On the convergence and the steady state in a delayed Solow model

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Abstract: The Solow-Swan model is analyzed with constant population growth rate and fix delay in the production process and in the depreciation. The linear stability of the trivial equilibrium and the steady state is investigated via stability charts after the identification of the effect of the delay on the level of the steady state. After that, the rate of convergence is approximated close to the steady state with the help of the estimation of the characteristic exponent with the largest real part. The identity of the convergence of the capital and the output in per capita is proven in the presence of delay too.

Keywords: Time delay, Economic growth, Solow-Swan, neoclassical economics, Delayed Cobb–Douglas production function.

1. INTRODUCTION

The investigation of stability of the steady state via mathematical methods is a critical question in economics since the appearance of the Solow (1956) model. Moreover, the occurrence of a dynamically unstable equilibrium with a stable periodic solution may explain the well known business cycles within the framework of a neoclassical macromodel. To find a phenomenon like this, existence of Hopf or so called Poincar-Andronov-Hopf bifurcation of the steady state is required. This can be found only in larger than one dimensional dynamical systems unlike the original Solow's one. But considering time delay, as a model of production processes from the installed capital, labor and technology, makes the governing equations infinite dimensional. In the literature this assumption is often referred as *"time to build"*. The analytical investigation of these simple models leads to closed form expressions on the steady state or even on the rate of convergence, which can qualitatively explain the disadvantage of the developing countries.

Time delay in economics was first presented in Kalecki (1935), most recently Asea and Zak (1999), Szydowski and Krawiec (2004) and Guerrini (2012) investigated the time to build approach. Guerrini had found Hopf bifurcation at negative rate of population growth with an emerging limit cycle. Here, this latest model is extended further by capital depreciation, which can be delayed or not depending on the following assumption: depreciation is applied to the working capital or to the existing one. During the analysis, the Cobb–Douglas production function is considered and the rate of convergence is also analyzed.

The rest of this paper is organized as follows: the fundamental equation is derived in Section 2. The steady state level of the capital per capita is determined in Section 3, while the stability property of this is investigated in Section 4. After that, the rate of convergence around this steady state is examined in terms of the delay. Finally, the results are concluded in Section 6.

2. MODEL OF CAPITAL ACCUMULATION

2.1 Delay in the production function

In our neoclassical model, the flow of output at time t

$$Y(t) = F\left(K(t-\tau_K), L(t-\tau_L), T\right) \tag{1}$$

depends on the existing capital K at a delayed time $t-\tau_K$, since the time τ_K is required to produce output. It also depends on a delayed version of the labor L as the people involved in production need τ_L time to be trained. The third term is the technology T, which is a nonrival good as opposed to the previous two. For the sake of simplicity we assume technology to be constant over time and independent from countries.

The function $F(.)$ is a neoclassical production function based on Barro and Sala-i Martin (2003) if it exhibits constant returns to scale

$$F\left(\gamma K, \gamma L, T\right) = \gamma F\left(K, L, T\right)\,, \tag{2}$$

and it also exhibits positive and diminishing marginal products with respect to each input

$$\frac{\partial F}{\partial K} > 0\,, \quad \frac{\partial^2 F}{\partial K^2} < 0\,, \quad \frac{\partial F}{\partial L} > 0\,, \quad \frac{\partial^2 F}{\partial L^2} < 0 \tag{3}$$

for all $K > 0$ and $L > 0$, while the Inada conditions

$$\begin{aligned} \lim_{K\to 0}\left(\frac{\partial F}{\partial K}\right) &= \lim_{L\to 0}\left(\frac{\partial F}{\partial L}\right) = \infty\,, \\ \lim_{K\to \infty}\left(\frac{\partial F}{\partial K}\right) &= \lim_{L\to \infty}\left(\frac{\partial F}{\partial L}\right) = 0 \end{aligned} \tag{4}$$

are satisfied too.

