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JOHN VON NEUMANN'S CONTRIBUTION TO MODERN GAME THEORY

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The paper gives a brief account of von Neumann's contribution to the foundation of game theory: definition of abstract games, the minimax theorem for two-person zero-sum games and the stable set solution for cooperative games with side payments. The presentation is self-contained, uses very little mathematical formalism and caters to the nonspecialist. Basic concepts and their implications are in focus. It is also indicated how von Neumann's groundbreaking work initiated further research, and a few unsolved problems are also mentioned.

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John von Neumann is rightly considered as one of the founding fathers of game theory. He addressed very fundamental issues both in non-cooperative and cooperative game theory and gave answers that left inerasable marks in the body of knowledge we call now modern game theory. His main contributions can be summarized as:

– mathematically correct definition and distinction between extensive and normal (strategic) form games;
– proof of the first existence theorem for the equilibrium of zero-sum two-person games;
– laying down the foundations of transferable utility cooperative games.

As in his other endeavors, von Neumann was interested in comprehensive theories and structures, and always focused on the core problems in the field. We shall first discuss his main accomplishments in non-cooperative and then in coop-

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erative game theory. The presentation tries to be as informal as possible without sacrificing the insight mathematical formalism can provide.

1. EQUILIBRIUM OF ZERO-SUM TWO-PERSON GAMES

There have been attempts to model situations calling for strategic thinking before von Neumann. The most famous precursor to modern game theory was Cournot (1838) who studied oligopolies where two firms make price through decisions about output. Von Neumann was the first, however, to define mathematically correctly the concept of an abstract, strategic game. In order to understand the significance and potential of what von Neumann did, we need a brief introduction to the basics of game theory.

1.1. A few basic concepts of game theory

Game theory, when defined in the broadest sense, is a collection of mathematical models formulated to study situations of conflict and cooperation. These situations are characterized by the interaction of several participants (players) whereby the outcome of the actions taken by the players not only depends on their own decisions but on the actions of the rest of the players as well. Conflict does not necessarily mean antagonism (though as a special case it may do so). The participants may have common interests and in certain cases can even cooperate to achieve preferable outcomes. Conflicts abound and we can experience them everywhere we go, and certain aspects of them are subjects of many fields such as law and psychology. Game theory restricts itself to study conflict by mathematical models and tries to find answers to why people and institutions behave the way they do and how they should behave if they followed certain patterns of rational decision making. It also engages in predicting outcomes of joint actions of the players, and stability of the situation thus realized is also of concern.

Two things must be made clear at this point. Though common language calls e.g. the game of dice as a game, we shall exclude it from consideration since the players in these games are passive, they are not making decisions relevant to final outcomes, their role is mechanical and is confined to simply throwing dice. We shall usually assume that there are at least two players, thus disregarding decision problems where there is just one decision-maker facing uncertainty as to certain constituents of the state of nature. Though, as a testing ground for certain solution concepts, especially in axiomatization, one-person decision problems will also be considered as a degenerate special case of many-player games. The methodology of game theory is also special: we shall study how "rational" players ought to act and not what real players in real life actually do in conflicts. The latter is the subject of experimental game theory or "gaming" as it is usually called. Gaming can be thought of as the closest relative to game theory and is very important in verifying theoretical models. If there is a big gap between reality and what theory suggests then theoretical models should be refined or completely abandoned and replaced by something better.

Since the very beginning, two major fields within game theory have been distinguished: cooperative and non-cooperative games. In the case of cooperative games, the rules of the game allow the players to make binding agreements for coordinating their actions in order to achieve favorable outcomes unattainable by non-cooperation. As an example, we can take a democratic parliament where two parties (players) may sign an agreement on how to vote together about certain issues. In the case of non-cooperative games the rules of the game do not allow explicitly to sign such agreements though other, more subtle means are available to achieve better outcomes.

