A generalized FKG-inequality for compositions

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Abstract. We prove a Fortuin-Kasteleyn-Ginibre-type inequality for the lattice of compositions of the integer n with at most r parts. As an immediate application we get a wide generalization of the classical Alexandrov-Fenchel inequality for mixed volumes and of Teissier's inequality for mixed covolumes.

1. Introduction

1.1. Consider a finite partially ordered set $(X, \preceq)$ and two non-decreasing (non-negative) functions, $f, g : X \to \mathbb{R}_{\geq 0}$. (Namely, for any $x, y \in X$, if $x \preceq y$ then one has $f(x) \leq f(y)$ and $g(x) \leq g(y)$.) The product function $f \cdot g : X \to \mathbb{R}_{\geq 0}$ is also non-decreasing. Take the arithmetic average

$$Av_X(f) := (\sum_{x \in X} f(x))/|X|.$$

A natural question is whether $Av_X(f) \cdot Av_X(g)$ can be compared with $Av_X(f \cdot g)$.

Example 1.1. Suppose that X is totally ordered. Then the non-decreasing functions are just the non-decreasing sequences of real numbers, $0 \leq a_1 \leq \cdots \leq a_n$ and $0 \leq b_1 \leq \cdots \leq b_n$. In this case the comparison of the averages is realized by the classical Chebyshev sum inequality: $(\sum_i a_i)(\sum_j b_j) \leq n(\sum_i a_i b_i)$.

On the other hand, if the order on X is not "strong enough" then the inequality utterly fails. Hence, the more precise question is:

(1) Which posets does $Av_X(f) \cdot Av_X(g) \leq Av_X(f \cdot g)$ hold for?

If $(X, \preceq)$ admits an action of some group G, then one can consider the "equivariant" version of this question by taking G-invariant functions f and g.

The fundamental Fortuin-Kasteleyn-Ginibre (FKG) inequality settles the question for a large class of lattices:

Theorem 1.2. [FKG-1971], *see also* [Bollobás, pg. 147, Theorem 5].
Let X be a finite distributive lattice. Consider a "measure", $X \stackrel{\mu}{\to} \mathbb{R}_{\geq 0}$, which is log-supermodular, i.e. $\mu(x \wedge y)\mu(x \vee y) \geq \mu(x)\mu(y)$ for any $x, y \in X$. Then $\Big(\sum_{x \in X} f(x)g(x)\mu(x)\Big) \cdot \sum_{x \in X} \mu(x) \geq \Big(\sum_{x \in X} f(x)\mu(x)\Big)\Big(\sum_{x \in X} g(x)\mu(x)\Big)$.

(The inequality of equation (1) is obtained for the constant measure, $\mu(x) = 1$, which is trivially supermodular.)

One of the interpretation of the FKG inequality is: "in many systems the increasing events are positively correlated" (while an increasing event and a decreasing event are negatively correlated). Hence, the applications of this inequality go far beyond the combinatorics and include e.g. statistical mechanics and probability.

1.2. The condition "X is a distributive lattice" in the above theorem is rather restrictive. Many of the natural posets appearing in arithmetics/algebra/geometry are not of this type. In the current work we establish the inequality of equation (1) for a particular poset $\mathcal{K}_{n,r}$ of ordered compositions, cf. Theorem 3.1. This poset appears frequently in the context of the Young diagrams (representation theory), complete intersections (algebraic geometry), mixed (co)volumes/multiplicities (integral geometry and commutative algebra).

It is known that lattices for which the FKG inequality holds (for any log-supermodular measure) are necessarily distributive. Theorem 3.1 gives an example of non-distributive lattice for which FKG holds for a particular measure.

For related inequalities see also [Cu.Gr.Sk].

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1.3. Sometimes one considers the geometric average, $Av^G_X(f) := (\prod_{x\in X} f(x))^{\frac{1}{|X|}}$ as well. Then it is natural to compare $(Av^G_X(f))^{Av_X(g)}$ to $Av^G_X(f^g)$. For example, given the real numbers $1 \le a_1 \le \dots \le a_n$ and $0 \le b_1 \le \dots \le b_n$, one has: $(\sqrt[n]{\prod a_i})^{\frac{\sum b_i}{n}} \le \sqrt[n]{\prod a_i^{b_i}}$.

One passes between Av_X and Av^G_X by using ln and exp. Thus a simple reformulation of Theorem 3.1 provides automatically the comparison of the geometric averages as well, cf. Corollary 3.4.

1.4. As an immediate application in §4.3 we prove a highly non-trivial convexity property for the mixed volumes of convex bodies in $\mathbb{R}^N$ and the parallel statement for the mixed volumes of Newton diagrams (i.e. co-volumes of the Newton polyhedra). These generalize the classical Alexandrov-Fenchel inequality and its Teissier's counterpart. In particular, it gives a partial answer to a question of [Gromov90].

Furthermore, in [Kerner-Némethi2014] we heavily use the inequality of averages to establish a bound on some topological invariants of singularities. In fact this was our initial motivation for the inequality of averages.

2. The poset $\mathcal{K}_{n,r}$

2.1. The set of compositions. Denote by $\mathcal{K}_{n,r}$ the set of the (ordered) compositions of the integer n into r summands,

$$\mathcal{K}_{n,r} := \{\underline{k} = (k_1,\dots,k_r) \,:\, k_i \ge 0 \text{ for all } i, \text{ and } \sum_i k_i = n\}. \tag{2}$$

This $\mathcal{K}_{n,r}$ can be thought of as the lattice points $\underline{k}$ of the simplex $\{\sum_i x_i = n\} \cap \mathbb{R}^r_{\ge 0}$. Its cardinality is $|\mathcal{K}_{n,r}| = \binom{n+r-1}{n}$.

The permutation group of r elements, Ξ_r, acts on $\mathcal{K}_{n,r}$ by $\sigma(\underline{k}) = (k_{\sigma(1)},\dots,k_{\sigma(r)})$. The quotient $\mathcal{K}_{n,r}/\Xi_r$ is the set of partitions into r summands. (In other words, a partition is an *unordered* composition.) For the general introduction see [Stanley, Chapter 7].

For convenience we put $\mathcal{K}_{n,r} = \varnothing$ when $r \le 0$ or $n < 0$.

2.2. The 'dominance' order on $\mathcal{K}_{n,r}$ can be defined as follows.
- Suppose for $\underline{k} = (k_1,k_2,\dots,k_r) \in \mathcal{K}_{n,r}$ one has $k_1 - 1 \ge k_2 + 1$. Then put $\underline{k} \succeq \underline{k}' := (k_1 - 1, k_2 + 1, k_3, \dots, k_r)$.
- Extend this by transitivity, i.e. if $\underline{k} \succeq \underline{k}'$ and $\underline{k}' \succeq \underline{k}''$ then $\underline{k} \succeq \underline{k}''$.
- Extend this by the action $\Xi_r \circlearrowright \mathcal{K}_{n,r}$, i.e. if $\underline{k} \succeq \underline{k}'$ then $\sigma(\underline{k}) \succeq \sigma(\underline{k}')$ for any $\sigma \in \Xi_r$.

