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THE IMPLICIT DYNAMICS OF THE VON NEUMANN GROWTH MODEL

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Von Neumann proved the theoretical existence of economic equilibrium. His assumptions explain also how and why cycles are generated. Two features of the model, reflecting the operation of the market, trigger these. The first feature is a non-focal equilibrium that takes place at a saddle-point, maximizing production while minimizing the rate of interest or monetary gain. The second is the inevitability of cross-regulation. Surplus gain boosts production while excess supply curbs prices.

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1. INTRODUCTION

The economic profession, if it did care about lasting imbalance, tried to explain it by a single cycle, "the" cycle, up till and including Keynes. The theoretical breakthrough came with Schumpeter. Investigating the business cycle (1939) he was led to the conclusion that at least three (but probably more) cycles are constantly at work. The imbalance may be caused by more than one and occasionally very many complexly intertwined fluctuations. He firmly believed that fluctuations and cycles are basic forms of economic motion. Yet he still maintained that it is right for the businessmen to look for the equilibrium behind this motion, as a theoretically sound, general, usual and normal state of the trade. At the same time he was showing convincingly why the economy is not found normally in this so-called normal state.

His argument was interspersed with caveats, excursions and qualifications, so it is extremely difficult to nail down its main tenets. What we know for sure is that he was keenly interested in the new mathematical developments emerging around

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the concept of equilibrium, balance and interdependence. But the essence of the work of John von Neumann on the delicate equilibrium of a growing economy eluded him. We also know that one of his best and mathematically most talented pupils, Richard Goodwin, when asked for help by Schumpeter, discouraged any further consideration of the model, branding it "bad economics". Later Goodwin revoked this rash judgment.

But originally von Neumann's achievement was received with deep silence and cold shoulders. His basic economic model caught attention only in the sixties when the success of operation research, linear programming and the theory of games showed promising novel mathematical possibilities. The growth model had been initially rejected because it had an implied edge against the prevalent economic paradigm. P. A. Samuelson who refreshed, refined and defended this paradigm still seems to have an account to settle with von Neumann.[1] Samuelson demonstrated how the maximization of profits and utilities balance economic systems. By attending to private interests equilibrium can be apparently accomplished and an optimal allocation of resources is also secured. Alas, all the actual economies we happen to know fail to satisfy this theoretical claim in practice.

Today, about 60 years later, we are in a position to appraise von Neumann's innovative efforts better. In the present paper two important facets will be mentioned. The first is a break with the idea of Leibniz. ("This world is the best of all the possible worlds because it includes the idea of and drive for an optimum.") The generally unjustified opinion of Leibniz was ridiculed already by Voltaire, who infused this naive optimism into the easygoing person of unforgettable doctor Pangloss. This first point is elaborated in the next chapter. The second point, cross regulation, is discussed in the chapter following thereafter.

2. THE SADDLE-POINT

Von Neumann, facing economic antagonism and turbulence, expressed equilibrium not as a simple optimum or maximum but as a saddle-point. This equilibrium point may be defined only by a special, double-edged, minimum-maximum principle. What is good for a given human being may be thoroughly detrimental to somebody else. Yet in spite of all the massively conflicting interests (so typical of everyday economic life) one can still find a meeting point that may be justly called equilibrium. Von Neumann conceived it (as the original title of his report indi-

[1] Both original opinions are described in Dore, Chakravarty and Goodwin (1989), where Goodwin confesses his erroneous first reaction (p. 126), but Samuelson still maintains that "Neumann's shade owes me a cigar" (p. 112).

cated) as a generalization of the Brouwer theorem for a fixed point. It is nothing more (and nothing less) but a fixed point of those economic transformations that go on repeatedly and incessantly. What he wanted to and did prove was only the abstract mathematical existence of such a point under certain assumptions.

The proof given by him made it unnecessary and superfluous to define some (always contestable and never exactly measurable) common weal. It does not require to prove that equilibrium is a universal attractor, a focal point of some perfect process that is actually going on in the real world. He defined a fixed point of economic transformations that is an important point, an ideal point, a point that merits to be called equilibrium, and still does not require to assume (or even prove) that the economy must or can ever reach it.

Beware. What he did prove was that there always exists at least one fixed point. There may be more than just one. If there are two, then all points of the line connecting them will be fixed, because of the linearity of the model. If there are even more, then the whole domain, spanned by their convex hull, will behave as a "fixed" domain. Nor did he prove that the fixed point is optimal in any sense. This question did not emerge in his discussion. It furnishes a maximal rate of growth but a minimal rate of interest. The latter may be interpreted as a necessary condition for a universal rate of surplus over and above the equilibrium cost of the inputs advanced.