Oligopoly is a good example of a strategic game. Consider a market with n firms producing a homogeneous product. The firms make decisions independently about production volumes. (Cartels are ruled out by anti-trust legislation.) Consumers clear the market at a certain price that is a monotone decreasing function of the production volume of the whole industry. Thus the profit a firm can make not only depends on its own decision and on its own cost of production, but also on the production decisions of the other firms.

In the oligopoly model it is quite plausible what decisions the players are supposed to make and how they evaluate a particular outcome if we assume that each firm is a profit maximizer. There are situations, however, where the evaluation of outcomes is not so straightforward and calls for a utility function for each player to help him assess how favorable an outcome for him is. Players are assumed to be rational: higher values of their utility functions are preferred to lower ones. If a player does not know exactly what the others will do but has a (subjective) probability distribution about the occurrence of the possible actions of the others, then it is assumed that the expected utility of the player is maximized. It should be noted that the mathematical foundation of expected utility theory is also due to John von Neumann and Oskar Morgenstern (1944).

1.2. Non-cooperative games

Von Neumann was the first to make a clear distinction between the extensive and the normal (strategic) form of non-cooperative games. The extensive form is a move-by-move detailed description of the game. The tool to describe a game in such a way is borrowed from graph theory and is called a finite rooted tree. A rooted tree looks like a real tree: branches go out of the root first and then more branches from the ends of previous ones until we reach the leaves of the tree from which no more branches emanate. This tree is given by its nodes and branches, and characterized by its connectedness (every node can be reached through branches) and by the lack of circles (we cannot return to any node by going through a series of nodes and branches). To every node of the tree we assign a player who is to decide along which branch he will proceed to the next node towards the leaves. It is allowed that at certain nodes a random mechanism (usually called "Chance") decides which way to go according to a known probability distribution. This is the case of most card games where the game usually begins with shuffling and dealing of cards. To every leaf of the tree n numbers are assigned representing the utilities of the players in the case if that particular terminal node is reached. In the game theory literature these utilities are mostly referred to as payoffs.

A path leading from the root to a leaf is called a play. The game just described is characterized by complete and perfect information, i.e. each player knows the whole tree, the payoffs and knows that all other players know the tree and everybody is a utility maximizer.

A good example of such a game is chess. The game of chess is determined by its rules. At the root of the game tree White is to move and can go in 20 different directions (each pawn can move by one or two squares and the two knights can go two ways). To the subsequent 20 nodes Black is assigned and can go 20 different ways either. The game goes on like this until termination which is assured in a finite number of steps by the draw-enforcing rules of chess. Payoffs can be the usual: 0, 1 and ½ points given for loss, win and draw, respectively. Because of the astronomical number of nodes this tree can only be drawn in principle but not in practice. (Even after two moves the tree has 421 nodes!)

In the normal (strategic) form of a game we assume that each player has a strategy set, i.e. n non-empty sets $S_1,\dots, S_n$ are given and the game is played as follows: every player, independently, selects an element from his strategy set resulting in a strategy profile $\mathbf{s} = (s_1,\dots,s_n)$. Furthermore, each player has a utility (payoff) function $f_1,\dots,f_n$, assigning to every strategy profile $\mathbf{s}$ a payoff (real number). The strategy sets and the payoff functions together are called the normal (strategic) form of a game. As already mentioned, von Neumann was the first to describe clearly how the strategic form can be obtained from the extensive form. A strategy of player *i* in the extensive form is a complete plan of action specifying what to do (in which direction to move) at each node player *i* is assigned to. In chess e.g. a strategy of White is a set of instructions telling White what to do in every situation when it is White's turn to move. If this set of instructions calls for a different move even at one of the nodes, then we have a different strategy. Obviously, Black's strategies can be defined similarly. The game can now be played by the players' simultaneously choosing a strategy from their strategy sets, which will unambiguously determine the outcome. Playing the game now reduces to the strategy choices which take place even before touching the pieces. The actual play in its everyday meaning becomes a mechanical task that can even be done by agents without the actual participation of the players. Needless to say that in competitive chess strategy has a different meaning and refers to how grandmasters line up their pieces in the early phase of the game in order to achieve positions they are comfortable with. Of course, listing the elements of the strategy sets is also beyond the capacity not only of any human being, but also that of the most powerful computers.