In this way we get a partially ordered set with the maximal elements, $(n,0,\dots,0)$ and its orbit under Ξ_r, and the minimal elements, $(\lfloor\frac{n}{r}\rfloor,\dots,\lfloor\frac{n}{r}\rfloor,\lceil\frac{n}{r}\rceil,\dots,\lceil\frac{n}{r}\rceil)$ and its orbit under Ξ_r. By construction, this partial order is Ξ_r invariant. In particular, any two different elements inside a Ξ_r orbit are incomparable. This order descends to the quotient $\mathcal{K}_{n,r}/\Xi_r$. (Indeed, for any two elements of $\mathcal{K}_{n,r}/\Xi_r$, suppose some of their representatives in $\mathcal{K}_{n,r}$ are comparable, $\underline{k} \preceq \underline{k}'$. Then put $[\underline{k}] \preceq [\underline{k}']$. By the Ξ_r-invariance this assignment is consistent: if some other preimages are comparable then they satisfy the same inequality.) The quotient poset $\mathcal{K}_{n,r}/\Xi_r$ has a unique minimal and a unique maximal element.

Example 2.1. 1. For $\mathcal{K}_{n,2}$ we get $(n,0) \succeq (n-1,1) \succeq \dots \succeq (\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor)$ and $(\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil) \preceq \dots \preceq (0,n)$. For $\mathcal{K}_{n,2}/\Xi_2$: $(n,0) \succeq (n-1,1) \succeq \dots \succeq (\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor)$. In particular, $\mathcal{K}_{n,2}/\Xi_2$ is *totally* ordered.
2. As mentioned above, $\mathcal{K}_{n,r}$ is never totally ordered for $r > 1$: the elements of any Ξ_r-orbit are incomparable. The quotient $\mathcal{K}_{n,r}/\Xi_r$ is not totally ordered for $r \ge 3$ and high enough n. For example, for $(n,r) = (6,3)$, the elements $(4,1,1)$ and $(3,3,0)$ are incomparable.
3. In the particular case $\mathcal{K}_{n,n}/\Xi_n$ coincides with the Young lattice of all the possible partitions of n, see e.g. [Brylawski], [Stanley, Chapter 7].
4. $\mathcal{K}_{n,r}/\Xi_r$ is always a lattice, but it is non-distributive for $n \ge 7$. It contains the non-distributive sublattice:

$$\begin{array}{ccccc} \overset{(*,4,2,1)}{\bullet} & \to & \overset{(*,4,1,1,1)}{\bullet} & & \\ \downarrow & & & \searrow & \\ \underset{(*,3,3,1)}{\bullet} & \to & \underset{(*,3,2,2)}{\bullet} & \to & \underset{(*,3,2,1,1)}{\bullet} \end{array} . \tag{3}$$

Here $*$ denotes the rest of the partition, some fixed tuple whose sum is $(n-7)$ and such that all the entries go in the non-increasing order. For this lattice one has:

$$(*,3,3,1) \wedge \Big((*,4,1,1,1) \vee (*,3,2,2)\Big) = (*,3,3,1), \tag{4}$$
$$\Big((*,3,3,1) \wedge (*,4,1,1,1)\Big) \vee \Big((*,3,3,1) \wedge (*,3,2,2)\Big) = (*,3,2,2).$$

2.3. Suppose that a set of objects is indexed by this set of compositions, $\{A_{\underline{k}}\}_{\underline{k}\in\mathcal{K}_{n,r}}$. We often use the standard set-theoretical inclusion-exclusion formula:

$$\sum_{\underline{k}\in\mathcal{K}_{n,r}} A_{\underline{k}} - \sum_{i=1}^{r} \sum_{\substack{\underline{k}\in\mathcal{K}_{n,r}\\ k_i=0}} A_{\underline{k}} + \sum_{1\le i_1<i_2\le r} \sum_{\substack{\underline{k}\in\mathcal{K}_{n,r}\\ k_{i_1}=k_{i_2}=0}} A_{\underline{k}} - \dots = \sum_{\substack{\underline{k}\in\mathcal{K}_{n,r}\\ k_1,\dots,k_r>0}} A_{\underline{k}}. \tag{5}$$

3. The inequality for averages over $\mathcal{K}_{n,r}$

Theorem 3.1. *Let $f,g:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq 0}$ be non-negative functions. Suppose f is Ξ_r-invariant, i.e. $f(\sigma(x))=f(x)$ for any $\sigma\in\Xi_r$.*
(a) If both functions are non-decreasing then $Av_{\mathcal{K}_{nr}}(f)\cdot Av_{\mathcal{K}_{n,r}}(g)\leq Av_{\mathcal{K}_{n,r}}(fg)$.
(b) If f is non-decreasing, while g is non-increasing then $Av_{\mathcal{K}_{n,r}}(f)\cdot Av_{\mathcal{K}_{n,r}}(g)\geq Av_{\mathcal{K}_{n,r}}(fg)$.
(c) In the statements above, the equality holds if and only if either f is constant, or the symmetrization of g, $\tilde{g}(x):=Av_{\Xi_r(x)}(g)$ is constant on $\mathcal{K}_{n,r}$.

Example 3.2. 1. This inequality was proved in [Kerner-Némethi2013, §1.3] in the particular case of $f(\underline{k}):=\prod\limits_{i=1}^{r}d_i^{k_i}$ and $g(\underline{k}):=\binom{n+r}{k_1+1,\dots,k_r+1}$, in which case it reads as

$$\Big(\sum_{\underline{k}\in\mathcal{K}_{n,r}}\binom{n+r}{k_1+1,\dots,k_r+1}\Big)\Big(\sum_{\underline{k}\in\mathcal{K}_{n,r}}(\prod_{j=1}^{r}d_j^{k_j})\Big)\geq|\mathcal{K}_{n,r}|\sum_{\underline{k}\in\mathcal{K}_{n,r}}\binom{n+r}{k_1+1,\dots,k_r+1}\Big(\prod_{j=1}^{r}d_j^{k_j}\Big). \tag{6}$$

This was the essential ingredient in establishing a bound for some topological invariants in Singularity Theory.

2. If one does not assume that at least one of the functions is Ξ_r-invariant then the inequality does not hold. For example, consider $\mathcal{K}_{2,2}=\{(0,2)\succeq(1,1)\preceq(2,0)\}$. The inequality is obviously violated for

$$f(x)=\begin{cases}1, & x=(0,2)\\ 0, & otherwise\end{cases},\qquad g(x)=\begin{cases}1, & x=(2,0)\\ 0, & otherwise\end{cases}.$$

3. The statement can be reformulated as averaging over ${}^{\mathcal{K}_{n,r}}\!/\!_{\Xi_r}$, with the weight function $\mu([\underline{k}])=|\Xi_r(\underline{k})|$.

Proof. **Step 1.** We start with some simplifying remarks.

We can (and will) assume that g is Ξ_r-invariant. Indeed, define $\tilde{g}:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq 0}$ by the averaging over the orbit, $\tilde{g}(x):=Av_{\Xi_r(x)}(g)$. By construction $\tilde{g}$ is non-negative, non-decreasing and Ξ_r-invariant. And $Av_{\mathcal{K}_{n,r}}(g)=Av_{\mathcal{K}_{n,r}}(\tilde{g})$, $Av_{\mathcal{K}_{n,r}}(fg)=Av_{\mathcal{K}_{n,r}}(f\tilde{g})$.

