the uniqueness nor the stability of the competitive solution is investi-

[11] Until a few decades ago the time horizon in intertemporal general equilibrium theory was assumed to be finite and, therefore, arbitrary. The study of intertemporal models with an infinite time horizon has begun with Bewley (1972). The introduction of an *infinite* horizon turned out to be critical (see also Burgstaller, 1994, pp. 43–48). It pushed the analysis towards steady-state analysis. (It ought to be stressed that the latter is only a special case of long-period analysis and must not be identified with it.) This was clearly spelled out, for instance, by Robert Lucas in a contribution to the theories of endogenous growth. Lucas observed that "for *any* initial capital $K(0) > 0$, the optimal capital-consumption path $(K(t), c(t))$ will converge to the balanced path asymptotically. That is, the balanced path will be a good approximation to any actual path 'most' of the time" and that "this is exactly the reason why the balanced path is interesting to us" (Lucas, 1988, p. 11).

gated in this paper" (Arrow and Debreu, 1954, p. 266; similarly Debreu, 1959). As Currie and Steedman (1990, p. 147) stressed, it makes no sense to think in terms of the "rates of return" in the context of the Arrow–Debreu model. This follows from the assumption that all transactions take place at the present instant. They added:

> "Meaningful asset equilibrium conditions – involving uniformity in appropriately defined rates of return – can be established for models which allow for changes in relative prices over time but *only* for those models with spot markets at each date, since, in such models, ownership of a durable good is a way of transferring wealth over time. In contrast, in the Arrow–Debreu model, the notion of transferring wealth over time has no real meaning."

Obviously, to take the capital endowment as given in kind implies that only "short-period" equilibria can be determined. Because firms prefer more profit to less, the size and composition of the capital stock will rapidly change. Thus, major factors which general equilibrium theory envisages as determining prices and quantities are themselves subject to quick changes. This, in turn, makes it difficult to distinguish them from those accidental and temporary factors, which, at any given moment of time, prevent the economy from settling in a position of equilibrium.

More important, the fast variation in relative prices necessitates the consideration of the influence of future states of the world on the present situation. The assumption that all intertemporal and all contingent markets exist, which has the effect of collapsing the future into the present, can be rejected on grounds of realism and economic reasoning (see, for example, Bliss, 1975, pp. 48 and 61). In addition, there is the following conceptual problem (Schefold, 1985). If in equilibrium some of the capital stocks turn out to be in excess supply, these stocks assume zero prices. This possibility appears to indicate that the expectations entrepreneurs held in the past when deciding to build up the present capital stocks are not realised. Hence, strictly speaking we are faced with a *dis*equilibrium situation because otherwise the wrong stocks could not have accumulated. Therefore, the problem arises how the past or, more exactly, possible discrepancies between expectations and facts influence the future.

5. VON NEUMANN'S MODEL AND NEOCLASSICAL THEORY

We have seen in *Section 3* above that the von Neumann model is a long-period model: It is concerned with determining the competitive rate of interest and normal prices and accomplishes this task because it does not start from a given vector of endowments of heterogeneous capital goods. It would therefore be quite point-

less to compare von Neumann's model with any of the short-period neoclassical models. The differences between the two types are all too obvious.[12] In fact, to the best of our knowledge none of the major neoclassical general equilibrium theorists have maintained that their models belong to the kind of models to which von Neumann contributed. However, in the literature one occasionally encounters sweeping statements as to the "neoclassical" derivation of the model. Von Neumann's contribution is typically considered a contribution to the so-called "Walras–Cassel" model as it was discussed by Karl Schlesinger and Abraham Wald in Menger's Vienna colloquium at the beginning of the 1930s. The original Cassel model however had no capital goods but only scarce natural resources, about which Cassel in a second model then assumed that they are all subject to a uniform increase. This assumption of an exogenously given uniform rate of growth of all productive resources seems to have prompted some observers to closely relate the von Neumann model and the Walras–Cassel tradition. Yet this is misleading for a variety of interconnected reasons.

First, as we have already seen, von Neumann is not at all concerned with the problem of *scarcity*, but only with that of *reproducibility*, whereas in the neoclassical model the problem of scarcity of resources occupies centre stage. There are no initial endowments of reproducible and accumulable inputs (capital goods) in the von Neumann model and all non-accumulable inputs (labour, land, etc.) are taken to adjust to the needs of economic growth (labour) or to be available in abundance. The difference between the two kinds of models can also be seen in the fact that von Neumann determines the (steady-state) rate of growth *endogenously*, whereas Cassel (and later Solow, who, unlike Cassel, in his growth model allowed for a choice of technique) took it as given from outside. *Second*, real wages in von Neumann's model are given from outside, while in neoclassical models they are determined by the model. Interest in the von Neumann model is consequently a residual income whose magnitude reflects the capacity of the system to generate a surplus. Interest in the neoclassical model is instead a scarcity price, just like wages.