The strategic form, as opposed to the often cumbersome extensive form, is usually more amenable to mathematical analysis and it also makes possible to formulate games that are very difficult or impossible to view as games in extensive form. The transition from the extensive to the normal form enabled von Neumann, the ingenious mathematician, to use the most powerful tools of contemporary mathematics in game theory.

1.3. Equilibrium of zero-sum two-person finite games

Around 1920, quite a few mathematicians got interested in a special game. In a certain sense, this is the simplest possible game: the number of players is *2*, both strategy sets are finite and the sum of the payoffs to the players is identically zero. The zero-sum property subsumes complete antagonism, the reward of a player is the loss of the other. Thus, it is sufficient to define the game by the payoffs of one of the players, the payoffs of the other one are simply obtained by taking the negatives. Therefore, in normal form, this game can be given by an m by n matrix $\mathbf{A}$. The element a_{ij} of this matrix represents the payoff Player 1 (the row player) gets from Player 2 (the column player) if Player 1 plays his i-th, whereas Player 2 plays his j-th strategy. As an example, consider the matrix of the game where Player 1 chooses between T and B, Player 2 chooses between L and R.

		Player 2	
		L	*R*
Player 1	*T*	1	2
	B	3	4

Let us see how rational players might think. If Player 1 chooses *T*, then he can expect payoff *1* if Player 2, being rational (loss minimizer) picks *L*. If he chooses *B*, then, by the same reasoning, he can expect *3* for payoff. Since Player 1 can freely choose between *T* and *B*, he will go for *B*. His security level, the payoff he can get no matter what the other player does, is *3*. *B*, in this case, can also be called a maximin strategy.

The same reasoning can be repeated for Player 2. By choosing strategy *L*, he minimizes the maximum loss he incurs if Player 1 does the most harm to him by playing *B*. Putting it in another way, *L* is Player 2's minimax strategy and payoff *3* is his security level. This thinking process leads to an equilibrium, where the players apply their maximin and minimax strategies, respectively, resulting in a common security level which is called the value of the game. Under the rationality assumptions made, this outcome can be expected, and can also be considered stable since it is in no player's interest to replace the equilibrium strategy if the other one sticks to his choice.

Not every matrix game is so simple to analyze as this one. Let us take another example. A soccer player is preparing to shoot a penalty kick. It is commonly known that the keeper has the most chance to save the goal if he makes up his mind early, mostly before the ball is actually kicked, in which direction he will move. It also helps a lot if the kicker decides where to direct the ball before the actual kick. Thus, the situation becomes a guessing game on the part of both players. Assume, for sake of simplicity, that the kicker has three choices: Right, Middle and Left indicating that part of the goal to where he intends to shoot the ball. The keeper also has three choices: move to Right, stay in the Middle and move to Left. (Middle also covers the case when he does not move until he sees where ball is going.) Viewing the game from the standpoint of the Kicker, his payoff will be the average number of successful penalties from ten attempts. The matrix below shows the payoffs (the numbers are not based on statistics but are not unrealistic either).

Obviously the Kicker is the maximizing and the Keeper is the minimizing player. If we follow the logic used in the previous example, then the Kicker must choose *R* or *L*, and his security level is *5*. The Keeper's minimax strategy is *M* and he can guarantee that no more than *8* goals will be scored. But these pairs of strategies are not stable. For example, it is not worth to kick the ball always to Right because the Keeper can then count on this and will move in time saving *5* penalties though he can secure only *2* saves himself.