Note that the statement (b) follows from (a). (Indeed, if g is non-increasing then consider the number $\underset{\mathcal{K}_{n,r}}{max}(g)$ and apply the first statement to the functions f and $\underset{\mathcal{K}_{n,r}}{max}(g)-g$.) In particular, we can assume that both f and g are non-decreasing and Ξ_r-invariant functions.

Suppose (f_1,g) and (f_2,g) satisfy the statement of the theorem. Then the pair $(a_1f_1+a_2f_2,g)$ satisfies the theorem too for any $a_1,a_2\in\mathbb{R}_{\geq 0}$. (Note that $a_1f_1+a_2f_2$ is Ξ_r-invariant and non-decreasing as well.) Moreover, the equality holds iff it holds for each pair separately: $Av_{\mathcal{K}_{n,r}}(f_i)Av_{\mathcal{K}_{n,r}}(g)=Av_{\mathcal{K}_{n,r}}(f_ig)$. On the other hand, any Ξ_r-invariant, non-decreasing function on $\mathcal{K}_{n,r}$ is presentable as a positive linear combination of certain $\{0,1\}$ valued functions, which are Ξ_r-invariant and non-decreasing as well. Therefore, to prove (a), we can assume that both f and g are of this form.

Therefore, we assume that f and g are the characteristic functions of some subsets. E.g., $f=\mathbb{1}_X$, where $X\subseteq\mathcal{K}_{n,r}$ is Ξ_r-invariant and "upward closed". This later condition means the following: if $\underline{k}\in X$ and $\underline{k}'\succeq\underline{k}$ then $\underline{k}'\in X$. First and last, it is enough to prove the inequality for cardinalities of "upward closed" and Ξ_r-invariant subsets:

$$Av_{\mathcal{K}_{n,r}}(X)\cdot Av_{\mathcal{K}_{n,r}}(Y):=\frac{|X|}{|\mathcal{K}_{n,r}|}\cdot\frac{|Y|}{|\mathcal{K}_{n,r}|}\leq\frac{|X\cap Y|}{|\mathcal{K}_{n,r}|}=Av_{\mathcal{K}_{n,r}}(X\cap Y). \tag{7}$$

Furthermore, we also verify that the equality holds iff either $X=\mathcal{K}_{n,r}$ or $X=\varnothing$.

Our proof generalizes and refines the proof of [Kerner-Némethi2013, §4].

Step 2. The proof below consists of some counting over the subsets of $\mathcal{K}_{n,r}$. First we define the stratification: $\mathcal{K}_{n,r}=\coprod\limits_{s=0,\dots,r-1}\mathcal{K}^s_{n,r}$, where $\mathcal{K}^s_{n,r}:=\{\underline{k}\in\mathcal{K}_{n,r}:\ |\{i:\ k_i=0\}|=s\}$. Note that $\mathcal{K}^s_{n,r}=\varnothing$ for $s<r-n$. We often use the expression for cardinality of these sets:

$$|\mathcal{K}^s_{n,r}|=\binom{r}{s}\binom{n-1}{r-s-1}. \tag{8}$$

If one thinks about $\mathcal{K}_{n,r}$ as a simplex (see §2.1) then $\mathcal{K}^s_{n,r}$ are collections of open cells/faces of codimension s.

Some strata of $\mathcal{K}_{n,r}$ are naturally isomorphic to (some strata of) $\mathcal{K}_{n',r'}$ with lower n',r'. For example: $\mathcal{K}^0_{n,r}\xrightarrow{\sim}\mathcal{K}_{n-r,r}$, by $(k_1,\dots,k_r)\to(k_1-1,\dots,k_r-1)$. Usually we identify $\mathcal{K}_{n-r,r}$ with its image $\mathcal{K}^0_{n,r}\subset\mathcal{K}_{n,r}$. For example, we write: $|X\cap\mathcal{K}^0_{n,r}|=|X\cap\mathcal{K}_{n-r,r}|$.

In the following we often split the sets into parts, e.g. $\mathcal{K}^s_{n,r}=\coprod\limits_{(i_1,\dots,i_s)\subset[1,\dots,r]}\mathcal{K}^s_{n,r}(k_{i_1}=\dots=k_{i_s}=0)$. Here $\mathcal{K}^s_{n,r}(k_{i_1}=\dots=k_{i_s}=0)=\{\underline{k}\in\mathcal{K}^s_{n,r}\mid k_j=0\text{ iff }j\in\{i_1,\dots,i_r\}\}$. As before, we get $\mathcal{K}^s_{n,r}(k_{i_1}=\dots=k_{i_s}=0)\xrightarrow{\sim}\mathcal{K}^0_{n,r-s}$.

Sometimes we write this in the sloppy way $\mathcal{K}^s_{n,r} = \coprod_{(i_1,\dots,i_s)\subset[1,\dots,r]} \mathcal{K}^0_{n,r-s}$. These formulas give immediate implications for the cardinality of the sets: $|\mathcal{K}^s_{n,r}| = \binom{r}{s}|\mathcal{K}^0_{n,r-s}| = \binom{r}{s}|\mathcal{K}_{n-r+s,r-s}|$.

Note that starting from a subset $X \subseteq \mathcal{K}_{n,r}$ we get the set $X \cap \mathcal{K}^s_{n,r}$ and its subsets $X \cap \mathcal{K}^s_{n,r} = \coprod_{(i_1,\dots,i_s)\subset[1,\dots,r]} X \cap \mathcal{K}^0_{n,r-s} = \coprod_{(i_1,\dots,i_s)\subset[1,\dots,r]} X \cap \mathcal{K}_{n-r+s,r-s}$. Moreover, if the initial X was Ξ_r-invariant and upward closed then so are all the above subsets of type $X \cap K_{n-r+s,r-s}$.

Step 3. Let a subset $X \subset \mathcal{K}_{n,r}$ be Ξ_r-invariant and upward closed, as above. Consider its averages over the strata, $\{Av_{\mathcal{K}^s_{n,r}} X := \frac{|X\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|}\}_{s=0,\dots,r-1}$. We prove:

$$Av_{\mathcal{K}^0_{n,r}} X \le Av_{\mathcal{K}^1_{n,r}} X \le \dots \le Av_{\mathcal{K}^{r-1}_{n,r}} X \tag{9}$$

and the equality holds iff $X = \mathcal{K}_{n,r}$ or $X = \varnothing$.

The proof goes by double induction on (n,r). Recall that $\mathcal{K}^s_{n,r} = \varnothing$ if $n<0$ or $r \le 0$. Note that if $r=1$ or $n=1$ then the statement is empty ($\mathcal{K}^s_{n,1} = \varnothing$ for $s>0$ and $\mathcal{K}^s_{1,r} = \varnothing$ for $s<r-1$). For $r=2$ the proof is trivial.

Fix some (n,r), suppose the statement holds for any (n',r',X), with $n'<n$ and $r'<r$ and X an Ξ_r-invariant set which is upward closed. First we reduce all the inequalities to $Av_{\mathcal{K}^0_{n,r}} X \le Av_{\mathcal{K}^1_{n,r}} X$. Indeed:

$$Av_{\mathcal{K}^s_{n,r}} X = \frac{|X\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|} = \sum_{(i_1,\dots,i_s)\subset[1,\dots,r]} \frac{|X\cap\mathcal{K}^s_{n,r}(k_{i_1}=\dots=k_{i_s}=0)|}{|\mathcal{K}^s_{n,r}|} \xlongequal{\Xi_r\circlearrowleft X} \binom{r}{s}\frac{|X\cap\mathcal{K}^0_{n,r-s}|}{|\mathcal{K}^s_{n,r}|}. \tag{10}$$

(Here in the last equality we used the Ξ_r-invariance of X.)

