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On the solution of a class of partial differential equations

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Abstract. In the paper we study the solution and smoothness of the solution of one class of partial differential equations of higher order in bounded domain $G \subset \mathbb{R}^n$ satisfying the flexible λ-horn condition.

Keywords: flexible λ-horn, generalized derivatives,generalized solution, smoothness of solution.

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1 Introduction

In the paper [7], Besov generalized spaces of the form

$$\bigcap_{i=0}^{n} \mathcal{L}^{<l^i>}_{p_i,\theta_i}(G), \tag{1.1}$$

$1 \le p^i \le \infty$, $1 \le \theta^i \le \infty$, $(i = 0,1,\dots,n)$, $l^i = (l^i_1,\dots,l^i_n)$, $l^0_j \ge 0$, $l^i_i > 0$, $l^i_j \ge 0$, $(j \ne i = 1,2,\dots,n)$, are introduced and studied. In the paper [13], the generalized spaces of Besov–Morrey type

$$\bigcap_{i=0}^{n} \mathcal{L}^{<l^i>}_{p_i,\theta_i,a,\varkappa,\tau}(G,\lambda), \tag{1.2}$$

with the finite norm

$$\|f\|_{\bigcap_{i=0}^{n} \mathcal{L}^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau}(G,\lambda)} = \sum_{i=0}^{n} \|f\|_{\mathcal{L}^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau}(G,\lambda)}, \tag{1.3}$$

$$\|f\|_{\mathcal{L}^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau}(G,\lambda)} = \left\{ \int_0^{h_0} \left[\frac{\left\| \Delta^{m^i}\left(h^{\lambda};G,\lambda\right) D^{k^i} f \right\|_{p^i,a,\varkappa,\tau}}{h^{(\lambda,l^i-k^i)}} \right]^{\theta^i} \frac{dh}{h} \right\}^{\frac{1}{\theta^i}}, \tag{1.4}$$

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$$\|f\|_{p^i,a,\varkappa,\tau;G} = \|f\|_{\mathcal{L}_{p^i,a,\varkappa,\tau}(G)} = \sup_{x\in G}\left\{\int_0^\infty \left[[t]_1^{-\frac{(\varkappa,a)}{p^i}} \|f\|_{p^i,G_{t^\varkappa}(x)}\right]^\tau \frac{dt}{t}\right\}^{\frac{1}{\tau}}, \tag{1.5}$$

where $1 \le p^i < \infty$, $1 \le \theta^i \le \infty$, $m^i \in \mathbb{N}^n$, $k^i \in \mathbb{N}_0^n$, $a \in [0,1]$, $\varkappa \in (0,\infty)^n$, $\tau \in [1,\infty]$, $l^i = (l_1^i,\dots,l_n^i)$, $l_j^0 \ge 0$, $l_i^i > 0$, $l_j^i \ge 0$, $(j \ne i = 1,2,\dots,n)$ are introduced, differential and difference-differential properties of functions from there spaces, determined in n-dimensional domains and satisfying the flexible horn condition are studied. In the case $l^0 = (0,\dots,0)$, $l^i = (0,\dots,l_i,\dots,0)$, $p^i = p$, $\theta^i = \theta$ space (1.1) coincides with the space $B^l_{p,\theta}(G)$ studied in [2], the space (1.2) coincides with the space $B^l_{p,\theta,a,\varkappa,\tau}(G)$ studied in [9], while in the case $a = 0$, $\tau = \infty$ it coincides with space (1.1). Note that consideration of such a space enables to study higher order differential equations of general form. In other words, the obtained imbedding theorems in the form of Sobolev type inequality in spaces (1.1) and (1.2) enable to estimate higher order generalized derivatives than in the case of spaces $B^l_{p,\theta}(G)$ and $B^l_{p,\theta,a,\varkappa,\tau}(G)$.

Example 1.1. Let us consider an equation of the form

$$u^{(5)}_{x^2y^3} + u^{(4)}_{x^2y^2} + u^{(3)}_{x^2y^1} + u^{(2)}_{xy} + u^{(1)}_{x} + u = f(x), \tag{1.6}$$

in our case the solution of this equation is sought in the space $L_2^{(0,0)} \bigcap L_2^{(3,1)} \bigcap L_2^{(1,4)}$. One can look for the solution of equation (1.6) in the space $W_2^{(6,3)}$ $(B_{2,2}^{(6,3)})$, but then this solution will require additional derivatives, in other words, in our case the solution belongs to a wider class.