To get the per capita form of the production function one must divide equation (1) by the current labor $L(t)$. Based on the definition of constant returns, it leads to

$$y(t) = \Lambda_L F\left(\frac{\Lambda_K}{\Lambda_L}\frac{K(t-\tau_K)}{L(t-\tau_K)}, 1, T\right) := \Lambda_L f\left(\frac{\Lambda_K}{\Lambda_L} k(t-\tau_K)\right), \tag{5}$$

where

$$y(t) := \frac{Y(t)}{L(t)}, \quad k(t) := \frac{K(t)}{L(t)} \tag{6}$$

are the output per worker and the capital per worker respectively. Expressions Λ_L and Λ_K are constants assuming constant increasing rate of the number of persons

$$n := \frac{\dot{L}(t)}{L(t)} \tag{7}$$

and they are

$$\Lambda_L := \frac{L(t-\tau_L)}{L(t)} = \mathrm{e}^{-n\tau_L}, \ \Lambda_K := \frac{L(t-\tau_K)}{L(t)} = \mathrm{e}^{-n\tau_K}. \tag{8}$$

2.2 The fundamental equation

A closed economy is assumed with one-sector production technology in which output is a homogeneous good that can be consumed, $C(t)$, or invested, $I(t)$. In this simple economy the amount saved $S(t)$ is equal the amount invested $I(t)$. The saving rate s – the fraction of output that is saved $(S(t) = sY(t))$ – is assumed to be constant $0 \le s \le 1$ as it was by Solow (1956) and Swan (1956) in their articles.

The capital is assumed to be homogeneous good that increases by the investments and depreciates at the constant rate δ, so the net increase in the stock of capital is

$$\dot{K}(t) = I(t) - \delta K(t-\tau_\delta) = sY(t) - \delta K(t-\tau_\delta), \tag{9}$$

If one choose the delay in the depreciating term to be equal to the delay of the capital in the production function ($\tau_\delta = \tau_K$) means only the working capital is depreciating. If τ_δ is set to be zero, then all the existing capital will be depreciating. Only these two cases are investigated in this paper.

The fundamental equation of the Solow-Swan model can be obtained by Eq. (9) and the introduction of the per capita variables:

$$\dot{k}(t) = sy(t) - \delta\Lambda_\delta k(t-\tau_\delta) - nk(t) = s\Lambda_L f\left(\frac{\Lambda_K}{\Lambda_L}k(t-\tau_K)\right) - \delta\Lambda_\delta k(t-\tau_\delta) - nk(t), \tag{10}$$

where

$$\Lambda_\delta := \frac{L(t-\tau_\delta)}{L(t)} = \mathrm{e}^{-n\tau_\delta}. \tag{11}$$

Hereafter, we use the Cobb–Douglas production function

$$Y = AK^\alpha L^{1-\alpha} \tag{12}$$

with the level of technology $A > 0$ and a constant $0 < \alpha < 1$.

The fundamental equation of our model whit this simple production function reads as

$$\dot{k}(t) = sA\Lambda_L\left(\frac{\Lambda_K}{\Lambda_L}\right)^\alpha k^\alpha(t-\tau_K) - \delta\Lambda_\delta k(t-\tau_\delta) - nk(t). \tag{13}$$

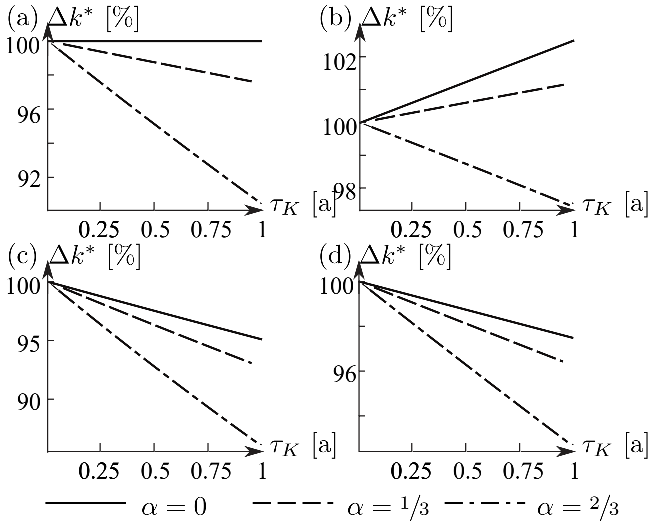

Fig. 1. The value of steady state respect to the delay free case. Panel (a) belongs to $\tau_K = \tau$, $\tau_\delta = 0$, $\tau_L = 0$; panel (b) shows the $\tau_K = \tau_\delta = \tau$, $\tau_L = 0$; panel (c) the $\tau_K = \tau_L = \tau$, $\tau_\delta = 0$, while panel (d) the $\tau_K = \tau_\delta = \tau_L = \tau$ cases.