In addition:

$$|X\cap\mathcal{K}^1_{n,r+1-s}| = \sum_{j=1}^{r+1-s} |X\cap\mathcal{K}^1_{n,r+1-s}(k_j=0)| \xlongequal{\Xi_r\circlearrowleft X} (r+1-s)|X\cap\mathcal{K}^0_{n,r-s}|. \tag{11}$$

Thus $Av_{\mathcal{K}^s_{n,r}} X = \frac{\binom{r}{s}}{r+1-s}\frac{|X\cap\mathcal{K}^1_{n,r+1-s}|}{|\mathcal{K}^s_{n,r}|}$, while $Av_{\mathcal{K}^{s-1}_{n,r}} X = \binom{r}{s-1}\frac{|X\cap\mathcal{K}^0_{n,r+1-s}|}{|\mathcal{K}^{s-1}_{n,r}|}$. Altogether:

$$Av_{\mathcal{K}^s_{n,r}} X - Av_{\mathcal{K}^{s-1}_{n,r}} X = \frac{\binom{r}{s}}{r+1-s}\frac{|\mathcal{K}^1_{n,r+1-s}|}{|\mathcal{K}^s_{n,r}|}\Big(Av_{\mathcal{K}^1_{n,r+1-s}} X - Av_{\mathcal{K}^0_{n,r+1-s}} X\Big). \tag{12}$$

(Here we used the equality of the coefficients: $\frac{\binom{r}{s}}{r+1-s}\frac{|\mathcal{K}^1_{n,r+1-s}|}{|\mathcal{K}^s_{n,r}|} = \binom{r}{s-1}\frac{|\mathcal{K}^0_{n,r+1-s}|}{|\mathcal{K}^{s-1}_{n,r}|}$.)

If $s>1$ then $Av_{\mathcal{K}^1_{n,r+1-s}} X \ge Av_{\mathcal{K}^0_{n,r+1-s}} X$ by the inductive assumption. Thus it remains to prove that $Av_{\mathcal{K}^1_{n,r}} X \ge Av_{\mathcal{K}^0_{n,r}} X$. Now we use the reduction:

$$\begin{aligned} |X\cap\mathcal{K}^1_{n,r}| &= \sum_{j=1}^{r} |X\cap\mathcal{K}^1_{n,r}(k_j=0)| = \sum_{j=1}^{r} |X\cap\mathcal{K}^0_{n,r-1}| = \sum_{j=1}^{r} |X\cap\mathcal{K}_{n-r+1,r-1}| = \\ &= \sum_{s=0}^{r-2}\sum_{j=1}^{r} |X\cap\mathcal{K}^s_{n-r+1,r-1}| = \sum_{s=0}^{r-2}(s+1)|X\cap\mathcal{K}^{s+1}_{n-r+1,r}| = \sum_{s=0}^{r-1} s|X\cap\mathcal{K}^s_{n-r+1,r}|. \end{aligned} \tag{13}$$

Similarly:

$$|X\cap\mathcal{K}^0_{n,r}| = \frac{1}{r}\sum_{j=1}^{r}|X\cap\mathcal{K}_{n-r+1,r}(k_j>0)| = \frac{1}{r}\sum_{j=1}^{r}\sum_{s=0}^{r-1}|X\cap\mathcal{K}^s_{n-r+1,r}(k_j>0)| = \frac{1}{r}\sum_{s=0}^{r-1}(r-s)|X\cap\mathcal{K}^s_{n-r+1,r}|. \tag{14}$$

Therefore:

$$Av_{\mathcal{K}^1_{n,r}} X - Av_{\mathcal{K}^0_{n,r}} X = \frac{\sum_{s=0}^{r-1} s|X\cap\mathcal{K}^s_{n-r+1,r}|}{|\mathcal{K}^1_{n,r}|} - \frac{\frac{1}{r}\sum_{s=0}^{r-1}(r-s)|X\cap\mathcal{K}^s_{n-r+1,r}|}{|\mathcal{K}^0_{n,r}|} = \sum_{s=0}^{r-1}\underbrace{\Big(\frac{s}{|\mathcal{K}^1_{n,r}|} - \frac{\frac{r-s}{r}}{|\mathcal{K}^0_{n,r}|}\Big)|\mathcal{K}^s_{n-r+1,r}|}_{\alpha_s}\underbrace{Av_{\mathcal{K}^s_{n-r+1,r}} X}_{\beta_s}. \tag{15}$$

Here by the inductive assumption: $\beta_{r-1} \ge \beta_{r-2} \ge \dots \ge \beta_0 \ge 0$. By direct check:

$$\alpha_s = \frac{1}{\binom{n-2}{r-2}(n-1)}\binom{n-r}{r-s-1}\Big(n\binom{r-1}{s-1} - (r-1)\binom{r}{s}\Big). \tag{16}$$

(We use the convention: $\binom{m}{n} = 0$ if $n<0$ or $m<n$.)

We claim that $\sum_{s=k}^{r-1}\alpha_s = \frac{1}{\binom{n-2}{r-2}(n-1)}\binom{n-r}{r-k}\binom{r-1}{k}k$, for $0\le k\le r-1$. (In particular, this sum is positive for $k>0$ and zero for $k=0$.)

The case $k=0$ follows from $n\sum_{s=0}^{r-1}\binom{n-r}{r-s-1}\binom{r-1}{s-1} = (r-1)\sum_{s=0}^{r-1}\binom{n-r}{r-s-1}\binom{r}{s}$, which in turn follows from the classical $\sum_{i=0}^{p}\binom{p}{i}\binom{q}{k-i}=\binom{p+q}{k}$.

Suppose the stated equality holds for some k, we prove it for $k+1$:

$$\sum_{s=k+1}^{r-1}\alpha_s = \sum_{s=k}^{r-1}\alpha_s - \alpha_k = \frac{\binom{n-r}{r-k}\binom{r-1}{k}k - \binom{n-r}{r-k-1}\Big(n\binom{r-1}{k-1} - (r-1)\binom{r}{k}\Big)}{\binom{n-2}{r-2}(n-1)} = \tag{17}$$
$$= \frac{\frac{(n-r)!}{(r-k)!(n-2r+k+1)!}\binom{r-1}{k}(k^2+k-2kr+r^2-r)}{\binom{n-2}{r-2}(n-1)} = \frac{\binom{n-r}{r-k-1}\binom{r-1}{k+1}(k+1)}{\binom{n-2}{r-2}(n-1)},$$

precisely as stated.