In this paper we study the existence, uniqueness and smoothness of one class of higher order partial differential equations. Earlier, a problem of smoothness of another kind equations was studied in [1,3–6,8,10–12].

Note that in this paper, as in the papers [9,12], unlike the previous papers for $|\alpha| = l^i$, $(i = 1,2,\dots,n)$ f_α belongs to a wider class. Furthermore, as in the papers [5,6,8,10–12,14,15] here the coefficients do not require smoothness.

2 Main results

At first we give two theorems proved in the paper [13].

Theorem 2.1 ([13])**.** *Let the open set $G \subset \mathbb{R}^n$ satisfy the flexible λ-horn condition [2], $\lambda = (\lambda_1,\dots,\lambda_n)$, $\lambda_j > 0$, $(j = 1,2,\dots,n)$, $1 \le p^i \le p \le \infty$, $1 \le \theta^i \le \infty$, $(i = 0,1,2,\dots,n)$; $\nu = (\nu_1,\nu_2,\dots,\nu_n)$, $\nu_j \ge 0$ be entire $(j = 1,2,\dots,n)$;*

1) $\nu_j \ge l_j^0$ $(j = 1,2,\dots,n)$;

2) $\nu_j \ge l_j^i$ $(j \ne i, j = 1,2,\dots,n)$, $\quad \nu_i < l_i^i$ $(j = i,\ i = 1,2,\dots,n)$;

$1 \le \tau_1 \le \tau_2 \le \infty$; $\overline{\varkappa} = c\varkappa$, $\frac{1}{c} = \max_{j=1,\dots,n} \frac{\varkappa_j}{\lambda_j}$; *$f \in \bigcap_{i=0}^n \mathcal{L}^{<l^i>}_{p^i,\theta^i a,\varkappa,\tau}(G,\lambda)$ and*

$$\mu^i = \sum_{j=1}^n \left[l_j^i\lambda_j - \nu_j\lambda_j - (\lambda_j - \varkappa_j a_j)\left(\frac{1}{p^i} - \frac{1}{p}\right)\right] > 0 \qquad (i = 1,\dots,n).$$

Then $D^\nu : \bigcap_{i=0}^n \mathcal{L}^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau_1}(G,\lambda) \hookrightarrow L_{p,b,\varkappa,\tau_2}(G)$. Precisely, for $f \in \bigcap_{i=0}^n L^{<l^i>}_{p^i,\theta^i a,\varkappa,\tau_1}(G,\lambda)$ in the domain G there exists the generalized derivative $D^\nu f$, for which the the following inequalities are valid:

$$\|D^{\nu}f\|_{p,G} \leq C_1 \sum_{i=0}^{n} T^{\mu^i} \|f\|_{L^{<l^i>}_{p^i,\theta^i a,\varkappa,\tau_1}(G,\lambda)}, \tag{2.1}$$

and

$$\|D^{\nu}f\|_{p,b,\varkappa,\tau_2;G} \leq C_2 \|f\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\theta^i a,\varkappa,\tau_1}(G,\lambda)} \qquad (p^i \leq p < \infty). \tag{2.2}$$

In particular, $\mu^{i,0} = \sum_{j=1}^{n}\left[l^i_j\lambda_j - \nu_j\lambda_j - \left(\lambda_j - \varkappa_j a_j\right)\frac{1}{p^i}\right] > 0$, $(i = 1,\ldots,n)$, *then* $D^{\nu}f$ *is continuous on* G *and*

$$\sup_{x\in G}|D^{\nu}f(x)| \leq C_1 \sum_{i=0}^{n} T^{\mu^{i,0}} \|f\|_{L^{<l^i>}_{p^i,\theta^i a,\varkappa,\tau_1}(G,\lambda)} \tag{2.3}$$

where T *is an arbitrary number from* $(0, \min(1, T_0)]$, $b = (b_1, b_2, \ldots, b_n)$, b_j *any numbers and satisfy the conditions*

$$\begin{aligned} &0 \leq b_j \leq 1, && \text{for } \mu^{i,0} > 0,\\ &0 \leq b_j < 1, && \text{for } \mu^{i,0} = 0,\\ &0 \leq b_j < a_j + \frac{\mu^i p\left(1-a_j\right)}{n\left(\lambda_j - \varkappa_j a_j\right)}, && \text{for } \mu^{i,0} < 0, \end{aligned} \tag{2.4}$$

but with $\varkappa$ *replaced by* $\overline{\varkappa}$, C_1 *and* C_2 *are constants independent of* f, *moreover* C_1 *is independent also of* T.

