Embedding of Classical Polar Unitals in $\mathrm{PG}(2, q^2)$

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Abstract

A unital, that is, a block-design $2-(q^3+1, q+1, 1)$ is embedded in a projective plane Π of order q^2 if its points and blocks are points and lines of Π. A unital embedded in $\mathrm{PG}(2, q^2)$ is Hermitian if its points and blocks are the absolute points and lines of a unitary polarity of $\mathrm{PG}(2, q^2)$. A classical polar unital is a unital isomorphic, as a block-design, to a Hermitian unital. We prove that there exists only one embedding of the classical polar unital in $\mathrm{PG}(2, q^2)$, namely the Hermitian unital.

1 Introduction

A *unital* is defined to be a set of $q^3 + 1$ points equipped with a family of subsets, each of size $q + 1$, such that every pair of distinct points are contained in exactly one subset of the family. Such subsets are usually called *blocks* so that unitals are block-designs $2-(q^3 + 1, q + 1, 1)$. A unital is *embedded* in a

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projective plane Π of order q^2, if its points are points of Π and its blocks are lines of Π. Sufficient conditions for a unital to be embeddable in a projective plane are given in [8]. Computer aided searches suggest that there should be plenty of unitals, especially for small values of q, but those embeddable in a projective plane are quite rare, see [1, 3, 10]. In the Desarguesian projective plane $\mathrm{PG}(2, q^2)$, a unital arises from a unitary polarity in $\mathrm{PG}(2, q^2)$: the points of the unital are the absolute points, and the blocks are the non-absolute lines of the polarity. The name of "Hermitian unital" is commonly used for such a unital since its points are the points of the Hermitian curve defined over $\mathrm{GF}(q^2)$. A *classical polar unital* is a unital isomorphic, as a block-design, to a Hermitian unital. By definition, the classical polar unital can be embedded in $\mathrm{PG}(2, q^2)$ as the Hermitian unital, and it has been conjectured for a long time that this is the unique embedding of the classical polar unital in $\mathrm{PG}(2, q^2)$. Our goal is to prove this conjecture. Our notation and terminology are standard. The principal references on unitals are [2, 6].

2 Projections and Hermitian unital

Let $\mathcal{H}$ be a Hermitian unital in the Desarguesian plane $\mathrm{PG}(2, q^2)$. Any non-absolute line intersects $\mathcal{H}$ in a *Baer subline*, that is a set of $q + 1$ points isomorphic to $\mathrm{PG}(1, q)$. Take any two distinct non-absolute lines ℓ and ℓ'. For any point Q outside both ℓ and ℓ', the projection of ℓ to ℓ' from Q takes $\ell \cap \mathcal{H}$ to a Baer subline of ℓ'. We say that Q is a *full point with respect to the line pair* (ℓ, ℓ') if the projection from Q takes $\ell \cap \mathcal{H}$ to $\ell' \cap \mathcal{H}$.

From now on, we assume that ℓ and ℓ' meet in a point P of $\mathrm{PG}(2, q^2)$ not lying in $\mathcal{H}$. We denote the polar line of P with respect to the unitary polarity associated to $\mathcal{H}$ by $P^\perp$. Then $P^\perp$ is a non-absolute line. We will prove that if q is even then $P^\perp \cap \mathcal{H}$ contains a unique full point. This does not hold true for odd q. In fact, we will prove that for odd q, $P^\perp \cap \mathcal{H}$ contains zero or two full points depending on the mutual position of ℓ and ℓ'.

To work out our proofs we need some notation and known results regarding $\mathcal{H}$ and the projective unitary group $\mathrm{PGU}(3, q)$ preserving $\mathcal{H}$.

Up to a change of the homogeneous coordinate system (X_1, X_2, X_3) in $\mathrm{PG}(2, q^2)$, the points of $\mathcal{H}$ are those satisfying the equation

$$X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0. \tag{1}$$

Since the unitary group $\mathrm{PGU}(3, q)$ preserving $\mathcal{H}$ acts transitively on the points of $\mathrm{PG}(2,q^2)$ not lying in $\mathcal{H}$, we may assume $P=(0,1,0)$. Then $P^\perp$ has equation $X_2=0$. Also, since the stabilizer of P in $PGU(3,q)$ acts transitively on the non-absolute lines through P, ℓ may be assumed to be the line of equation $X_3=0$.