Therefore:

$$Av_{\mathcal{K}^1_{n,r}}X - Av_{\mathcal{K}^0_{n,r}}X = \sum_{s=0}^{r-1}\alpha_s\beta_s = \underbrace{\alpha_{r-1}(\beta_{r-1}-\beta_{r-2})}_{\ge 0} + \underbrace{(\alpha_{r-1}+\alpha_{r-2})(\beta_{r-2}-\beta_{r-3})}_{\ge 0} + \tag{18}$$
$$+\cdots+\underbrace{(\sum_{i=1}^{r-1}\alpha_i)(\beta_1-\beta_0)}_{\ge 0} + \underbrace{(\sum_{i=0}^{r-1}\alpha_i)\beta_0}_{=0}.$$

As each term is non-negative we get: $Av_{\mathcal{K}^1_{n,r}}X \ge Av_{\mathcal{K}^0_{n,r}}X$. Furthermore, the equality holds iff $\{\beta_{i+1}=\beta_i\}_i$, i.e. either $\{X\cap\mathcal{K}^i_{n,r}=\mathcal{K}^i_{n,r}\}_i$ or $X=\varnothing$.

Step 4. Finally, using $Av_{\mathcal{K}^0_{n,r}}X \le Av_{\mathcal{K}^1_{n,r}}X \le \cdots \le Av_{\mathcal{K}^{r-1}_{n,r}}X$, we prove $(Av_{\mathcal{K}_{n,r}}X)(Av_{\mathcal{K}_{n,r}}Y)\le Av_{\mathcal{K}_{n,r}}(X\cap Y)$. As above, this is done by the double induction on (n,r). Note that the statement holds trivially for $r=1$ or $n=1$. Suppose it holds for any (n',r') with $n'<n$ and $r'<r$.

First observe:

$$|\mathcal{K}_{n,r}|\cdot Av_{\mathcal{K}_{n,r}}(X\cap Y) = |X\cap Y\cap\mathcal{K}_{n,r}| = \sum_{s=0}^{r-1}\sum_{(i_1,\dots,i_s)\subset[1,\dots,r]}|X\cap Y\cap\mathcal{K}^0_{n,r-s}(k_{i_1}=0=\cdots=k_{i_s})|. \tag{19}$$

Further:

$$|X\cap Y\cap\mathcal{K}^0_{n,r-s}(k_{i_1}=0=\cdots=k_{i_s})| = |X\cap Y\cap\mathcal{K}_{n-r+s,r-s}(k_{i_1}=0=\cdots=k_{i_s})| \ge \tag{20}$$
$$\ge \frac{|X\cap\mathcal{K}_{n-r+s,r-s}(k_{i_1}=0=\cdots=k_{i_s})|\cdot|Y\cap\mathcal{K}_{n-r+s,r-s}(k_{i_1}=0=\cdots=k_{i_s})|}{|\mathcal{K}_{n-r+s,r-s}|}.$$

The last inequality here is the induction assumption.

As was mentioned above, the cardinalities of the sets are related by: $|\mathcal{K}_{n-r+s,r-s}| = |\mathcal{K}^0_{n,r-s}| = \frac{|\mathcal{K}^s_{n,r}|}{\binom{r}{s}}$. Similarly (as X,Y are Ξ_r-invariant): $|X\cap\mathcal{K}_{n-r+s,r-s}| = |X\cap\mathcal{K}^0_{n,r-s}| = \frac{|X\cap\mathcal{K}^s_{n,r}|}{\binom{r}{s}}$. Therefore:

$$|\mathcal{K}_{n,r}|Av_{\mathcal{K}_{n,r}}(X\cap Y) \ge \sum_{s=0}^{r-1}\sum_{(i_1,\dots,i_s)\subset[1,\dots,r]}\frac{|X\cap\mathcal{K}^s_{n,r}|\cdot|Y\cap\mathcal{K}^s_{n,r}|}{\binom{r}{s}|\mathcal{K}^s_{n,r}|} = \sum_{s=0}^{r-1}\underbrace{\frac{|X\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|}}_{\alpha_s}\cdot\underbrace{\frac{|Y\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|}}_{\beta_s}\cdot\underbrace{|\mathcal{K}^s_{n,r}|}_{\gamma_s}. \tag{21}$$

Here we have $\alpha_0\le\alpha_1\le\cdots\le\alpha_{r-1}$ and $\beta_0\le\beta_1\le\cdots\le\beta_{r-1}$, by *Step 3*, and $\gamma_s>0$.

Thus we can use the following generalization of Chebyshev's sum inequality

$$\big(\sum_s\gamma_s\alpha_s\big)\big(\sum_{s'}\gamma_{s'}\beta_{s'}\big) \le \big(\sum_{s'}\gamma_{s'}\big)\big(\sum_s\alpha_s\beta_s\gamma_s\big), \tag{22}$$

which basically is the summation $\sum_{s,s'}\gamma_s\gamma_{s'}(\alpha_s-\alpha_{s'})(\beta_s-\beta_{s'})\ge 0$, see [Hardy-Littlewood-Pólya, p. 43].

Hence:

$$(23)\quad |\mathcal{K}_{n,r}|^2\cdot Av_{\mathcal{K}_{n,r}}(X\cap Y)\geq\Big(\sum_{s'=0}^{r-1}\mathcal{K}^{s'}_{n,r}\Big)\sum_{s=0}^{r-1}\underbrace{\frac{|X\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|}}_{\alpha_s}\cdot\underbrace{\frac{|Y\cap\mathcal{K}^s_{n,r}|}{|\mathcal{K}^s_{n,r}|}}_{\beta_s}\cdot\underbrace{|\mathcal{K}^s_{n,r}|}_{\gamma_s}\geq$$

$$\geq\Big(\sum_{s=0}^{r-1}|X\cap\mathcal{K}^s_{n,r}|\Big)\Big(\sum_{s'=0}^{r-1}|Y\cap\mathcal{K}^{s'}_{n,r}|\Big)=|X\cap\mathcal{K}_{n,r}|\cdot|Y\cap\mathcal{K}_{n,r}|.$$

This finishes the proof. ■

This theorem can be extended to the functions whose "pushforwards" are monotonic on ${}^{\mathcal{K}_{n,r}}\!/\!_{\Xi_r}$. Given $f:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$ define $[f]:{}^{\mathcal{K}_{n,r}}\!/\!_{\Xi_r}\to\mathbb{R}_{\geq0}$ by: $[f]([x])=Av_{\Xi_r(x)}f$. (Note that this expression does not depend on the choice of the representative of $[x]$.)

Corollary 3.3. *Given functions $f,g:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$. Suppose f is Ξ_r invariant and $[f],[g]:{}^{\mathcal{K}_{n,r}}\!/\!_{\Xi_r}\to\mathbb{R}_{\geq0}$ are monotonic. Then $Av_{\mathcal{K}_{n,r}}(f)\cdot Av_{\mathcal{K}_{n,r}}(g)\leq Av_{\mathcal{K}_{n,r}}(fg)$, with equality iff one of $[f]$, $[g]$ is constant.*

To prove this, define $\tilde{g}:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$ by $\tilde{g}(x)=Av_{\Xi_r(x)}(g)$. Then $Av_{\mathcal{K}_{n,r}}(g)=Av_{\mathcal{K}_{n,r}}(\tilde{g})$ and $Av_{\mathcal{K}_{n,r}}(fg)=Av_{\mathcal{K}_{n,r}}(f\tilde{g})$. And by construction f, $\tilde{g}$ are monotonic on $\mathcal{K}_{n,r}$. Now, apply the theorem.

We can reformulate theorem 3.1 to compare the geometric averages.