Let γ be an n-dimensional vector.

Theorem 2.2 ([13]). *Let the conditions of Theorem 2.1 be fulfilled. Then for* $\mu^i > 0$ $(i = 1, 2, \ldots, n)$ *the derivative* $D^{\nu}f$ *satisfies on* G *the Hölder condition in the metrics* L_p *with the exponent* σ, *more exactly,*

$$\|\Delta(\gamma, G)\, D^{\nu}f\|_{p,G} \leq C \|f\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau}(G,\lambda)} |\gamma|^{\sigma}, \tag{2.5}$$

here σ *is any number satisfying the inequalities:*

$$\begin{aligned} &0 \leq \sigma \leq 1, && \text{for } \frac{\mu_0}{\lambda_0} > 1,\\ &0 \leq \sigma < 1, && \text{for } \frac{\mu_0}{\lambda_0} = 1,\\ &0 \leq \sigma \leq \frac{\mu_0}{\lambda_0}, && \text{for } \frac{\mu_0}{\lambda_0} < 1, \end{aligned} \tag{2.6}$$

where $\mu_0 = \min \mu^i$ $(i = 1, 2, \ldots, n)$, $\lambda_0 = \max \lambda_j$ $(j = 1, 2, \ldots, n)$, *while* C *is a constant independent of* f *and* $|\gamma|$.

In particular, if $\mu^{i,0} > 0$ $(i = 1, 2, \ldots, n)$, *then*

$$\sup_{x\in G}|\Delta(\gamma, G)\, D^{\nu}f(x)| \leq C \|f\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\theta^i,a,\varkappa,\tau}(G,\lambda)} |\gamma|^{\sigma^0}, \tag{2.7}$$

σ^0 *satisfies the same conditions that* σ *satisfies, but with* μ^i *replaced by* $\mu^{i,0}$.

Let us consider the Dirichlet problem for a higher order partial differential equation, i.e. consider a problem of the form

$$\sum_{\substack{|\alpha|\leq|l^i|,\\ |\beta|\leq|l^i|\\ i=1,2,\ldots,n}} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta}u(x)) = \sum_{\substack{|\alpha|\leq|l^i|,\\ i=1,2,\ldots,n}} D^{\alpha}f_{\alpha}(x), \tag{2.8}$$

$$D^{\nu}u|_{\partial G} = \varphi_{\nu}|_{\partial G}, \tag{2.9}$$

where it is assumed that G is a bounded n-dimensional domain with piecewise-smooth boundary ∂G, $\nu = (\nu_1, \ldots, \nu_n)$, where $|\nu| < |l^i|$, $i = 1, 2, \ldots, n$ while $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\beta = (\beta_1, \ldots, \beta_n)$; $\alpha_j, \beta_j \geq 0$ are integer $(j = \overline{1, n})$. We assume that the coefficients $a_{\alpha\beta}(x)$ are bounded measurable functions in the domain G, $a_{\alpha\beta}(x) \equiv a_{\beta\alpha}(x)$ and for $\xi \in R^n$

$$\sum_{\substack{|\alpha| \leq |l^i|,\\ |\beta| \leq |l^i|\\ i=1,2,\ldots,n}} (-1)^{|\alpha|} a_{\alpha,\beta}(x)\xi_\alpha \xi_\beta \geq C_0 \sum_{\substack{|\alpha| \leq |l^i|\\ i=1,2,\ldots,n}} |\xi_\alpha|^2, \qquad C_0 = \text{const} > 0, \tag{2.10}$$

we also assume that $f_\alpha \in L_2(G)$, for all $\alpha = (\alpha_1, \ldots, \alpha_n)$.