In the affine plane $\mathrm{AG}(2,q^2)$ arising from $\mathrm{PG}(2,q^2)$ with respect to the line $X_3=0$, we use the coordinates (X,Y) where $X=X_1/X_3$ and $Y=X_2/X_3$. Then the points of $\mathcal{H}$ in $\mathrm{AG}(2,q^2)$ have affine coordinates (X,Y) that satisfy the equation

$$X^{q+1}+Y^{q+1}+1=0,$$

whereas the points of $\mathcal{H}$ at infinity are the $q+1$ points $M=(1,m,0)$ with $m^{q+1}+1=0$. In this setting the line ℓ' is a vertical line and hence it has equation $X-c=0$ where $c^{q+1}+1\neq 0$ as ℓ' is a non-absolute line. In the following, we will use ℓ_c to denote the line with equation $X-c=0$.

Fix a point Q of $\mathcal{H}$ lying on $P^\perp$. Then $Q=Q(a,0)$ with $a^{q+1}+1=0$. Take a point $M=(1,m,0)$ at infinity lying in $\mathcal{H}$, and project it to ℓ_c from Q. If the point $T=(c,t)$ is the result of the projection then $t=(c-a)m$. Therefore, T lies on $\mathcal{H}$ if and only if $ca^q+ac^q+2=0$.

2.1 The case q odd

Let q be a odd prime power. As $a^q=-a^{-1}$, $ca^q+ac^q+2=0$ can also be written in the form

$$a^2c^q+2a-c=0. \tag{2}$$

By abuse of notation, let $\sqrt{1+c^{q+1}}$ and $-\sqrt{1+c^{q+1}}$ denote the roots of the equation $Z^2=1+c^{q+1}$. Then the solutions of (2) are

$$a_{1,2}=\tfrac{-1\pm\sqrt{1+c^{q+1}}}{c^q}. \tag{3}$$

Here, $\sqrt{1+c^{q+1}}\in \mathrm{GF}(q)$ if and only if $1+c^{q+1}$ is a (non-zero) square element in $\mathrm{GF}(q)$. Actually, this case cannot occur. In fact, (2) together with $\sqrt{1+c^{q+1}}\in\mathrm{GF}(q)$ yield $c^qa+1=\pm\sqrt{1+c^{q+1}}$ whence

$$(c^qa+1)^{q+1}=(\sqrt{1+c^{q+1}})^{q+1}=(\sqrt{1+c^{q+1}})^2=1+c^{q+1}.$$

Expanding the left hand side and using $a^{q+1}=-1$ we obtain $ca^q+c^qa=2c^{q+1}$ whence $-c+c^qa^2-2ac^{q+1}=0$. Subtracting (2) gives either $1+c^{q+1}=0$, or $a=0$. The former case cannot occur by the choice of ℓ_c. In the latter case, $Q=(0,0)$ but the origin does not lie in $\mathcal{H}$.

Therefore, $\sqrt{1+c^{q+1}} \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. Hence $\sqrt{1+c^{q+1}} = iu$, with $u \in \mathrm{GF}(q)$ where $\mathrm{GF}(q^2)$ is considered as the quadratic extension of $\mathrm{GF}(q)$ by adjunction of a root i of the polynomial $X^2 - s$ with a fixed non-square element $s \in \mathrm{GF}(q)$. From $i^q = -i$, we get $(\sqrt{1+c^{q+1}})^q = -\sqrt{1+c^{q+1}}$. Hence

$$a_1^{q+1} = a_1^q a_1 = -a_1 a_2 = -\tfrac{(\sqrt{1+c^{q+1}}-1)(\sqrt{1+c^{q+1}}+1)}{c^{q+1}} = -1.$$

This shows that $Q_1 = (a_1, 0)$ lies in $\mathcal{H}$. Similarly, $Q_2 = (a_2, 0) \in \mathcal{H}$.