3. STEADY STATE LEVEL OF CAPITAL

The nonlinear differential equation (13) has two equilibriums where

$$\dot{k}(t) \equiv 0 \quad \text{and} \quad k(t) \equiv k(t-\tau_K) \equiv k(t-\tau_\delta) \equiv k^*. \tag{14}$$

First one is the trivial one $k^* = 0$, and the other one is the so-called steady state level of capital

$$k^* = \left(\frac{\Lambda_K^\alpha}{\Lambda_L^\alpha}\frac{sA\Lambda_L}{\delta\Lambda_\delta + n}\right)^{\frac{1}{1-\alpha}}. \tag{15}$$

To investigate the effect of the delay on the steady state, the ratio of the steady state and its delay free version

$$\Delta k^* = \frac{k^*}{k^*|_{\tau_K=0,\tau_L=0,\tau_\delta=0}}. \tag{16}$$

is used in Fig.1.

The delayed steady state can be greater than the delay free in the vicinity of zero delays if and only if

$$\alpha < \frac{\tau_L(\delta+n) - \delta\tau_\delta}{(\tau_L - \tau_K)(\delta+n)}\tau_K, \tag{17}$$

based on Taylor series expansion around zero delays. Note that, k^* goes to zero as τ_K goes to infinity.

The effect of the delay on the steady state is plotted in Fig. 1 for four different cases. One can see, that the level of steady state is greater than the delay free in our case if $\tau_\delta > 0$, otherwise it is 2-10 % smaller depending on the value of the delay and α.

4. STABILITY OF THE STEADY STATE

To investigate the stability of different equilibriums, one must produce the linearized version of the fundamental equation (13) respect to small perturbations in the capital k and its delayed versions around the investigated equilibrium:

$$
\begin{aligned}
\dot{k}(t) =& s\Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k^*\right)\left(k(t-\tau_K)-k^*\right) \\
&- \delta\Lambda_\delta k\left(k(t-\tau_\delta)-k^*\right) - n\left(k(t)-k^*\right) ,
\end{aligned}
\tag{18}
$$

where prime denotes derivative of a function respect to its variable. Applying the Cobb-Douglas production function, we get

$$
\begin{aligned}
\dot{k}(t) =& sA\Lambda_L\left(\frac{\Lambda_K}{\Lambda_L}\right)^{\alpha} k^{*\alpha-1}\left(k(t-\tau_K)-k^*\right) \\
&- \delta\Lambda_\delta\left(k(t-\tau_\delta)-k^*\right) - n\left(k(t)-k^*\right) .
\end{aligned}
\tag{19}
$$

4.1 Stability of the trivial solution

First, we deal with the stability of the trivial solution, which is unstable in the traditional delay free Solow-Swan model. Although, the linearizion around the zero steady-state does not exist, the right hand side limit to reach the linearized version of the fundamental equation (13)

$$
\begin{aligned}
\dot{k}(t) =& sA\Lambda_L\left(\frac{\Lambda_K}{\Lambda_L}\right)^{\alpha} \lim_{k^*\to 0+}\left(k^{*\alpha-1}\right) k(t-\tau_K) \\
&- \delta\Lambda_\delta k(t-\tau_\delta) - nk(t) .
\end{aligned}
\tag{20}
$$

Where the coefficient of $k(t-\tau_K)$ is $+\infty$ as saving rate is positive ($s>0$), beside the positivity of the level of technology ($A>0$), and all the other parameters including the delays are finite. Thus we can say, that this equilibrium remains unstable after the introduction of the delay, which result was verified via numerical simulations.