Corollary 3.4. *Given two non-decreasing functions, $f:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq1}$ and $g:\mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$. Suppose at least one of them is Ξ_r-invariant. Then*

$$\Big(Av^G_{\mathcal{K}_{n,r}}(f)\Big)^{Av_{\mathcal{K}_{n,r}}(g)}\geq Av^G_{\mathcal{K}_{n,r}}(f^g).$$

To prove this take the logarithm of both sides. Note that $ln(Av^G_{\mathcal{K}_{n,r}}(f))=Av_{\mathcal{K}_{n,r}}(ln(f))$ where $ln(f):\mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$ is still a non-decreasing function. Similarly $ln(Av^G_{\mathcal{K}_{n,r}}(f^g))=Av_{\mathcal{K}_{n,r}}(g\cdot ln(f))$. Now, apply the theorem.

4. An application to mixed (co-)volumes

4.1. Newton polyhedra. Let $f(x_1,\dots,x_N)=\sum\limits_I a_I\underline{x}^I$ be a power series over some field $\Bbbk$. Consider the support of its monomials, $Supp(f):=\{I\in\mathbb{Z}^N_{\geq0}|\ a_I\neq0\}$. The Newton polyhedron is defined as the convex hull,

$$(24)\qquad \Gamma^+=\Gamma^+_f:=Conv(Supp(f)+\mathbb{R}^N_{\geq0}).$$

It has compact faces and unbounded faces. The Newton polyhedron is called convenient if Γ^+ intersects all the coordinate axes (i.e. f contains monomials of type $x_1^{m_1},\dots,x_N^{m_N}$). We always assume Γ^+ to be convenient. (Though in the present note we do not study the analytic properties of the power series related with their Newton diagrams, by the above definition we wish to emphasize the main motivation supported by algebraic geometry and complex analysis.)

4.2. Mixed (co)volumes. Given several convex bodies $A_1,\dots,A_r$ in $\mathbb{R}^N$, consider their scaled Minkowski sum, $\lambda_1A_1+\dots+\lambda_rA_r$. The mixed volumes are defined as the coefficients in the expansion:

$$(25)\qquad Vol_N(\lambda_1A_1+\dots+\lambda_rA_r)=\sum_{\underline{k}\in\mathcal{K}_{N,r}}\binom{N}{k_1,\dots,k_r}Vol\Big((A_1)^{k_1},\dots,(A_r)^{k_r}\Big)(\prod_{i=1}^r\lambda_i^{k_i}).$$

Dually, given a convenient Newton polyhedron, $\Gamma^+\subset\mathbb{R}^N_{\geq0}$, consider its covolume, i.e. the volume of the complement: $coVol(\Gamma^+):=Vol_N(\mathbb{R}^N_{\geq0}\setminus\Gamma^+)$. Given a collection of Newton polyhedra, $\{\Gamma^+_i\}^r_{i=1}$, consider their scaled Minkowski sum, $\lambda_1\Gamma^+_1+\dots+\lambda_r\Gamma^+_r$. The covolume of this sum is a polynomial in $\{\lambda_i\}$, see e.g. [Kaveh-Khovanskii-2013-2, Theorem 10.4]:

$$(26)\qquad coVol(\lambda_1\Gamma^+_1+\dots+\lambda_r\Gamma^+_r)=\sum_{\underline{k}\in\mathcal{K}_{N,r}}\binom{N}{k_1,\dots,k_r}coVol\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)(\prod_{i=1}^r\lambda_i^{k_i}).$$

The mixed covolumes are the (positive) coefficients $coVol\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)$.

Here $coVol\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)$ is a shorthand for $coVol\Big(\underbrace{\Gamma^+_1,\dots,\Gamma^+_1}_{k_1},\dots,\underbrace{\Gamma^+_r,\dots,\Gamma^+_r}_{k_r}\Big)$, for $k_1+\dots+k_r=N$, similarly to $Vol\Big((A_1)^{k_1},\dots,(A_r)^{k_r}\Big)$.

We list some of the basic properties of the mixed (co)volumes, see e.g. [Kaveh-Khovanskii-2013-1, §2]:

- They are symmetric and multilinear, e.g. $coVol(\Gamma^+_{11}+\Gamma^+_{12},\Gamma^+_2,\dots,\Gamma^+_N)=coVol(\Gamma^+_{11},\Gamma^+_2,\dots,\Gamma^+_N)+coVol(\Gamma^+_{12},\Gamma^+_2,\dots,\Gamma^+_N)$.

• If $\Gamma^+_i = d_i\Gamma^+$ for any $i = 1, \dots, r$ then
(27)
$$coVol_N(\sum_i \lambda_i\Gamma^+_i) = coVol_N(\sum_i \lambda_i d_i\Gamma^+) = (\sum_i \lambda_i d_i)^N coVol_N(\Gamma^+) = \sum_{\underline{k}\in\mathcal{K}_{n,r}} \binom{N}{k_1,\dots,k_r}(\prod_{i=1}^r(\lambda_i d_i)^{k_i})coVol_N(\Gamma^+).$$

A similar statement holds for the convex bodies: $Vol_N(\sum_i \lambda_i A_i) = Vol_N(\sum_i \lambda_i d_i A) = (\sum_i d_i\lambda_i)^N Vol_N(A)$.

• The mixed volumes of the convex bodies in $\mathbb{R}^N$ satisfy the Alexandrov-Fenchel inequality, [Alexandrov-1937], [Alexandrov-1938], [Fenchel-1936]:

$$Vol(A_1, \dots, A_N)^2 \geq Vol(A_1, A_1, A_3 \dots, A_N)Vol(A_2, A_2, A_3 \dots, A_N). \quad (28)$$

The dual property of the mixed covolumes of Newton polyhedra was proved in [Teissier1978], [Rees-Sharp-1978], [Katz1988], [Teissier2004, Appendix], [Kaveh-Khovanskii-2013-2, Theorem 10.5]:

$$coVol(\Gamma^+_1, \Gamma^+_2, \cdots, \Gamma^+_N)^2 \leq coVol(\Gamma^+_1, \Gamma^+_1, \Gamma^+_3, \cdots, \Gamma^+_N)coVol(\Gamma^+_2, \Gamma^+_2, \Gamma^+_3, \cdots, \Gamma^+_N). \quad (29)$$

4.3. A generalization of the Alexandrov-Fenchel-Teissier inequalities. The inequalities of equations (28) and (29) correspond to the triples of points on a segment in $\mathcal{K}_{n,r}$: $\{k_1+2, k_2, k_3, \dots, \}$, $\{k_1+1, k_2+1, k_3, \dots\}$, $\{k_1, k_2+2, k_3, \dots\}$. One would ask for the general property, the Jensen-type inequality:

Suppose a collection of lattice points $\underline{k}^{(1)}, \underline{k}^{(2)}, \dots, \underline{k}^{(s)} \in \mathcal{K}_{N,r}$, *satisfy:* $\underline{k} := \frac{\underline{k}^{(1)}+\cdots+\underline{k}^{(s)}}{s} \in \mathcal{K}_{N,r}$ *(i.e.* $\underline{k}$ *defined in this way is a lattice point too). Then* $coVol\big((\Gamma^+)^{\underline{k}}\big)^s \leq \prod_{i=1}^s coVol\big((\Gamma^+)^{\underline{k}^{(i)}}\big)$ *and* $Vol\big(\underline{A}^{\underline{k}}\big)^s \geq \prod_{i=1}^s Vol\big(\underline{A}^{\underline{k}^{(i)}}\big)$.