Since a_1 and a_2 do not depend on the choice of M, both points Q_1 and Q_2 are full points with respect to the line pair (ℓ, ℓ_c). The projection φ with center Q_1 which maps ℓ to ℓ_c takes the point $M = (1, m, 0)$ to the point $T' = (c, m(c - a_1))$, and the projection φ' with center Q_2 mapping ℓ_c to ℓ takes the point $T = (c, t)$ to the point $M' = (1, m', 0)$ with $m' = t(c - a_2)^{-1}$. Therefore, the product $\psi = \varphi' \circ \varphi$ is the automorphism of the line ℓ_c with equation

$$m' = d\,m, \tag{4}$$

where $d = \frac{c-a_1}{c-a_2} = -\frac{1-\sqrt{1+c^{q+1}}}{1+\sqrt{1+c^{q+1}}}$. We show that ψ^{q+1} is the identity automorphism of ℓ. From (4), ψ^{q+1} takes the point $M = (1, m, 0)$ to the point $\bar{M}(1, \bar{m}, 0)$, where $\bar{m} = d^{q+1} m$ with

$$d^{q+1} = \left(-\tfrac{1-\sqrt{1+c^{q+1}}}{1+\sqrt{1+c^{q+1}}}\right)^{q+1} = \left(-\tfrac{1-\sqrt{1+c^{q+1}}}{1+\sqrt{1+c^{q+1}}}\right)^{q} \left(-\tfrac{1-\sqrt{1+c^{q+1}}}{1+\sqrt{1+c^{q+1}}}\right).$$

Since $\sqrt{1+c^{q+1}}^{\,q} = -\sqrt{1+c^{q+1}}$ this yields $d = 1$.

Now we count the automorphisms ψ when c ranges over $\mathrm{GF}(q^2)$.

We show that each $u \in \mathrm{GF}(q)^*$ produces such an automorphism. Observe that $(iu)^2 = su^2$ is a non-square element in $\mathrm{GF}(q)$. As the norm function $x \mapsto x^{q+1}$ from $\mathrm{GF}(q^2)^*$ in $\mathrm{GF}(q)^*$ is surjective, $\mathrm{GF}(q^2)$ contains an a nonzero element c such that $su^2 = 1 + c^{q+1}$. Therefore, either $iu = \sqrt{1+c^{q+1}}$, or $iu = -\sqrt{1+c^{q+1}}$. With this notation,

$$m' = -\tfrac{1-iu}{1+iu} m. \tag{5}$$

Any two different choices of u in $\mathrm{GF}(q)^*$ produce two different automorphisms of ℓ. In fact, if $u, v \in \mathrm{GF}(q)^*$,

$$-\tfrac{1-iu}{1+iu} = -\tfrac{1-iv}{1+iv}$$

then $u = v$.

Therefore, we have produced as many as $q-1$ pairwise distinct nontrivial automorphisms ψ_u. A further nontrivial automorphism of ℓ preserving $\ell \cap \mathcal{H}$ is ψ_0 of equation $m' = -m$ which is the restriction on ℓ of the linear collineation $(X_1, X_2, X_3) \mapsto (X_1, -X_2, X_3)$ belonging to PGU$(3, q)$. In fact, ψ_0 occurs for $u = 0$ in (5). Furthermore, ψ_0 is an involution, and hence its $q+1$-st power is the identity. All these automorphisms together with the identity ψ_∞ form a set of $q+1$ automorphisms of ℓ which preserve $\ell \cap \mathcal{H}$. To show that they form a group Ψ, replace u with v/s in (5). Then (5) reads

$$m' = \tfrac{1-iv}{1+iv} m, \tag{6}$$

and the claim follows from the fact that the product of two such maps takes m to

$$\tfrac{1-iv}{1+iv} \tfrac{1-iw}{1+iw} m = \tfrac{1-iz}{1+iz} m$$

with

$$z = \tfrac{v+w}{1+svw}.$$

On other hand, the cyclic automorphism group of ℓ consisting of all maps of equation $m' = hm$ with $h \in GF(q^2)^*$ fixes $P = (0, 1, 0)$ and $R = (1, 0, 0)$. Therefore its subgroup Ψ is also cyclic, and leaves $\ell \cap \mathcal{H}$ invariant acting on it regularly.