He was apparently not interested in actually computing it, or inquiring into the limits to the computability of a fixed point. That came much later when he was asked for help by Dantzig in the latter's quest for finding an algorithm to solve his much simpler linear programming model. But, in essence, such a dynamic motion that may (or may not) lead to equilibrium already lurked behind the very definition of the general model of von Neumann. We shall return to this question later.

3. THE CROSS-REGULATION

Von Neumann wanted mainly to explain which products would become free goods (goods that in equilibrium do fetch no prices) and why a theoretically and practically feasible technology would be applied or rejected (under the same equilibrium conditions). He wanted to exhibit all the dual economic decisions that serve as a basis for the choice of prices and technology. They render certain products and certain technologies redundant and worthless. This setup has far-reaching consequences in shaping the motion of the economic system, its practical possibilities for control and thus also for the mathematical algorithm that may be used to compute its fixed point. It entails the cross-regulation of prices and quantities. It also hinders the direct adaptation of the production of a single commodity to its own prevailing price.

Thus the second important feature of his economic model was to drop the simple assumption of those productive units, enterprises or sectors that turn out a single product with a well-defined price. He considered the economic system as consisting of a multitude of feasible processes that all consume, and also produce a variety of products. In this respect his approach was more realistic than the former theoretical (and later also practically pursued) simpler categorization that unequivocally identifies each and every sector with an individual and unique output. In the von Neumann system both input and output matrices are rectangular but they are not necessarily square. This renders it impossible but also superfluous to assign a given row and column to the same singular product. While the number of products and prices is naturally equal, the number of processes might surpass that of the prices, or the number of prices might surpass that of processes.

Von Neumann returned to the classical explanation of economic adjustment. The producer has to assess all the changes in all the prices of his various inputs and outputs to decide whether to start and then step up or step down his own activity. The apparent gain (or profit) of a given process regulates the extent of the process. On the other side the imbalance emerging on a given market (the apparent surplus of demand or supply) regulates the prices of the individual products.

He did not use the term "quantity of production" because that would presuppose a single kind of product. If the output is manifold then the "amount of output" must be a weighed sum of the prices and quantities of more than just one product. The weights should naturally be the equilibrium proportions. Yet the producer will never exactly know these ratios. The relevant equilibrium prices or quantities are unknown to him at the time he has to take his decisions, thus they cannot regulate all these diverse decisions.

What the producer does (or may) know at any given moment is the off-equilibrium price vector of the instant and, of course, the extent of his own activities. These prices do determine his momentary gain or loss. For the same reason the "quantity" of his production is indeterminate. It would be an average weighed with unknown equilibrium prices. Von Neumann defined instead something he called "intensities" of the various productive processes. Prices and intensities are the practical decision variables of the producer and they are influenced by the off-equilibrium prices and intensities. Von Neumann thus assumed that it is always the extent of the whole process that is or must be changed to arrive at equilibrium.

The separate and individual inputs and outputs of a process will therefore keep their original setup intact, and if their proportions must be and can be changed, then a new and different process has to be defined and started. This observation is required when the producer may have the possibility and opportunity to change the proportion of the various products his activity or activities turn out.

A real enterprise or a real process namely often turns out twin or joint products. For instance raw iron and combustible gas, which are both further utilized in a metallurgical plant; or a chemical process may also yield two or more compounds. In such a case it is the joint economic result (gain or loss) that will (or may) regulate his production. A pig, when slaughtered, yields a rich variety of useful end-products but it would be imprudent to single out, say, the price of pork ribs to regulate the intensity of slaughtering.[2]

Mathematically we possess *n* by *m* matrices as our initial data of outputs and inputs. That means we have *n* prices (rows) and *m* processes (columns). For control purposes we may find out (or compute, by multiplication with prices) *m* data that indicate profit or loss of the individual processes and may be therefore used to correct their extent. We may step up or down the *m* processes but we are unable to use this same information for doing anything to the *n* prices. For control purposes we may equally well compute *n* data (by multiplying the same matrices from the other side by quantities) that indicate the market surplus or deficit of the *n* individual products. We may thus alter their prices upward or downward. But we shall be equally unable to use this information for doing anything to the *m* processes. The gains or losses of the processes cannot act on prices, and the surplus or deficit of the products cannot act on processes because in both cases there is a mismatch between the amount of required information.