In particular, Gromov in [Gromov90] asked whether this holds at least for points $\underline{k}^{(1)}, \underline{k}^{(2)}, \dots, \underline{k}^{(s)}$ sitting inside a low dimensional linear subspace of $\mathcal{K}_{n,r}$. As it was shown by Burda in [Burda-2012] such a property in general fails, e.g. he provided an example with

$$Vol_3(A_1, A_2, A_3)^3 < Vol_3(A_1, A_1, A_2)Vol_3(A_2, A_2, A_3)Vol_3(A_3, A_3, A_1). \quad (30)$$

Yet, Theorem 3.1 and Corollary 3.4 allow us to establish the 'corrected version' of the above question (see Corollary 4.1).

Given a pair (n', r') fix some $1 \leq r \leq r'$ and denote $n := n'+r-r'$. Fix some Newton polyhedra, $\Gamma^+_1, \dots, \Gamma^+_r, \Gamma^+_{r+1}, \dots, \Gamma^+_{r'} \subset \mathbb{R}^{n'}$. Define the function $Mix.coVol : \mathcal{K}_{n,r} \to \mathbb{R}_{\geq 0}$ by $\underline{k} \to coVol\Big((\Gamma^+_1)^{k_1}, \dots, (\Gamma^+_r)^{k_r}, \Gamma^+_{r+1}, \dots, \Gamma^+_{r'}\Big)$. Fix some convex bodies $A_1, \dots, A_r, A_{r+1}, \dots, A_{r'} \subset \mathbb{R}^{n'}$ and define the function

$$Mix.Vol: \ \mathcal{K}_{n,r} \to \mathbb{R}_{\geq 0}, \quad \underline{k} \to Vol\Big(A_1^{k_1}, \dots, A_r^{k_r}, A_{r+1}, \dots, A_{r'}\Big). \quad (31)$$

These functions correspond to the particular embedding $\mathcal{K}_{n,r} \hookrightarrow \mathcal{K}_{n',r'}$ by $(k_1, \dots, k_r) \to (k_1, \dots, k_r, 1, \dots, 1)$.

Corollary 4.1. *Let* $C : \mathcal{K}_{n,r} \to \mathbb{R}_{\geq 0}$ *be a non-decreasing* Ξ_r*-invariant function.*
1. $Av_{\mathcal{K}_{n,r}}(C) \cdot Av_{\mathcal{K}_{n,r}}(Mix.coVol) \leq Av_{\mathcal{K}_{n,r}}(C \cdot Mix.coVol)$.
2. Suppose all the values of $Mix.coVol$ *and* $Mix.Vol$ *are* ≥ 1. *Then* $\Big(Av^G_{\mathcal{K}_{n,r}}(Mix.coVol)\Big)^{Av_{\mathcal{K}_{n,r}}(C)} \leq Av^G_{\mathcal{K}_{n,r}}(Mix.coVol^C)$ *and* $\Big(Av^G_{\mathcal{K}_{n,r}}(Mix.Vol)\Big)^{Av_{\mathcal{K}_{n,r}}(C)} \geq Av^G_{\mathcal{K}_{n,r}}(Mix.Vol^C)$.

Proof. **Part 1.**
Step 1. First, using the inequality $\sqrt{ab} \leq \frac{a+b}{2}$ one gets from equation (29) the weaker convexity property:

$$coVol(\Gamma^+_1, \Gamma^+_2, \cdots, \Gamma^+_N) \leq \frac{coVol(\Gamma^+_1, \Gamma^+_1, \Gamma^+_3, \cdots, \Gamma^+_N) + coVol(\Gamma^+_2, \Gamma^+_2, \Gamma^+_3, \cdots, \Gamma^+_N)}{2}. \quad (32)$$

The function $Mic.coVol$ is not Ξ_r-invariant, for Ξ_r acting on the first r indices. Consider its symmetrization:

$$[Mix.coVol]: \ \mathcal{K}_{n,r} \to \mathbb{R}_{\geq 0}, \qquad \underline{k} \to Av_{\Xi_r(\underline{k})}(Mix.coVol) = \frac{\sum_{\sigma\in\Xi_r} coVol\Big((\Gamma^+_{\sigma(1)})^{k_1}, \dots, (\Gamma^+_{\sigma(r)})^{k_r}, \Gamma^+_{r+1}, \dots, \Gamma^+_{r'}\Big)}{|\Xi_r(\underline{k})|}. \quad (33)$$

We prove that this symmetrization is a non-decreasing function, i.e. if $\underline{k} \succeq \underline{k}' \in \mathcal{K}_{n,r}$ then $[Mix.coVol](\underline{k}) \geq [Mix.coVol](\underline{k}')$.

It is enough to check the inequality for elementary steps: if $k_1 + 1 \leq k_2 - 1$ then $[Mix.coVol]([k_1, k_2, k_3, \dots]) \geq [Mix.coVol]([k_1+1, k_2-1, k_3, \dots])$.

Suppose $k_1+k_2=2l\in 2\mathbb{Z}$. Then the convexity property of the mixed volumes reads:

$$(34)\quad [Mix.coVol](l,l,k_3,\dots)=\sum_{\sigma\in\Xi_r}\frac{coVol\Big((\Gamma^+_{\sigma(1)})^{l},(\Gamma^+_{\sigma(2)})^{l},(\Gamma^+_{\sigma(3)})^{k_3}\dots,(\Gamma^+_{\sigma(r)})^{k_r},\dots\Big)}{r!}\overset{eq.(32)}{\leq}$$

$$\leq\sum_{\sigma\in\Xi_r}\frac{coVol\Big((\Gamma^+_{\sigma(1)})^{l-1},(\Gamma^+_{\sigma(2)})^{l+1},(\Gamma^+_{\sigma(3)})^{k_3}\dots,(\Gamma^+_{\sigma(r)})^{k_r},\dots\Big)+coVol\Big((\Gamma^+_{\sigma(1)})^{l+1},(\Gamma^+_{\sigma(2)})^{l-1},(\Gamma^+_{\sigma(3)})^{k_3}\dots,(\Gamma^+_{\sigma(r)})^{k_r},\dots\Big)}{2r!}=$$

$$=\sum_{\sigma\in\Xi_r}\frac{coVol\Big((\Gamma^+_{\sigma(1)})^{l-1},(\Gamma^+_{\sigma(2)})^{l+1},(\Gamma^+_{\sigma(3)})^{k_3}\dots,(\Gamma^+_{\sigma(r)})^{k_r},\dots\Big)}{r!}=[Mix.coVol](l-1,l+1,k_3,\dots).$$

Similarly, $[Mix.coVol](l-1,l+1,k_3,\dots)\leq\frac{[Mix.coVol](l,l,k_3,\dots)+[Mix.coVol](l-2,l+2,k_3,\dots)}{2}$. Combining with the previous one it gives $[Mix.coVol](l-2,l+2,k_3,\dots)\geq[Mix.coVol](l-1,l+1,k_3,\dots)$. One continues in the same way.

The case $k_1+k_2=2l+1\in 2\mathbb{Z}+1$ is treated similarly.