2.2 The case q even

Let $q = 2^e \geq 4$. From $a^{q+1} + 1 = 0$ and $t = (a + c)m$, we have $a = \sqrt{\frac{c}{c^q}}$. Therefore, $T \in \mathcal{H}$ if and only if $a = \sqrt{\frac{c}{c^q}}$. This shows that a is independent of the choice of M on ℓ. Thus, Q is a full point for the line pair (ℓ, ℓ_c). It is easily seen that Q is also a full point for the pair (ℓ_c, ℓ).

Take two distinct non-absolute lines ℓ_{c_1} and ℓ_{c_2} through P with $c_1 \neq 0 \neq c_2$, and let

$$\gamma(c_1, c_2) = c_2(1 + c_1^{q+1}) + c_1(1 + c_2^{q+1}).$$

A straightforward computation shows that $Q = (a, 0)$ with $a^{q+1} + 1 = 0$ is the full point for the line pair (ℓ_{c_1}, ℓ_{c_2}) if and only if

$$a = \sqrt{\frac{\gamma(c_1, c_2)}{\gamma(c_1, c_2)^q}}. \tag{7}$$

Furthermore, the projection with center Q which maps ℓ_{c_1} to ℓ_{c_2}, takes the point $M = (c_1, m)$ to the point $T = (c_2, m(a + c_1)/(a + c_2))$.

Take an element $s \in \mathrm{GF}(q)$ with absolute trace 1, and look at $\mathrm{GF}(q^2)$ as the quadratic extension of $\mathrm{GF}(q)$ arising from the (irreducible) polynomial $X^2 + X + s = 0$. Let i be one of the roots of this polynomial. Then the other root is i^q, and hence $i^q = 1 + i$. Furthermore, any element α of $\mathrm{GF}(q^2)$ is uniquely written as $x + iy$ with $x, y \in \mathrm{GF}(q)$, giving $\alpha^q = x + y + iy$ and $\alpha^{q+1} = x^2 + xy + sy^2$.

Lemma 2.1. *For any given $c_1 \in \mathrm{GF}(q^2)^*$, with $c_1^{q+1} \neq 1$, there exists only one further $c_2 \in \mathrm{GF}(q^2)^*$, with $c_2^{q+1} \neq 1$ such that*

$$\gamma(c_1, c_2) = c_2(1 + c_1^{q+1}) + c_1(1 + c_2^{q+1}) = 0. \tag{8}$$

In particular, $c_2 = c_1 t$, for some $t \in \mathrm{GF}(q)^*$.

Proof. Let $c_1 = x_1 + iy_1$ and $c_2 = x_2 + iy_2$. Then, $c_1^{q+1} = x_1^2 + x_1y_1 + sy_1^2$ and $c_2^{q+1} = x_2^2 + x_2y_2 + sy_2^2$.

Since

$$c_2(1 + c_1^{q+1}) = x_2(1 + x_1^2 + x_1y_1 + sy_1^2) + iy_2(1 + x_1^2 + x_1y_1 + sy_1^2)$$

and

$$c_1(1 + c_2^{q+1}) = x_1(1 + x_2^2 + x_2y_2 + sy_2^2) + iy_1(1 + x_2^2 + x_2y_2 + sy_2^2),$$

equation (8) holds if and only if

$$\begin{cases} x_2(1 + x_1^2 + x_1y_1 + sy_1^2) + x_1(1 + x_2^2 + x_2y_2 + sy_2^2) & = & 0 \\ y_2(1 + x_1^2 + x_1y_1 + sy_1^2) + y_1(1 + x_2^2 + x_2y_2 + sy_2^2) & = & 0. \end{cases}$$

If $x_1 = 0$ then $c_1 = iy_1$ with $sy_1 \neq 1$, and from the above equations, $x_2 = 0$ and y_2 is a root of the polynomial in ξ

$$sy_1\xi^2 + (1 + sy_1^2)\xi + y + 1. \tag{9}$$

Since y_1 is also a root of (9), y_1 and y_2 are the two roots and the assertion is proven in this case. If $y_1 = 0$, a similar argument can be used to prove the assertion.