4. ABOUT REGULATION IN GENERAL

Leaving mathematics and prices for a while, the basic problem of regulating economic cooperation is the following. Economic production is changing through time. If it lacks the needed balance and regularity, then some inventories go needlessly up or down. This signals the lack of balance. We are able to observe inventories of products, that is, stocks, and their unusual, unwanted or unneeded in-

2 The term *intensity* is not correctly used here. Yet it reveals that the setup has been inspired by thermodynamics, more specifically by the approach of Gibbs (1906). In thermodynamics it is usual to distinguish intensive and extensive variables. The distinction between these two kinds of variables is made easy by the following mental exercise: Let us consider two strictly identical and equilibrated systems that are characterized by appropriate parameters. Let us now unite the two systems. In the resulting system some magnitudes will keep their original values (these are the intensive variables) and some others will take on a double value (these are the extensive variables). Thus, in this respect the correct designation would have been not "intensity" but some extensive measure of production, say, its volume or bulk or extent or size. In an economic model it is the price that acts as an intensive variable.

crease or decrease. To redress the balance by stepping production up or down requires not only lengthy and circular calculations (or, if prices exist, then market adjustments: changed prices that then change the anticipation of profits of the individual processes), but also a certain amount of time. It is impossible to alter and rectify an unbalanced situation instantaneously. It should be evident that balancing stocks by changing the respective product flows requires often very long time spans, gestation periods, "roundabout ways of production" as the Austrian school maintained, and so on. The process of adjustment or regulation will therefore become prone to delays and thus to fluctuations.

A better and more flexible regulation, if it exists, may keep the fluctuations down and perhaps try to dampen them. It would also try to avoid the most dangerous and critical frequencies. But it will never be able to dismiss temporary imbalance. It is impossible to smooth out every bump, to reach and maintain perfect equilibrium all the time. What we have to investigate therefore is exactly how the system behaves in the vicinity of equilibrium. Our key problem then will be timing. It is not enough to know how a change of some flow influences equilibrium itself. The same change, applied with different timing, may have an entirely different, unwanted, possibly opposite effect on the system. By analyzing an equation of general equilibrium we may derive for instance that a cut of consumption, by boosting accumulation, does indeed increase the rate of growth. But decreasing consumption in a trough has an entirely different effect than the same decrease, at the time of peak performance. What may improve or remedy an economy in a downswing might be detrimental, or even ruinous in an upswing and *vice versa*. The practically relevant question is mostly not the change of the equilibrium point itself but the change of the prevailing transient path.

Multisectoral equilibrium analysis is indispensable for understanding and teaching economics. But it is only a snapshot, highly stylized as an anatomic atlas of the human body. It does not breathe, pulsate, move, change, as the living body does. Comparing various equilibria as they shift when data and parameters are modified does help understanding basic facts and tendencies of economic life. But the comparison, more often than not, neglects or blurs how changes arise and propagate. Simple comparison is unable to trace and sketch a movement as it takes place. The comparison of different equilibria may be misleading and is unfit to yield a realistic and feasible time path. It does not answer our questions about timing, direction and secondary movements set in motion by economic interference or specific actions.

The fluctuating and cyclic models offered in the literature are inevitably of a preliminary and stylized character. Yet they address cycles not as annoyances but as parts of a regulating mechanism that keeps the economy going. Without acknowledging and describing the actual motion in the vicinity of equilibrium, our efforts to improve economic relations are necessarily doomed. It turns out that the strict and sharp theoretical equilibrium is not and hardly can be reached by any real economic process. Those imagined controls that automatically lead the economy back to its (incessantly changing) equilibrium point or path do not exist, and are thus not feasible.

Neither free markets nor central directives were ever able to secure a growth path where the yearly deviations from the average rate of growth had not regularly reached or surpassed the size of this very rate. The dispersion had been usually of the order of the rate itself, often greater, and sometimes a tough multiple. The jagged outline and the zigzag of yearly or quarterly growth rates reveal and demonstrate the basic nature of all present economic systems.

5. COMPUTING A SOLUTION

Now it only remains to show that the von Neumann model, when properly set up, leads necessarily to cyclic paths, when we take into account the Huyghens principle that no action may travel with infinite speed. The original axiomatic rules namely hide actual economic (or computational) processes that, on one hand, explain the cyclic character of economic motion and, on the other hand, permit the approximation of its fixed point.