After having established that von Neumann's model is classical and not neoclassical in spirit, we conclude by showing that a false analogy has been drawn between von Neumann's assumption (1f) (see *Section 3* above) and a crucial assumption of the Arrow–Debreu model.

Initially, it should be stressed that neither Arrow and Debreu (1954) nor Debreu (1959) asserted that the von Neumann model is connected with

[12] For a discussion of some additional aspects of the relationship between von Neumann's model and neoclassical theory, see the paper by Zalai (2004) in this volume.

theirs.[13] The only references to von Neumann (1945) are technical. In Arrow and Debreu (1954, p. 270, fn. 6) it is asserted that the second half of their *assumption IV.a* (on the same page) "plays the same role as the one made by Professor von Neumann in his study of a dynamic model of production ... that each commodity enters into every production process either as an input or as an output".[14] And in the Preface to Debreu (1959) it is asserted that "A little earlier J. von Neumann ... had begun to develop, in different contexts, a mathematical tool which was eventually to play an essential role in that area under the definite form as a fixed point theorem it received from S. Kakutani" (p. ix). Hence the relationship between Arrow and Debreu and Debreu on the one side, and von Neumann on the other, is relegated to formal aspects only.

It may be of some interest to identify the formal similarities between the two frameworks. In order to do so, the Arrow–Debreu theorems can be seen as giving sufficient conditions of the existence of vectors $\mathbf{y}_j^*$, $\mathbf{x}_i^*$, $\mathbf{p}^*$ such that

$$\mathbf{y}_j^* \in Y_j \tag{2a}$$

$$\mathbf{y}_j^* \mathbf{p}^* \geqq \mathbf{y}_j \mathbf{p}^* \qquad \forall\, \mathbf{y}_j \in Y_j \tag{2b}$$

$$\mathbf{x}_i^* \in \{\mathbf{x}_i \in X_i \,|\, \mathbf{x}_i \mathbf{p}^* \leqq \boldsymbol{\zeta}_i \mathbf{p}^* + \sum_{j=1}^{n} \alpha_{ij} \mathbf{p}^* \mathbf{y}_j^* \} \tag{3a}$$

$$\mathbf{x}_i^* \gtrsim \mathbf{x}_i \quad \forall\, \mathbf{x}_i^* \in \{\mathbf{x}_i \in X_i \,|\, \mathbf{x}_i \mathbf{p}^* \leqq \boldsymbol{\zeta}_i \mathbf{p}^* + \sum_{j=1}^{n} \alpha_{ij} \mathbf{p}^* \mathbf{y}_j^* \} \tag{3b}$$

$$\mathbf{p}^* \geqq \mathbf{0} \tag{4a}$$

$$\mathbf{e}\mathbf{p}^* = 1 \tag{4b}$$

$$\sum_{i=1} \mathbf{x}_i - \sum_{j=1} \mathbf{y}_j - \sum_{i=1} \boldsymbol{\zeta}_i \leqq \mathbf{0} \tag{5a}$$

$$\mathbf{p}^* [\sum_{i=1} \mathbf{x}_i - \sum_{j=1} \mathbf{y}_j - \sum_{i=1} \boldsymbol{\zeta}_i] = 0 \tag{5b}$$

where Y_j and X_i are closed and convex sets such that X_i is bounded from below (cf. Arrow and Debreu, 1954, p. 268), Y_j has no semipositive element and satisfies the property that

[13] See, however, Arrow (1989, p. 17), who relates the von Neumann model closely to that of Cassel. Similarly, Weintraub (1985, p. 77), McKenzie (1987, p. 500), Dore, Chakravarty and Goodwin (1989, p. 2) and Punzo (1989; 1991).

[14] Assumption IV.a is reported in the following as condition (6). We shall comment on this statement below.

$$\mathbf{y}_j \in Y_j \Rightarrow -\mathbf{y}_j \notin Y_j$$

(ibid.); the preference ordering $\succsim$ is such that a utility function can be built on X_i and such a function is continuous and has a maximum in X_i and is monotonic on each segment connecting two elements of X_i (cf. *ibid.,* p. 269); and the given constants $\boldsymbol{\zeta}_i$ and α_{ij} are such that

$$\exists \mathbf{x}_i \in X_i \,|\, \mathbf{x}_i < \boldsymbol{\zeta}_i \tag{6}$$