		Keeper		
		R	M	L
	R	5	8	9
Kicker	M	8	3	8
	L	9	8	5

This instability and the associated indecision can be resolved if we modify the game itself by allowing probabilistic mixtures of the original strategies as strategies in a new game called the mixed extension of the matrix game **A**. In this new game the payoffs are defined as expected payoffs and for Player it can be given by the bilinear form **xAy,** where **x** is the probability vector (strategy in the mixed extension) chosen by Player 1, and **y** is the probability vector selected by Player 2. The original strategies are also strategies in the mixed extension (choose one strategy with probability *1* and the others with probability *0*) and called pure strategies in this context.

As we saw, allowing only pure strategies, equilibrium may not exist. What is the situation with the mixed extension? The famous French mathematician, Emile Borel was the first to realize the significance of pure and mixed strategies in matrix games. He made efforts to prove the existence of equilibrium (i.e. to prove maxmin = minmax) for the mixed extension, but failed. He even expressed doubts about the validity of the maxmin = minmax equality. This was the point where von Neumann joined in. In 1928, using a powerful fixed-point theorem, he proved the existence of equilibrium strategies for the mixed extension. In fact, he proved a more general theorem which has been quoted as the first minimax theorem ever since. Though it is not completely in line with the informal style of presentation, the significance of this theorem warrants precise language.

Von Neumann's minimax theorem (1928)

Let X and Y be unit simplexes of finite dimensional Euclidean spaces, and f a jointly continuous real-valued function defined on $X \cdot Y$. Suppose that f is quasiconcave on X, that is to say, for all $y \in Y$ the upper level sets of f are convex, and f is quasiconvex on Y, that is to say, for all $x \in X$ the lower level sets of f are convex. Then

$$\min_{Y} \max_{X} f = \max_{X} \min_{Y} f.$$

This result was later extended by von Neumann (1937) himself by replacing unit simplexes with nonempty compact, convex subsets of Euclidean spaces. In both cases, von Neumann used topological and fixed-point arguments. It turned out later that fixed-point theorems are not necessary for the proof, convex analysis, in particular linear separation is enough to prove even more general results where the continuity of f is replaced by partial upper and lower semicontinuity in the respective variables, Sion (1958). Theorem 1 has opened a whole avenue of research about minimax theorems and their various generalizations, and has been applied in many fields inside and outside of mathematics. (For a good overview of minimax theorems we recommend Simons 1995.)

Return now to our penalty kick game. In the mixed extension of this game the Kicker's equilibrium (optimal) strategy is as follows (results have been obtained by linear programming):

Right with probability	0.416;
Middle with probability	0.168;
Left with probability	0.416.

The Keeper's minimax strategy is as follows:

Right with probability	0.416;
Middle with probability	0.168;
Left with probability	0.416.

As a result, the expected number of goals is 0.717.

2. VON NEUMANN'S CONTRIBUTION TO *N*-PERSON GAME THEORY

2.1. Nash equilibrium of *n*-person games

Inspired by the equilibrium concept developed by von Neumann for two-person zero-sum games, John Nash (1950) defined equilibrium for general-sum n-person games which proved to be so successful in various applications, especially in economics, that this definition and the proof of the existence of equilibrium earned him the Nobel Prize in Economics in 1994.

Given an n-person non-cooperative game in normal form

$$G = \{S_1, \dots S_n; f_1, \dots f_n\},$$

a strategy profile $\mathbf{s} = (s_1, \dots s_n)$ is said to be an (Nash) equilibrium point if

$$f_i(\mathbf{s}) = f_i(s_1, \dots s_{i-1}, t_i, s_{i+1} \dots, s_n)$$

holds for all $t_i \in S_i$ and $i = 1, \dots, n$.

The Nash equilibrium expresses a certain kind of stability: it is no player's interest to deviate from his equilibrium strategy provided the rest of the players will not deviate. In other words, if a player knows what the others are going to play, then his best response to this is his equilibrium strategy. It is easy to see that this definition reduces to von Neumann's equilibrium (maxmin and minmax strategies) when applied to two-person, zero-sum games. This definition is only useful if equilibria exist for a wide range of games. Nash proved in 1950 that the mixed extension of finite n-person games do have at least one equilibrium point. Mixed extension is defined in the spirit of von Neumann: strategy sets are probability distributions over the finite set of pure strategies, and payoffs are expected ones. The proof is also inspired by von Neumann's proof of the minimax theorem since it uses a fixed-point theorem: that of Brouwer (1912).