The function $u \in \bigcap_{i=0}^n L_2^{<l^i>}(G)$ is called a generalized solution of problem (2.8)–(2.9) in the domain G, if $D^\nu u - \varphi_\nu \in \bigcap_{i=0}^n \mathring{L}_2^{<m^i>}(G)$, when $|m^i + \nu| < |l^i|, (i = 1, 2, \ldots, n)$ and for any function $\vartheta(x) \in \bigcap_{i=0}^n \mathring{L}_2^{<l^i>}(G)$ the following integral identity is valid:

$$\sum_{\substack{|\alpha| \leq |l^i|,\\ |\beta| \leq |l^i|,\\ i=1,2,\ldots,n}} \int_G a_{\alpha\beta}(x) D^\beta u(x) D^\alpha \vartheta(x) dx = \sum_{|\alpha| \leq |l^i|} (-1)^{|\alpha|} \int_G f_\alpha D^\alpha \vartheta(x) dx. \tag{2.11}$$

The space $\bigcap_{i=0}^n \mathring{L}_2^{<l^i>}(G)$ is completion of $C_0^\infty(G)$ in the metric $\bigcap_{i=0}^n L_2^{<l^i>}(G)$. Prove that there exists a unique generalized solution of problems (2.8) and (2.9). Consider for $\phi, \psi \in \bigcap_{i=0}^n L_2^{<l^i>}(G)$ the bilinear functional

$$\begin{aligned} F(\phi, \psi) = &\sum_{\substack{|\alpha| \leq |l^i|,\\ |\beta| \leq |l^i|,\\ i=1,2,\ldots,n}} (-1)^{|\alpha|} \int_G a_{\alpha\beta}(x) D^\beta \phi(x) D^\alpha \psi(x) dx \\ &- \sum_{|\alpha| \leq |l^i|} (-1)^{|\alpha|} \int_G f_\alpha D^\alpha \psi(x) dx - \sum_{|\beta| \leq |l^i|} (-1)^{|\beta|} \int_G f_\beta D^\beta \phi(x) dx \\ = &\, I(\phi, \psi) - (f_\alpha, \psi) - (f_\beta, \phi), \end{aligned}$$

where

$$I(\phi, \phi) = I(\phi) \geq \|\phi\|^2_{\bigcap_{i=0}^n L_2^{<l^i>}(G)}.$$

The variational problem is stated as follows: it is required to find the function $\phi \in \bigcap_{i=0}^n L_2^{<l^i>}(G)$, $l = (l_1, \ldots, l_n)$, $l_j > 0$, $j = 1, \ldots, n$ are entire, that gives the least value to the integral $F(\phi)$ and is unique. Equation (2.8) is the Euler equation for the considered variational problem.

$$\begin{aligned} F(\phi, \phi) = F(\phi) &= I(\phi) - (f_\alpha + f_\beta, \phi) \\ &\geq I(\phi) - \sum_{|\alpha| \leq |l^i|} (-1)^{|\alpha|} \left(\int_G |f_\alpha|^2 dx + \int_G |D^\alpha \phi(x)|^2 dx \right) \\ &\quad - \sum_{|\alpha| \leq |l^i|} (-1)^{|\beta|} \left(\int_G |f_\beta|^2 dx + \int_G |D^\beta \phi(x)|^2 dx \right) \\ &\geq \|\phi\|_{\bigcap_{i=0}^n L_2^{<l^i>}(G)} - 2\|\phi\|_{\bigcap_{i=0}^n L_2^{<l^i>}(G)} - d = -d_1. \end{aligned}$$