$$\alpha_{ij} \geqq 0, \ \sum_i \alpha_{ij} = 1$$

(cf. *ibid.,* p. 270).[15] The constant $\boldsymbol{\zeta}_i$ denotes the initial endowment of agent *i* and the constant α_{ij} denotes the share that agent *i* has in firm *j*. It is easily recognised that conditions (2) correspond to condition 1 stated by Arrow and Debreu (*ibid.,* p. 268): "$\mathbf{y}^*_j$ maximises $\mathbf{p}^* \mathbf{y}_j$, over the set Y_j, for each *j*". Similarly, conditions (3) correspond to condition 2 stated by Arrow and Debreu (*ibid.,* p. 271): "$\mathbf{x}^*_j$ maximises $u_i(\mathbf{x}_i)$ over the set $\{\mathbf{x}_i \in X_i \,|\, \mathbf{x}_i\mathbf{p}^* \leqq \boldsymbol{\zeta}_i\mathbf{p}^* + \sum_{j=1}^{n} \alpha_{ij}\mathbf{p}^*\mathbf{y}^*_j\}$", and conditions (4) and (5) correspond to properties 3 and 4 stated on the same page. This way of stating the problem of the existence of a competitive equilibrium allows for a comparison with the problem studied by von Neumann.

In the von Neumann model there is no consumption choice and "Consumption of goods takes place only through the processes of production which include necessities of life consumed by workers and employees" (von Neumann, 1945, p. 2). Hence in the von Neumann model there is no analogue of condition (3): it is as if all the $\mathbf{x}_i$'s were set equal to $\mathbf{0}$ and the "necessities of life" were inserted in the $\mathbf{y}_j$'s. Then condition (2) is the analogue of conditions (1b), (1d), and the first of conditions (1e); condition (4) is the analogue of the second of conditions (1e); condition (5) is the analogue of conditions (1a) and (1c), paying attention to the fact that $\mathbf{x}_i = \mathbf{0}$ for each *i*, and that in von Neumann the $\boldsymbol{\zeta}_i$'s are not given as in Arrow and Debreu, but are endogenously determined in order to maintain a steady state. These formal similarities allow one to use the same topological tool to solve the problem.[16]

As we have seen, Arrow and Debreu assumed that each agent has an initial endowment that suffices to guarentee the agent's survival until the very end of the

[15] In sections 4 and 5 of their paper property (6) is partially weakened.

[16] Now we know that this tool is not necessary to solve the von Neumann problem, but it is in order to solve the Arrow–Debreu problem; and this is so exactly because of condition (3).

time horizon of the model even if the agent should not participate in production and exchange, that is, even if the agent would just live on his own endowments (see condition (6) above). Arrow and Debreu argued that this assumption "plays the same role as the one made by Professor von Neumann in his study of a dynamic model of production ... that each commodity enters into every production process either as an input or as an output". This contention deserves to be carefully scrutinised. We ask, what can be meant by the "same role"? As a matter of fact, both assumptions are used to show that some sets are not empty (cf. Arrow and Debreu, 1954, p. 278 and von Neumann, 1945, p. 4). But the conditions are totally different from other points of view. The condition stated by Arrow and Debreu is a continuity condition of the budget set and may be interpreted as a viability condition of the consuming agent: there is a feasible consumption path for each and every agent such that the agents initial endowments are sufficient for their survival even if they do not enter into any exchange or cooperation with one another, including any production activity. The assumption thus guarantees that the number of agents defining the economy will remain the same over the entire time horizon contemplated. If that number were to change, a new model would have to be formulated any time that happens. (This indicates anew that the Arrow–Debreu model is essentially a short-period model.) On the contrary, the condition stated by von Neumann is a typical simplifying assumption which can be dropped at the cost of some complication – without, however, affecting the model as such. This has been shown by Kemeny, Morgenstern and Thompson (1956).

Despite the mathematical similarities, the two dissimilarities we have mentioned, the one relative to consumption and the other one relative to the endowments are enough to consider the two models as belonging to two different traditions of economic thought.

6. CONCLUDING REMARKS

In this paper we compared John von Neumann's growth model with the classical economists' approach to the theory of value and distribution on the one hand and to the neoclassical approach on the other. It was shown that von Neumann's model shares all important features of the former in terms of method of analysis, analytical structure and economic content, and differs in important respects from the latter. In particular, like the classical economists, von Neumann does not start from given endowments of the economy of "capital" as the neoclassicists, and unlike the latter takes the real wage rate(s) as given in accordance with the classical authors. His main concern is the problem of reproducibility of the system under con-

sideration, while he sets aside the problem of scarcity. Interest, that is the only share of income other than wages in von Neumann's case, is explained in a full-fledged classical manner in terms of a surplus product, or residual. Other than the models of steady-state growth of Gustav Cassel (and later Robert Solow), von Neumann determines the rate of expansion *endogenously* – reflecting the surplus-creating capacity of the economic system. It is also shown that setting aside some purely formal aspects there are no connections between the von Neumann model and the Arrow–Debreu model. A claim according to which an assumption of the latter plays the "same role" as an assumption of the former is scrutinised and rejected.

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