4.2 Stability of the steady state with immediate depreciation

The linearized fundamental equation around the steady state with immediate depreciation $\tau_\delta = 0$ is

$$
\dot{\tilde{k}}(t) = \alpha\left(n+\delta\right)\tilde{k}(t-\tau_K) - \left(n+\delta\right)\tilde{k}(t) . \tag{21}
$$

Note that, here a new variable is used describing the difference between the actual capital and its steady state

$$
\tilde{k}(t) = k(t) - k^* . \tag{22}
$$

The stability investigation of linear delayed equations can be handled by means of Laplace transformation see Stepan (1989) as an equivalent method of the exponential trial solution. The corresponding characteristic equation:

$$
D(\lambda) := \lambda - \alpha\left(n+\delta\right)\mathrm{e}^{-\lambda\tau_K} + \left(n+\delta\right) = 0 , \tag{23}
$$

where λ is the characteristic exponent.

Two different type of stability boundaries can be distinguished: a fold (F) bifurcation, where both the real and imaginary parts of the characteristic exponent λ are zero, and Hopf bifurcation belongs to the pure complex pair of characteristic roots. In this model fold (F) bifurcation occurs if

$$
D(\lambda=0) = -\alpha\left(n+\delta\right) + \left(n+\delta\right) , \tag{24}
$$

resulting

$$
\alpha_{\mathrm{F}} = 1 \quad \text{and} \quad \left(n+\delta\right)_{\mathrm{F}} = 0 . \tag{25}
$$

boundaries. Note that, without the time delay only these two boundaries arise in the system, thus there is no chance to dynamic loss of stability or existence of limit cycles as it is usual in 1 dimensional systems.

But in a 1 dimensional delayed system dynamic loss of stability arises for parameters satisfying equation $D(\lambda = i\omega) = 0$, where $\omega \in \mathbb{R}$ is the frequency of the possibly arising periodic solution. According to the D-subdivision method, this complex characteristic equation can be separated into real and imaginary parts

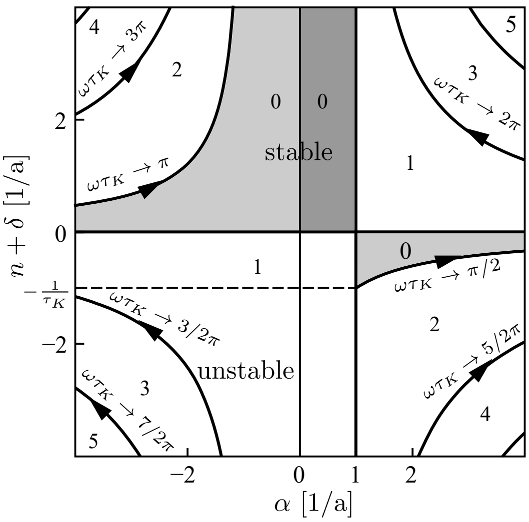

Fig. 2. Stability chart of the delayed Solow-Swan model for $\tau_\delta = 0$. The numbers indicate the number of unstable characteristic exponents, and the economically relevant regime ($0 < \alpha < 1$) is highlighted.

$$
\begin{aligned}
R(\omega) &= \left(n+\delta\right)\left(1-\alpha\cos\left(\omega\tau_K\right)\right) , \\
S(\omega) &= \omega + \alpha\left(n+\delta\right)\sin\left(\omega\tau_K\right) ,
\end{aligned}
\tag{26}
$$

respectively, from where the critical parameters can be expressed:

$$
\alpha_{\mathrm{H}} = \frac{1}{\cos\left(\omega\tau_K\right)} , \text{ and } \left(n+\delta\right)_{\mathrm{H}} = -\frac{\omega}{\tan\left(\omega\tau_K\right)} . \tag{27}
$$

Based on expressions (25) and (27) one can construct the stability chart in the plane of α and $(n+\delta)$, see Fig. 2. The the number of the unstable characteristic roots, (N, ie. with positive real part) indicated in the figure. These are computed based on the formula

$$
N = \frac{1}{2} + \sum_{j=1}^{r-1}(-1)^j \operatorname{sign}\left(R(\sigma_j)\right) + \frac{1}{2}(-1)^r \operatorname{sign}\left(R(0)\right) \tag{28}
$$

form Stepan (1989), where σ_j, $j = 1..r$ are the $S = 0$ equation negative real roots for ω. The results were checked via the semi discretization method see Insperger and Stepan (2011).