Nash always thought very highly of von Neumann who was among the first ones to whom Nash showed his proof of equilibrium and asked for advice.

2.2. Foundation of the theory of cooperative games

As we pointed out in the introduction, cooperative games arise when binding agreements among players are allowed. Groups of players, coalitions, may form and distribute commonly achieved gains among its members. Distribution of wealth within a coalition is only possible if a linearly transferable commodity (such as money) is available. Depending on whether there exists such a commodity which enables the players to compensate each other for sacrifices in pursuing a common goal, different methods of analysis are required. We shall only consider games where side-payments are possible and call them TU- (transferable utility)-games.

Not only non-cooperative game theory received the initiating impetus from von Neumann but the cooperative theory as well. In the seminal book written together with Morgenstern (1944) he set up the model of a TU-game most commonly used for analysis ever since.

Given a finite set of players $N = \{1,\ldots,n\}$, a pair $G = (N,v)$ is defined an n-person TU-game in characteristic function form if v is a real valued function defined on the subsets (coalitions) of N. The function v assigns a real number $v(S)$ to every coalition, with the convention $v(\emptyset) = 0$. The value $v(S)$ represents the transferable utility coalition S can achieve on its own when its members fully cooperate. The theory is mostly concerned with how the utility $v(N)$ achievable by the grand coalition N can be distributed taking into account the power, as expressed by the characteristic function the coalitions have. Von Neumann and Morgenstern consider only essential, constant-sum games in their book. A game $G = (N,v)$ is essential if

$$\sum_{i \in N} v(\{i\}) < v(N)$$

and it is constant-sum if

$$v(S) + v(N \setminus S) = v(N)$$

holds for any coalition S. We call the game $G=(N,v)$ superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$

holds for all disjoint coalitions S, T.

For a given game $G=(N,v)$ let S be a coalition and $\mathbf{x}=(x_1,\ldots,x_n)$ a real n-vector. Define

$$x(S) = \sum_{i \in S} x_i\,.$$

An n-vector $\mathbf{x} = (x_1,\ldots,x_n)$ is called an imputation if it is individually rational, i.e.

$$x_i \geq v(\{i\}) \text{ for all } i \in N$$

and Pareto-optimal or efficient, i.e.

$$x(N) = v(N).$$

An imputation represents a distribution of $v(N)$ among players in such a way that no player will get less than his own value $v(\{i\})$.

We arrive at one the most important concepts of modern economics if we not only require individual rationality but also coalitional rationality. This amounts to requiring

$$x(S) \geq v(S)$$

for any coalition *S*. The set of imputations that are coalitionally rational is called the core of the game. The core represents a particular kind of stability. If an imputation **x** is proposed, then there is no coalition that can successfully upset the grand coalition by leaving it. The core can also be characterized by a special relation called dominance first introduced by von Neumann and Morgenstern.

An imputation **x** is said to dominate imputation **y,** in notation **x** *dom* **y,** if there is a coalition *S* such that $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$. If **x** *dom* **y**, then coalition *S* can block the imputation **y** since it can give more to its members and also has the power to achieve this. It is easy to prove that for superadditive games the core is the set of undominated imputations. One of the problems with the core is that it can be empty for large classes of games, such as e.g. constant-sum games, the class of games von Neumann and Morgenstern study in their famous book. This is the reason why they proposed an alternative solution concept which captures stability in a different way.

Von Neumann and Morgenstern define a "solution" to a game $G=(N,v)$ as a subset V of the set of all imputations that is both internally and externally stable, i.e.