Step 2. Now we have two Ξ_r-invariant, non-decreasing functions, then theorem 3.1 gives:

$$(35)\qquad Av_{\mathcal{K}_{n,r}}(C)Av_{\mathcal{K}_{n,r}}([Mix.coVol])\leq Av_{\mathcal{K}_{n,r}}(C\cdot[Mix.coVol]).$$

It remains to add that $Av_{\mathcal{K}_{n,r}}([Mix.coVol])=Av_{\mathcal{K}_{n,r}}(Mix.coVol)$ and $Av_{\mathcal{K}_{n,r}}(C\cdot[Mix.coVol])=Av_{\mathcal{K}_{n,r}}(C\cdot Mix.coVol)$.

Part 2. Define the function $f:\ \mathcal{K}_{n,r}\to\mathbb{R}_{\geq0}$ by $f(\underline{k})=ln\Big(coVol\big(A_1^{k_1},\dots,A_r^{k_r},A_{r+1},\dots,A_{r'}\big)\Big)$. Equation (29) implies: $f(k_1,k_2,k_3,\dots)\leq\frac{f(k_1,k_1,k_3,\dots)+f(k_2,k_2,k_3,\dots)}{2}$. Then, repeat Part 1 for the functions f, C to get:

$$(36)\qquad Av_{\mathcal{K}_{n,r}}\Big(ln\big(coVol(\dots)^{Av_{\mathcal{K}_{n,r}}(C)}\big)\Big)=Av_{\mathcal{K}_{n,r}}(C)Av_{\mathcal{K}_{n,r}}(f)\leq Av_{\mathcal{K}_{n,r}}(C\cdot f)=Av_{\mathcal{K}_{n,r}}\Big(ln\big(coVol(\dots)^{C(\dots)}\big)\Big).$$

Now take the exponent.

The proof of the inequality for mixed volumes is the same, it is based now on equation (28). ■

Example 4.2. • The case $r=1$ is 'empty' as $\mathcal{K}_{n,1}=\{n\}$.
• As the simplest case, suppose $r=2\leq r'$, $n=2\leq n'$. Then part 2 of the corollary states just the ordinary Alexandrov-Fenchel/Teissier inequalities.
• An extremal case is $r=r'$, then $n=n'$. For $n=r=3$ fix the numbers $C_{(3,0,0)}=a\geq C_{(2,1,0)}=b\geq C_{(1,1,1)}=c\geq0$. Extend this to the function $C:\mathcal{K}_{3,3}\to\mathbb{R}_{\geq0}$ by Ξ_3-action. We get a Ξ_3-invariant non-decreasing function. Then Part 2 of the proposition gives:

$$(37)\qquad Vol(A_1,A_2,A_3)^{3a+6b-9c}\geq\Big(\prod_{i=1}^{3}Vol(A_i,A_i,A_i)\Big)^{7a-6b-c}\Big(\prod_{j\neq i}Vol(A_i,A_i,A_j)\Big)^{4b-3a-c}$$

(This formula becomes equality in the simplest case $\{A_i=\lambda_iA\}$. Then $Vol_3(A_i,A_i,A_i)=d_i^3Vol_3(A)$, $Vol_3(A_i,A_i,A_j)=d_i^2d_jVol_3(A)$, $Vol_3(A_1,A_2,A_3)=d_1d_2d_3Vol_3(A)$.)

Example 4.3. One can get a bigger class of inequalities by multiplying the inequalities of part 2 of the corollary for different embeddings $\mathcal{K}_{n,r}\hookrightarrow\mathcal{K}_{n',r'}$. (We do not know whether they contain/generate all the possible inequalities on mixed volumes.) For example, fix $r=2=n<r'=3=n'$. Consider the embeddings $(k_1,k_2)\to(k_1,k_2,1)$, $(k_1,k_2)\to(1,k_1,k_2)$, $(k_1,k_2)\to(k_1,1,k_2)$. Each of them gives the ordinary Aleandrov-Fenchel inequality. The product of these inequalities is:

$$(38)\quad Vol_3(A_1,A_2,A_3)^6\geq$$
$$Vol_3(A_1,A_1,A_2)Vol_3(A_1,A_2,A_2)Vol_3(A_2,A_2,A_3)Vol_3(A_2,A_3,A_3)Vol_3(A_3,A_3,A_1)Vol_3(A_3,A_1,A_1).$$

This can be considered as a 'corrected' version of equation (30).

Example 4.4. In [Kerner-Némethi2014] we use this corollary in the particular case of $C(\underline{k})=\binom{n+r}{k_1+1,\dots,k_r+1}:=\frac{(n+r)!}{(k_1+1)!\cdots(k_r+1)!}$. (Note that this expression is Ξ_r-invariant and non-decreasing on $\mathcal{K}_{n,r}$.) We claim that:

$$(39)\quad\Big(\sum_{\underline{k}\in\mathcal{K}_{n,r}}\binom{n+r}{k_1+1,\dots,k_r+1}\Big)\sum_{\substack{\underline{k}\in\mathcal{K}_{n+r,r}\\k_1,\dots,k_r\geq1}}coVol_{n+r}\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)\geq$$
$$\geq\binom{n+r-1}{n}\sum_{\substack{\underline{k}\in\mathcal{K}_{n+r,r}\\k_1,\dots,k_r\geq1}}\binom{n+r}{k_1+1,\dots,k_r+1}coVol_{n+r}\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)$$

and equality occurs iff $\Gamma^+_1=\dots=\Gamma^+_r$.

Note that $\mathcal{K}_{n+r,r}\cap\{k_1,\dots,k_r\ge 1\}=\mathcal{K}^0_{n+r,r}\xrightarrow{\sim}\mathcal{K}_{n,r}$ in the notations of the proof of theorem 3.1. Therefore the inequality is the comparison of averages of the numbers $\{coVol_{n+r}\Big((\Gamma^+_1)^{k_1+1},\dots,(\Gamma^+_r)^{k_r+1}\Big)\}_{\underline{k}\in\mathcal{K}_{n,r}}$ and $\{\binom{n+r}{k_1+1,\dots,k_r+1}\}_{\underline{k}\in\mathcal{K}_{n,r}}$.

Note that $\{\binom{n+r}{k_1+1,\dots,k_r+1}$ is symmetric under permutations of $\underline{k}$ and gives a non-increasing function on $\mathcal{K}_{n,r}/\Xi_r$.

Thus, corollary 4.1 gives:

$$Av_{\mathcal{K}_{n,r}}\Big\{\binom{n+r}{k_1+1,\dots,k_r+1}\Big\}Av_{\mathcal{K}_{n,r}}\Big\{coVol_{n+r}\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)\Big\}\ge \\ \ge Av_{\mathcal{K}_{n,r}}\Big\{\binom{n+r}{k_1+1,\dots,k_r+1}coVol_{n+r}\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)\Big\} \tag{40}$$

which is precisely the inequality (39).

Moreover, as $\binom{n+r}{k_1+1,\dots,k_r+1}$ is not constant on $\mathcal{K}_{n,r}/\Xi_r$, the equality occurs iff the symmetrization of the function $coVol_{n+r}\Big((\Gamma^+_1)^{k_1},\dots,(\Gamma^+_r)^{k_r}\Big)$ is constant on $\mathcal{K}_{n,r}$; which means $\Gamma^+_1=\dots=\Gamma^+_r$.

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