In Fig. 2 the steady sate is stable if the parameters are chosen from the shaded domain, where the economically relevant regime (ie. $0 < \alpha < 1$) is highlighted. Based on this, one can see, that the steady state of a real economy cannot go through a Hopf bifurcation if capital is depreciating immediately. The number of unstable roots and the frequency of the arising periodic motion (ω) are also denoted in Fig. 2.

4.3 Stability of the steady state with delayed depreciation

In this subsection, we investigate the case, where the capital depreciates by using, so

$$
\tau_\delta = \tau_L . \tag{29}
$$

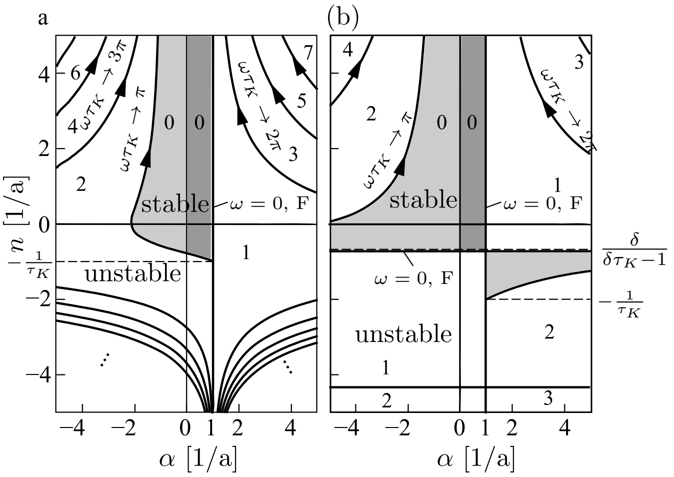

Fig. 3. Stability chart of the delayed Solow-Swan model for $\tau_\delta = \tau_K$ and the used parameters are $\delta = 0.5$ 1/a and $\tau_K = 1$ a for panel (a) and $\delta = 0.5$ 1/a and $\tau_K = 0.5$ a for panel (b). The numbers indicate the number of unstable characteristic exponents, and the economically relevant regime is highlighted.

The linearized equation around the steady state is

$$\dot{\tilde{k}}(t) = \left((\alpha-1)\,\delta\Lambda_\delta + \alpha n\right)\tilde{k}(t-\tau_K) - n\tilde{k}(t) \qquad (30)$$

and the according characteristic equation is

$$D(\lambda) := \lambda - \left((\alpha-1)\,\delta\Lambda_\delta + \alpha n\right)\mathrm{e}^{-\lambda\tau_K} + n = 0\,. \qquad (31)$$

Fold bifurcation may occur at

$$\alpha_\mathrm{F} = 1 \quad \text{and} \quad \left(n + \delta\mathrm{e}^{-n\tau_K}\right)_\mathrm{F} = 0\,, \qquad (32)$$

while the dynamic (Hopf) boundary characterized by

$$\begin{aligned} \alpha_\mathrm{H} &= 1 - \frac{\omega\tan\left(\frac{\omega\tau_K}{2}\right)}{n_\mathrm{H} + \delta\mathrm{e}^{-n_\mathrm{H}\tau_K}}\,, \\ n_\mathrm{H} &= -\frac{\omega}{\tan(\omega\tau_K)}\,, \end{aligned} \qquad (33)$$

based on the real and imaginary parts of the characteristic equation $D(\lambda = i\omega) = 0$

$$\begin{aligned} R(\omega) &:= n - \left((\alpha-1)\,\delta\mathrm{e}^{-n\tau_K} + \alpha n\right)\cos(\omega\tau_K) = 0\,, \\ S(\omega) &:= \omega + \left((\alpha-1)\,\delta\mathrm{e}^{-n\tau_K} + \alpha n\right)\sin(\omega\tau_K)\,. \end{aligned} \qquad (34)$$

The second Fold boundary, relating to the increasing rate of the number of people, does not exist if

$$\tau_K > \frac{1}{\mathrm{e}\delta}\,, \qquad (35)$$

so basically two topologically different stability chart can be constructed as it is presented in Fig. 3. Based on Taylor series expansion around zero delay, a linear approximation can be given for this fold bifurcation:

$$n_{\mathrm{F},1} \approx \frac{\delta}{\delta\tau_K - 1}\,. \qquad (36)$$

Note that, for realistic parameters (i.e. $\delta = 0.02$ 1/a and the delay is 1/4 a) the stability charts looks similar as it is shown by Fig. 4. Accordingly, there is no chance for Hopf bifurcation or periodic motion even for decreasing population.