Then means that $F(\phi)$ is lower bounded on $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$. Show that there exists $\phi_0 \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ such that $F(\phi_0) = \min_{\phi \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)} F(\phi)$. Indeed, let $k = \min_{\Phi \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)} F(\phi)$. Fix some sequence $\phi_m \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ such that $\lim_{m\to\infty} F(\phi_m) = k$ and let $\varepsilon > 0$. Choose m_ε so that for $m \geq m_\varepsilon$ and $\mu = 0, 1, 2, \dots$ would hold $F(\phi_{m+\mu}) < k + \varepsilon$. Then noting $\frac{1}{2}(\phi_{m+\mu} + \phi_m) \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$, we have $F(\frac{\phi_{m+\mu}+\phi_m}{2}) \geq k$. Further, by direct calculations we show that $I(\frac{\phi_{m+\mu}-\phi_m}{2}) < 4\varepsilon$. From the ellipticity condition (2.10) it follows that $\|\phi_{m+\mu} - \phi_m\|_{\bigcap_{i=0}^{n} L_2^{<l^i>}(G)} < 2\sqrt{\frac{\varepsilon}{c_0}}$. It means that the sequence $\{\phi_m\}$ in fundamental in the space $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$. Therefore, because of completeness in the space $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ there exists the function $\phi_0 \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ such that $\lim_{m\to\infty} \|\phi_m - \phi_0\|_{\bigcap_{i=0}^{n} L_2^{<l^i>}(G)} = 0$. By [7, Theorem 1], it is proved that $|F(\phi_m) - F(\phi_0)| \leq C\|\phi_m - \phi_0\|_{\bigcap_{i=0}^{n} L_2^{<l^i>}(G)}$, and hence it follows that $k = \lim_{m\to\infty} F(\phi_m) = F(\phi_0)$. Show that the function delivering minimum to the functions $F(\phi)$ in the space $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ is unique and satisfies

$$D^\nu u|_{\partial G} = \varphi_\nu|_{\partial G}.$$

Indeed, $\phi \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ and $F(\phi_0) = k$, we have:

$$0 \leq I\left(\frac{\phi - \phi_0}{2}\right) = \frac{1}{2}F(\phi) + \frac{1}{2}F(\phi_0) - F\left(\frac{\phi + \phi_0}{2}\right) \leq \frac{k}{2} + \frac{k}{2} - k = 0,$$

$$I(\phi - \phi_0) = 0,$$

then again by the ellipticity condition (2.10), $\|\phi_m - \phi_0\|_{\bigcap_{i=0}^{n} L_2^{<l^i>}(G)} \overset{m\to 0}{\longrightarrow} 0$, hence it follows that ϕ coincides with ϕ_0 as an element of $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$. By Theorem 1 in [7] we have:

$$\|D^\nu(\phi_m - \phi_0)|_{\partial G}\|_{L_2(\partial G)} \leq \|\phi_m - \phi_0\|_{\bigcap_{i=0}^{n} L_2^{<l^i>}(G)} \to 0,$$

$m \to \infty$, $|\nu| < |l^i|$ $(i = 1, 2, \dots, n)$, as

$$\|D^\nu \phi_m|_{\partial G} - \phi_\nu|_{\partial G}\|_{L_2(G)} \to 0,$$

$m \to \infty$, $|\nu| \leq |l^i|$ $(i = 1, 2, \dots, n)$, therefore

$$\|D^\nu \phi_0|_{\partial G} - \phi_\nu|_{\partial G}\|_{L_2(G)} = 0,$$

$|\nu| \leq |l^i| (i = 1, 2, \dots, n)$. Taking into account the conditions $\left(\frac{d}{d\lambda}F(\phi_0 + \lambda\psi)\right)_{\lambda=0} = 0$, show that the function $\phi_0 \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$, minimizing the integral $F(\phi)$, satisfies the following equation:

$$I(\phi_0, \psi) - (f_\alpha, \psi) = 0. \tag{2.12}$$

Now prove that the function $\phi_0 \in \bigcap_{i=0}^{n} L_2^{<l^i>}(G)$, minimizing the integral $F(\phi)$ is the solution (generalized) of problem (2.8)–(2.9). For that we suppose that $a_{\alpha,\beta}(x)$ are bounded in absolute value in the domain G together with its derivatives and the function f_α has derivatives belonging to the space $L_2(G)$. Denote by $\Theta(t)$ some monotonically decreasing function on the interval $\frac{1}{2} \leq t \leq 1$, and possessing the following properties:

$$\Theta\left(\frac{1}{2} + 0\right) = 1; \qquad \Theta(1 - 0) = -1;$$

$$\Theta^{(s)}\left(\frac{1}{2}+0\right)=\Theta^{(s)}(1-0)=0 \quad \text{for any } s>0.$$