By economic necessity every process that operates at a loss will be decreased. Von Neumann stipulates that if loss prevails at equilibrium then the process has to be abandoned. He just speeds up the steps of regulation. The same economic force dictates that an oversupplied product must suffer price cuts. This is stated in the static final solution as fetching no price at all and turning the product into a free good. The double rule, seen as a gradient method for changing prices and quantities according to (positive or negative) surplus supply and surplus profit generates a skew-symmetric process. Market regulation (and the computational imitation of the latter) generates cyclic paths around the equilibrium point.

This cyclic path driven by the markets may be turned into a convergent iteration on the computer. It may be done, for instance, by zeroing out (canceling) a variable when it crosses zero from above. This way the computer is constrained to operate in the positive orthant. The market heeds this rule. If shrinking of some price or process is persisting long enough it will be wiped out. Yet the cyclic path of all the remaining positive prices and quantities may cause some processes to switch back into operation and some free goods to obtain again (at least transiently) a positive price.

Thus to find the fixed point the cycles have to be dampened. On the computer this may be done again easily. One simply averages their value through their apparent cycle, or – even better – takes their median. (The latter is less sensitive if ephemeral positive values re-appear.)

These latter exercises are not done on the market, thus in a real economy a cycle once started (by chance, shock, foreign trade or political turmoil, and so on) tends to retain a given amplitude. Even on the computer the power of resolution will depend on the remaining amplitude of the cycle. Thus if there are neighbor processes (with small differences in their coefficients), some switching back and forth is unavoidable.

This is the reason then, why (as with coexisting species in ecological systems) neither economics nor evolution produces sharp equilibria. Instead of the survival of only the fittest, the different species live habitually in a fuzzy and pulsating coexistence. Such coexistence certainly puts the fitter into a better position but it usually permits also the somewhat less fit to survive. The surrounding allows moderate elbowroom, which may be used to improve oneself.

This is a bit of luck for mankind. It would be otherwise extinct by now.

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APPENDIX

This is an iteration to select goods and processes and determine the fixed point. The program is described as a MATLAB.m file. This is the closest language to actual mathematical notation I know. It seems to work well and fast for random matrices. "%" indicates comments. Warnings (for incorrect inputs, looping, multiple solutions, and so on) should certainly be added to the final program, and a final check for the solution.

```
function [rho,p,x] = neumann(in,out)
% the function [rho,p,x] = neumann(in,out) computes a solution
% rho,p,x (growth factor, prices and quantities),
% for rectangular matrices in and out(input and output).
% Preliminaries:
% initial vectors, scaling, flag for indicating selection,
```

```
% initial rate, and select = 1 for going on with the process

[m,n] = size(out);
p = ones(1,m)/m;
x = ones(1,n)/n;

scale = m;
b = in/scale;
a = out/scale;

flag = [0 0 0 0 0];

rho = (p*a*x')/(p*b*x');

select = 1;

while select
      c = (a-rho*b);
      for i = 1:100

      dx = p*c;     %surplus gain
      dp = x*c';    %excess product
      p = p-dp;
      x = x+dx;

      p = p.*(p>0); %selecting valuable goods
      p = p/sum(p); %normalizing
      x = x.*(x>0); %selecting applied processes
      x = x/sum(x); %normalizing

      R(i,:) = p;
      Y(i,:) = x;

      end
p = median(R); %taking median price
x = median(Y); %taking median quantity
p = p.*(p>0);
p = p/sum(p);
x = x.*(x>0);
x = x/sum(x);
```

```
rho = (p*a*x')/(p*b*x');

% adjusting flag
flag(1:4) = flag(2:5);
flag(5) = sum(p>0) == sum(x>0); %indicating squareness

% testing if the matrix was square for the last 5 iterations
    if sum(flag) == 5
    select = 0;
    end
end

%Selection finished. Finding equilbrium

pindex = find(p>0); %indexes of selected goods
xindex = find(x>0); %indexes of selected processes

A = a(pindex,xindex); %square matrix of outputs
B = b(pindex,xindex); %square matrix of inputs

%Inverting an almost singular matrix to improve exactness

for k = 1:3
    s = inv(A-rho*B);
    P = sum(s)/sum(s(:));
    X = sum(s')/sum(s(:));
    rho = P*A*X'/(P*B*X');
end

%Final values and exactness

p = zeros(1,m);
p(pindex) = P;
x = zeros(1,n);
x(xindex) = X;

sistem = rho*in-out;
error = num2str(p*sistem*x');
disp(['Exactness  ',error])
```