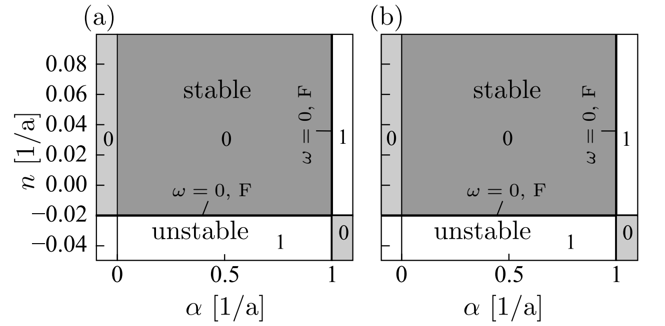

Fig. 4. Stability chart of the delayed Solow-Swan model for realistic parameters ($\delta = 0.02$ 1/a and $\tau_K (= \tau_\delta) = 1/4$ a). Panel (a) belongs to the immediate and panel (b) to the delayed depreciation. The numbers indicate the number of unstable characteristic exponents, and the economically relevant regime is highlighted.

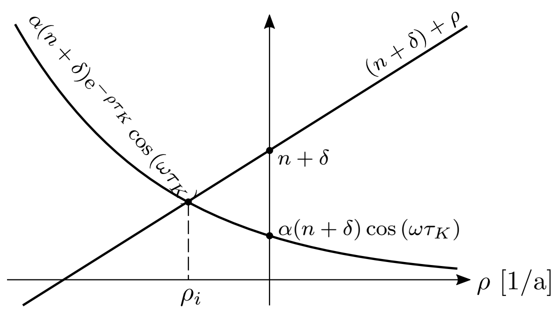

Fig. 5. Graphical solution: explanation for the solution of the real part of equation (39).

5. RATE OF CONVERGENCE IN TERMS OF THE DELAY

In this section, the convergence rate (λ) will be approximated, since one criticism of the Solow-Swan model is its predicted too fast convergence according to Mankiw et al. (1992).

To determine the rate of convergence

$$\lambda_k = -\frac{\dot{k}(t)}{k(t) - k^*} \equiv -\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \qquad (37)$$

in case of a delayed equation, the characteristic exponent ($\lambda = \rho + i\omega$, $\rho, \omega \in \mathbb{R}$) with the largest real part (ρ) must be found. The solution can be approximated with

$$\tilde{k}(t) = \sum_{i=1}^{\infty} C_i \mathrm{e}^{\lambda_i t} \approx C_1 \mathrm{e}^{\lambda_1 t}\,, \qquad (38)$$

since the terms with smaller real part (larger absolute value) decays faster.

5.1 Convergence with immediate depreciation

To find this characteristic exponent in case of the system with immediate depreciation, one must solve the real and imaginary part of the characteristic equation (23):

$$\begin{aligned} R(\rho + i\omega) &= \rho - \alpha(n+\delta)\mathrm{e}^{-\rho\tau_K}\cos(\omega\tau_K) + (n+\delta) = 0\,, \\ S(\rho + i\omega) &= \omega + \alpha(n+\delta)\mathrm{e}^{-\rho\tau_K}\sin(\omega\tau_K) = 0\,. \end{aligned} \qquad (39)$$

Although the equation on the real part cannot be solved explicitly, it can be done graphically for a given ω see Fig. 5, where the parameters are form the stable regime of Fig. 2, namely $n+\delta > 0$ and $0 < \alpha < 1$. One can see, that the value of ρ is negative (belonging to a stable equilibrium), and it is the largest if $\omega = 2l\pi/\tau_K$, $l \in \mathbb{Z}$. But from these, only $\omega = 0$ satisfies the equation $S(\rho+i\omega)$. Thus, the solution

$$\tilde{k}(t) \approx C_1 \mathrm{e}^{\rho_1 t} \tag{40}$$

is decaying rather than oscillatory.