The function

$$\begin{cases}\Theta'(t), & \frac{1}{2}\le t\le 1,\\ 0, & -\infty<t\le\frac{1}{2},\ 1\le t<\infty\end{cases}$$

is infinitely differentiable and finite over the whole axis. Let $\eta>0$ and $G_\eta=\{y:\rho(y,R^n\backslash G)>\eta\}$, x be an arbitrary point of domain G and $r=\rho(x,y)$. Following S. L. Sobolev [16] we introduce the function

$$\psi(x)=\gamma\left(\frac{r}{h_1}\right)-\gamma\left(\frac{r}{h_2}\right).$$

For $0<h_1<h_2<\eta$. Obviously, ψ is infinitely differentiable finite function with support of annular domain $\frac{h_1}{2}<r<\frac{h_2}{2}$. Therefore, $\psi\in\bigcap_{i=0}^{n}L_2^{<l^i>}(G)$, and $D^{(s)}\psi|_{\partial G}=0$ for any $s>0$. Then from the expression (2.12) by definition of the generalized derivative it follows that,

$$\int_G\omega\left(\frac{r}{h_1}\right)\phi(x)dx=\int_G\omega\left(\frac{r}{h_2}\right)\phi(x)dx,$$

where

$$\omega\left(\frac{r}{h_i}\right)=D^\beta\left(a_{\alpha,\beta}(x)D^\alpha\gamma\left(\frac{r}{h_i}\right)\right)-(-1)^{|\alpha|}f_\alpha D^\alpha\gamma\left(\frac{r}{h_i}\right),\qquad (i=1,2).$$

The function $\omega(\frac{r}{h_i})$ possesses all the properties of a kernel. Then for the function ϕ_0 (the solution of the variational problem) we can construct the Sobolev averaging $\phi_{0,h_i}(x)$, $i=1,2$ over the ball h_i, $(i=1,2)$ centered at the point x:

$$\phi_{0,h_i}(x)=\frac{1}{\sigma_n h_i^n}\int_{\mathbb{R}^n}\omega\left(\frac{|z-x|}{h_i}\right)\phi_0(z)dz,\qquad i=1,2.$$

Then we can rewrite equality (2.12) in the form $\phi_{0,h_1}(x)=\phi_{0,h_2}(x)$. Consequently, for $h<\eta$

$$\phi_{0,h}(x)=\phi_0(x).$$

As the average function $\phi_{0,h}(x)$ is continuous and has any order continuous derivatives, then $\phi_0(x)$ also possesses these properties. Making integration in parts in the equality $I(\phi_{0,h},\psi)-(f_\alpha,\psi)=0$, in the limiting case

$$\int_G\sum_{\substack{|\alpha|\le|l^i|,\\|\beta|\le|l^i|\\ i=1,2,\dots,n}}\psi(x)\left[D^\alpha\left(a_{\alpha,\beta}(x)D^\beta\phi_0(x)\right)-D^\alpha f_\alpha(x)\right]dx=0.$$

Hence, by arbitrariness of the function $\psi(x)$ it follows

$$\sum_{\substack{|\alpha|\le|l^i|,\\|\beta|\le|l^i|\\ i=1,2,\dots,n}}D^\alpha\left(a_{\alpha,\beta}(x)D^\beta\phi_0(x)\right)=\sum_{\substack{|\alpha|\le|l^i|,\\ i=1,2,\dots,n}}D^\alpha f_\alpha(x).$$

Thus the solution of the variational problem from the class $\bigcap_{i=0}^{n}L_2^{<l^i>}(G)$ is also the solution of problem (2.8)–(2.9) and this solution is unique.

Assume also that $f_\alpha \in L_{2,\alpha,\varkappa}(G)$ for $|\alpha| = |l^i|$ $(i = 1, 2, \ldots, n)$, $0 < d < 1$, $d = \text{const}$, $b \leq d$, $x_0 \in G_d$; G_d is a subdomain of the domain G such that

$$G_d = \{x'' : |x'' - x'_j| > d^{\lambda_j},\ x' \in \partial G,\ j = 1, 2, \ldots, n\},$$

and

$$\Pi_b(x_0) = \{x : |x_j - x_{j,0}| > b^{\lambda_j},\ j = 1, 2, \ldots, n\}.$$

Theorem 2.3. *If* $\frac{|\lambda|}{2} \leq (\lambda, l^i)$ $(i = 1, 2, \ldots, n)$, *then any generalized solution of equation* (2.8) *from* $\bigcap_{i=0}^{n} L_2^{<l^i>}(G)$ *is continuous in* G *and satisfies the Hölder condition in any subdomain compactly imbedded into* G.