(1) there is no $\mathbf{x}, \mathbf{y} \in V$ such that **x** *dom* **y,**
(2) if $\mathbf{y} \notin V$, then there is an $\mathbf{x} \in V$ such that **x** *dom* **y**.

To distinguish from other solution concepts that have emerged since the groundbreaking work of von Neumann and Morgenstern, the above "solution" is usually referred to as stable-set or von Neumann–Morgenstern solution. They interpreted stable sets as a standard of behavior within a society and left open the question of which imputation in V will actually be realized. This is assumed to be determined by the bargaining ability of the players, outside forces, chance, etc. They were not disturbed at all by the fact that in many games there is a multitude of different stable sets. They considered each as a standard of behavior and did not consider part of their model which one of these will be realized. They were, however, deeply concerned with the existence of stable sets. They proved in their book the following theorem.

Von Neumann–Morgenstern Theorem (1944)

Every superadditive, essential, three-person game has at least one stable set.

Although some special classes of games were shown by von Neumann and Morgenstern to have stable sets, they were unable to prove a general existence theorem. To settle the existence of stable sets has proved to be a hard problem over the years. Lucas constructed a 10-person, non-constant-sum game in 1968 which had no stable sets and Bondareva et al. (1979) proved that all 4-person games do have stable sets. The question of existence for general games is unsettled for 5 to 9-persons. No one could prove or disprove the original conjecture of von Neumann and Morgenstern that every constant-sum game has at least one stable set. It is also conjectured that games with no solutions are "rare"; known examples are unstable in the sense that minor changes in the characteristic function value make them have stable sets.

Stable sets have a surprisingly rich mathematical structure and give rise to extremely difficult problems. Though there are several other solution concepts of cooperative game theory (such as bargaining sets, the kernel, different kinds of nucleoli, the Shapley value, etc.), the von Neumann–Morgenstern solution remains one of the powerful tools to analyze games with empty cores. An excellent overview of the developments in stable set theory from von Neumann and Morgenstern to the early nineties is given by Lucas (1992).

REFERENCES

Bondareva, O. N., Kulakovskaya, T. E. and Naumova, N. I. (1979): The Solution of Four-person Games (in Russian). *Leningrad Universitet Vestnik* 6: 104–105.

Brouwer, L. E. J. (1912): Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen,* 71: 97–115.

Cournot, A. A. (1838): *Recherches sur les principes mathématiques de la théorie des richesses.* Paris: Hachette. (In English: *Researches into the Mathematical Principles of the Theory of Wealth*. New York: Macmillan, 1929.)

Forgó, F., Szép, J. and Szidarovszky, F. (1999): *Introduction to the Theory of Games: Concepts, Methods, Applications*. Dordrecht: Kluwer Academic Publishers.

Kakutani, S. (1941): A Generalization of Brouwer's Fixed Point Theorem. *Duke Journal of Mathematics,* 8: 457–459.

Lucas, W. F. (1968): A Game with no Solution. *Bulletin American Mathematics Society*, 74: 237–239.

Lucas, W. F. (1992): Von Neumann–Morgenstern Stable Sets. In: Ausmann, R. J. and Hart, S. (eds): *Handbook of Game Theory with Economic Applications* 1, 543–590.

Nash, J. (1950): Equilibrium Points in *n*-Person Games. *Proceedings of the National Academy of Sciences USA,* 36: 48–49.

Neumann, J. von (1928): Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100: 295–320.

Neumann, J. von (1937): Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. *Ergebnisse eienes Mathematische Kolloquiums* (Wien), 8: 73–83.

Neumann, J. von and Morgenstern, O. (1944): *Theory of Games and Economic Behavior*. Princeton: Princeton University Press.

Simons, S. (1995): *Minimax Theorems and their Proofs, in Minimax and Applications.* (eds: Du, D. Z. and Pardalos, P. M.). Dordrecht: Kluwer Academic Publishers, 1–24.

Sion, M. (1958): On General Minimax Theorems*.* *Pacific Journal of Mathematics,* 8: 171–176.