Using the approximated solution (40) and substituted back into the linearized equation of motion (21), one can approximate the rate of convergence as

$$\lambda_k \approx \left(1 - \alpha \mathrm{e}^{-\rho\tau_K}\right)(n+\delta) \ . \tag{41}$$

Note that, this value is always smaller than the rate of convergence without delay, since ρ is negative.

The value of $\mathrm{e}^{-\rho\tau_K}$ can be approximated linearly by means of Taylor expansion of $R(\rho + i0)$ from equation (39) with respect to $\mathrm{e}^{-\rho\tau_K}$ around 1, (where $\rho = -\log\left(\mathrm{e}^{-\rho\tau_K}\right)/\tau_k$). This results

$$\mathrm{e}^{-\rho\tau_K} \approx \frac{1+\tau_K(n+\delta)}{1+\alpha\tau_K(n+\delta)} \ , \tag{42}$$

leading to

$$\lambda_k \approx \frac{(1-\alpha)(n+\delta)}{1+\alpha\tau_K(n+\delta)} \ . \tag{43}$$

The effect of delay on the rate of convergence is analyzed via the relative deviation from the delay free case:

$$\Delta\lambda_k = \frac{\lambda_k}{\lambda_k|_{\tau_K=0}} \ . \tag{44}$$

The delay makes the convergence slower approximately 1-2 % for realistic parameters (n=δ =2 %/a) as it shown in Fig. 6.(a), where the results are validated via numerical approximation of the characteristic exponent with the largest real part.

5.2 Convergence with delayed depreciation

The same derivation can be repeated for delayed depreciation, so only the key points and the results are published here.

The real and imaginary part of the characteristic equation (31) with a general characteristic exponent

$$R(\rho+i\omega) := \rho + n - \left((\alpha-1)\,\delta \mathrm{e}^{-n\tau_K} + \alpha n\right) \times \mathrm{e}^{-\rho\tau_K}\cos(\omega\tau_K) \tag{45}$$

$$S(\rho+i\omega) := \omega + \left((\alpha-1)\,\delta \mathrm{e}^{-n\tau_K} + \alpha n\right)\mathrm{e}^{-\rho\tau_K}\sin(\omega\tau_K) \ ,$$

from where the exponential term is approximately

$$\mathrm{e}^{-\rho\tau_K} \approx \frac{1+n\tau_K}{1+\tau_K\left(\alpha\left(\delta\mathrm{e}^{-n\tau_K}+n\right) - \delta\mathrm{e}^{-n\tau_K}\right)} \ . \tag{46}$$

Finally, one can approximate the rate of convergence

$$\lambda_k \approx \frac{(1-\alpha)\left(\delta\mathrm{e}^{-n\tau_K}+n\right)}{1+\tau_K\left(\alpha\left(\delta\mathrm{e}^{-n\tau_K}+n\right)-\delta\mathrm{e}^{-n\tau_K}\right)} \tag{47}$$

for the case where delayed depreciation is assumed. The delay cause similar deceleration in the convergence, this is plotted in Fig. 6.(b) by means of the relative deviation $\Delta\lambda_k$.

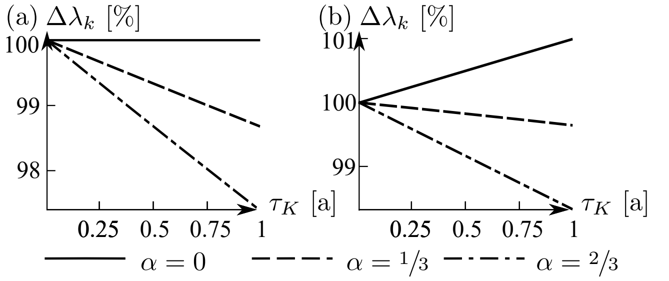

Fig. 6. Relative rate of convergence respect to the delay free case. Panel (a) belongs to the immediate and panel (b) to the delayed depreciation.

Note that, the changing of the rate of convergence due to the delay is significant for large delays, which was typical before the industrial revolution explaining the slow economic growth at that time.