Proof. Let at first all $a_{\alpha\beta}(x) \equiv 0$, except the ones for which $|\alpha| = |\beta| = |l^i|$ $(i = 1, 2, \ldots, n)$ and the left hand side equals zero. For any $\Theta(x) \in \Pi_b(x_0)$, such that $\Theta \equiv 1$ in the vicinity of $\partial\Pi_b(x_0)$ any polynomial

$$P(x) = \sum_{\substack{|\alpha|=|l^i|,\\ i=1,2,\ldots,n}} C_\alpha x^\alpha$$

and for arbitrary solution $u(x)$ from the variational principle it follows that

$$\begin{aligned}
&\int_{\Pi_b(x_0)} \sum_{\substack{|\alpha|=|\beta|=|l^i|,\\ i=1,2,\ldots,n}} (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\beta \left(\Theta(x)(u(x) - P(x))\right) D^\alpha \left(\Theta(x)(u(x) - P(x))\right) dx \\
&\quad \geq \int_{\Pi_b(x_0)} \sum_{\substack{|\alpha|=|\beta|=|l^i|,\\ i=1,2,\ldots,n}} (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\beta \left((u(x) - P(x))\right) D^\alpha \left((u(x) - P(x))\right) dx \\
&\quad = A(u(x) - P(x), \Pi_b(x_0)),
\end{aligned} \tag{2.13}$$

moreover, $\theta(x) = 1 - \prod_{i=1}^{n} \omega_j\left(\frac{x - x_0}{b^{\lambda_j}}\right), x \in G$, where $\omega_j(t) \in C^\infty(\mathbb{R})$ is such that $\omega_j(t) \equiv 1$ for $|t| < 2^{-\lambda_j}, \omega_j(t) \equiv 0$ for $|t| > 1$, $0 \leq \omega_j(t) \leq 1$. It is seen that $\theta(x) \equiv 0$ in $\Pi_{\frac{b}{2}}(x_0)$, $\theta(x) \equiv 1$ in the vicinity of $\partial\Pi_b(x_0)$, and the coefficients $P(x)$ are chosen so that

$$\int_{(\Pi_b(x_0))\setminus\left(\Pi_{\frac{b}{2}}(x_0)\right)} (u - p(x))\, x^\alpha dx = 0.$$

By means of (2.1) and (2.2) we get

$$\begin{aligned}
&A\left(u(x) - P(x), \Pi_b(x_0)\right) \\
&\quad \leq A\left(u(x) - P(x); \Pi_b(x_0) \setminus \left(\Pi_{\frac{b}{2}}(x_0)\right)\right) \\
&\qquad + \int_{(\Pi_b(x_0))\setminus\left(\Pi_{\frac{b}{2}}(x_0)\right)} \sum_{|\alpha|<|l^i|} b^{2|\alpha|-2|l^i|} D^\alpha \left(u(x) - P(x)\right)^2 dx \\
&\quad \leq qA\left(u(x) - P(x),\ \Pi_b(x_0) \setminus \left(\Pi_{\frac{b}{2}}(x_0)\right)\right).
\end{aligned} \tag{2.14}$$

As $A\left(u(x) - p(x), G\right) = A\left(u(x), G\right)$, then in view of (2.14) by induction we get

$$A\left(u(x), \Pi_{\frac{b}{2^k}}(x_0)\right) \leq \left(1 - \frac{1}{q}\right)^k A\left(u(x), \Pi_b(x_0)\right).$$

Let $0 < \delta < \frac{b}{2^k}$, $\zeta = 1 - \frac{1}{q}$, then

$$A\left(u(x), \Pi_\delta\left(x_0\right)\right) \leq \left(\frac{\delta}{b}\right)^{\left|\frac{\ln\zeta}{\ln 2}\right| - \left|\frac{\ln\zeta}{\ln b/\delta}\right|} A\left(u(x), G\right) = \left(\frac{\delta}{b}\right)^{\xi-\sigma} A\left(u(x), G\right) \tag{2.15}$$

for any $\delta \leq b$, and consequently,

$$\int_0^1 \left[\eta^{-\xi} \int_{\Pi_\eta(x_0)} u^2 dx\right]^{\frac{1}{2}} \frac{d\eta}{\eta} \leq C \int_0^1 \frac{db}{b^{1-\frac{1}{2}\sigma}} < \infty.$$