5.3 Convergence of the outcome

The convergence of the outcome can be explained via the linear estimation of the dynamics of the outcome near to the steady state. Taking the derivative of the intensive form of the outcome (5) respect to time results

$$\dot{y}(t) = \Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k(t-\tau_K)\right)\Bigg(s\Lambda_L f\left(\frac{\Lambda_K}{\Lambda_L}k(t-2\tau_K)\right) - \delta\Lambda_\delta k(t-\tau_K-\tau_\delta) - nk(t-\tau_K)\Bigg) \ , \tag{48}$$

a function of $k(t-\tau_K)$, $k(t-2\tau_K)$ and $k(t-\tau_K-\tau_\delta)$.

Achieving the linear estimation via the Taylor series expansion around the steady state k^*, we found

$$\begin{aligned}\dot{y}(t) \approx \dot{y}^* &+ \left.\frac{\partial\dot{y}(t)}{\partial k(t-\tau_K)}\right|_{k^*}(k(t-\tau_K)-k^*) \\ &+ \left.\frac{\partial\dot{y}(t)}{\partial k(t-2\tau_K)}\right|_{k^*}(k(t-2\tau_K)-k^*) \\ &+ \left.\frac{\partial\dot{y}(t)}{\partial k(t-\tau_K-\tau_\delta)}\right|_{k^*}(k(t-\tau_K-\tau_\delta)-k^*) \ ,\end{aligned} \tag{49}$$

where $\dot{y}^*$ is zero, since the steady state is an equilibrium and

$$\begin{aligned}\left.\frac{\partial\dot{y}(t)}{\partial k(t-\tau_K)}\right|_{k^*} &= -n\Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k^*\right) \ , \\ \left.\frac{\partial\dot{y}(t)}{\partial k(t-2\tau_K)}\right|_{k^*} &= s\Lambda_K^2 f'^2\left(\frac{\Lambda_K}{\Lambda_L}k^*\right) \quad \text{and} \\ \left.\frac{\partial\dot{y}(t)}{\partial k(t-\tau_K-\tau_\delta)}\right|_{k^*} &= -\delta\Lambda_\delta\Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k^*\right) \ .\end{aligned} \tag{50}$$

Moreover, based on the Taylor series expansion of the intensive outcome itself (5) around the steady state k^*, the following result is obtained:

$$y(.) \approx y^* + \Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k^*\right)(k(.)-k^*) \ , \tag{51}$$

where the steady state of the GDP per capita is

$$y^* := \Lambda_L f\left(\frac{\Lambda_K}{\Lambda_L}k^*\right) \ . \tag{52}$$

With the results of expressions (48)-(52), the governing equation of the outcome around the steady state can be given as

$$\begin{aligned}\dot{y}(t) \approx & s\Lambda_K f'\left(\frac{\Lambda_K}{\Lambda_L}k^*\right)(y(t-\tau_K)-y^*) \\ & -\delta\Lambda_\delta k\,(y(t-\tau_\delta)-y^*) - n\,(y(t)-y^*)\ ,\end{aligned} \tag{53}$$

which is formally identical with the linearized version of the fundamental equation of the delayed Solow-Swan model (18). Thus, the rate of convergence of the outcome behaves exactly the same way as the capital's one does. To reach these results, the same procedure can be repeated.

6. CONCLUSION

This paper analyze a neoclassical growth model with delayed Cobb-Douglas production function. The steady state usually decreases as the delay increases. It has been proved, that it can increases for small delays if and only if the share of the capital is less than a critical value.

The steady state with immediate depreciation (also with zero depreciation) can lose its stability just through fold bifurcation as opposed to Guerrini (2012) found it. Hopf bifurcation can take place if and only if the depreciation is delayed. Generally, one can say, that although the delay allows Hopf bifurcation it does not appears for realistic parameters, even for negative population growth.

It had been shown, that the linear estimation of the convergence rate of the capital and the output is the same. The delay generally makes the convergence slower, resulting a more realistic model, since the too fast convergence was a main criticism against the original Solow-Swan model according to Mankiw et al. (1992).

This model, among others, can explain the slower economic growth of the countries before the industrial revolution, since the time delay, according to the time to build phenomenon, was larger than nowadays. And for huge delays both the steady state is small and the convergence is slow, resulting tiny economic growth.

For further research, this model can be extended with technological progress or human capital. Moreover, the AK model can be investigated similarly, which provides endogenous growth by avoiding diminishing returns to capital in the long run.

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