From $0 < \xi = (\varkappa, a) < 1$, it follows that $u(x) \in L_{2,a,\varkappa,1}\left(G_d\right) \subset L_{2,a,\varkappa,\tau}\left(G_d\right)$ and from the condition of Theorem 2.3 it follows that $\mu_i > 0$, $\mu_{i,0} > 0$ $(i = 1, 2, \ldots, n)$, i.e. the conditions of Theorems 2.1 and 2.2 are fulfilled. Thus, by Theorem 2.1, $u(x)$ is continuous, by Theorem 2.2, $u(x)$ satisfies the Hölder condition on G_d.

Let $a_{\alpha\beta} = 0$, except $a_{\alpha\beta}$, for which $|\alpha| = |\beta| = |l^i|$, and the right hand sides of equation (2.8) be nonzero. Let u_{b,x_0} be a generalized solution of this equation in $\Pi_b\left(x_0\right)$ from $\bigcap_{i=0}^n \mathring{L}_2^{<l^i>}\left(\Pi_b\left(x_0\right)\right)$. Existence of such a solution is proved by the functional method. Put in (2.11) $\vartheta \equiv u_{b,x_0}$ then from (2.10) we get

$$\int_{\Pi_b(x_0)} \sum_{\substack{|\alpha|=|l^i|,\\ i=1,2,\ldots,n}} \left(D^\alpha u_{b,x_0}\right)^2 dx \leq \sum_{\substack{|\alpha|=|l^i|,\\ i=1,2,\ldots,n}} b^{2|l^i|-2|\alpha|} \int_{\Pi_b(x_0)} f_\alpha^2 dx + \sum_{\substack{|\alpha|=|l^i|,\\ i=1,2,\ldots,n}} \int_{\Pi_b(x_0)} f_\alpha^2 dx \leq C_1 b^r,$$

if

$$r = \min_{\substack{|\alpha|=|l^i|,\\ i=1,2,\ldots,n}} \left\{2|l^i| - 2|\alpha|, (\varkappa, a)\right\} > 0,$$

here C_1 and r are independent of u and x_0.

Hence it follows that

$$A\left(u_{b,x_0}, \Pi_b\left(x_0\right)\right) \leq C_1 b^r. \tag{2.16}$$

As $\overline{u} = u - u_{b,x_0}$ is the solution of homogeneous equation (2.8) then the following inequality is valid for it:

$$A\left(\overline{u}, \Pi_b\left(x_0\right)\right) \leq C_2 \left(\frac{\delta}{b}\right)^{\xi-\sigma} A\left(\overline{u}, G\right). \tag{2.17}$$

From inequalities (2.16) and (2.17) we get

$$\begin{aligned} A\left(u, \Pi_b\left(x_0\right)\right) &\leq C_3 A\left(\overline{u}, \Pi_b\left(x_0\right)\right) + C_3 A\left(u_{b,x_0}, \Pi_b\left(x_0\right)\right) \\ &\leq C_4 \left(\frac{\delta}{b}\right)^{\xi-\sigma} A\left(\bar{u}, G\right) + C_5 b^r \leq C_6 \left(\frac{\delta}{b}\right)^{\xi-\sigma}, \end{aligned}$$

and hence we get

$$\int_0^1 \left(\eta^{-\xi} \int_{\Pi_b(x_0)} u^2 dx\right)^{\frac{1}{2}} \frac{d\eta}{\eta} \leq C \int_0^1 \frac{db}{b^{1-\frac{1}{2}\sigma}} < \infty.$$

Here using Theorems 2.1 and 2.2 we get that $u(x)$ is continuous and satisfies the Hölder condition on G_d.

Finally, we consider equations (2.8) where there are nonzero coefficients at minor derivatives of the solution. Then we take these terms to the right hand side of the equation and in this case we get the desired result. □

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