Therefore $x_1 \neq 0 \neq y_1$ may be assumed. From

$$\begin{cases} y_1x_2(1 + x_1^2 + x_1y_1 + sy_1^2) + y_1x_1(1 + x_2^2 + x_2y_2 + sy_2^2) & = & 0 \\ x_1y_2(1 + x_1^2 + x_1y_1 + sy_1^2) + x_1y_1(1 + x_2^2 + x_2y_2 + sy_2^2) & = & 0 \end{cases} \tag{10}$$

we infer $y_1x_2 = x_1y_2$, that is, $y_2 = y_1x_2x_1^{-1}$. Replacing y_2 by $y_1x_2x_1^{-1}$ in the first equation of (10) shows that x_2 is a root of the polynomial in ξ

$$(x_1^2 + y_1x_1 + sy_1^2)x_1^{-1}\xi^2 + (1 + x_1^2 + x_1y_1 + sy_1^2)\xi + x_1 = 0. \qquad (11)$$

Since x_1 is another root of (11), x_1 and x_2 are the roots, and the assertion is proven. □

For the rest of this section, let

$$a_i = \sqrt{\frac{c_i}{c_i^q}}, \qquad i = 1, 2.$$

Project ℓ to ℓ_{c_1} from $Q_1(a_1, 0)$, then project ℓ_{c_1} to ℓ_{c_2} from $Q = (a, 0)$, and finally project ℓ_{c_2} to ℓ. The result is the automorphism ψ_{c_1,c_2} of the line ℓ, viewed as $\mathrm{PG}(1, q^2)$, defined by the equation

$$\psi_{c_1,c_2}((1, m, 0)) = (1, d(c_1, c_2)m, 0)$$

where

$$d(c_1, c_2) = \frac{(a + c_2)(a_1 + c_1)}{(a + c_1)(a_2 + c_2)}.$$

Using the definition of a, a_1, a_2, a straightforward computation gives $d(c_1, c_2)^2$ as a rational function of c_1 and c_2:

$$d(c_1, c_2)^2 = \frac{c_1c_2^q(1 + c_1^qc_2)}{c_1^qc_2(1 + c_1c_2^q)},$$

whence

$$d(c_1, c_2) = \sqrt{\frac{c_1c_2^q(1 + c_1^qc_2)}{c_1^qc_2(1 + c_1c_2^q)}}.$$

This also shows that $d(c_1, c_2)$ is of the form $\alpha^q/\alpha = \alpha^{q-1}$ for some $\alpha \in \mathrm{GF}(q^2)$. Hence $d^{q+1} = 1$.

Lemma 2.2. *Let $\alpha, \beta \in GF(q^2)^*$ with $\alpha + \alpha^{q+1} \neq 0 \neq \beta + \beta^{q+1}$. Then there exists $\delta \in \mathrm{GF}(q^2)^*$ such that*

$$\frac{\alpha^q + \alpha^{q+1}}{\alpha + \alpha^{q+1}} \cdot \frac{\beta^q + \beta^{q+1}}{\beta + \beta^{q+1}} = \frac{\delta^q + \delta^{q+1}}{\delta + \delta^{q+1}}.$$

Proof. If $\delta = a + ib$, then there exist $c, d \in \mathrm{GF}(q)$ such that

$$\frac{\delta^q + \delta^{q+1}}{\delta + \delta^{q+1}} = \frac{c + d + id}{c + id}.$$

Let $\alpha = x + iy$ and $\beta = u + iv$, with $x, y, u, v \in \mathrm{GF}(q)$. Then

$$\begin{aligned}(\alpha^q + \alpha^{q+1})(\beta^q + \beta^{q+1}) &= (x + y + x^2 + xy + sy^2)(u + v + u^2 + uv + sv^2) + svy \\ &\quad +i[(x + x^2 + xy + sy^2)v + (u + u^2 + uv + sv^2)y + yv]\end{aligned}$$

and the expression on the right hand side is equal to

$$\begin{aligned}&(x + x^2 + xy + sy^2)(u + u^2 + uv + sv^2) + svy \\ &+i[(x + x^2 + xy + sy^2)v + (u + u^2 + uv + sv^2)y + yv].\end{aligned}$$

Therefore,

$$\begin{aligned}&(x + x^2 + xy + sy^2)(u + u^2 + uv + sv^2) + svy \\ &+(x + x^2 + xy + sy^2)v + (u + u^2 + uv + sv^2)y + yv = \\ &(x + x^2 + xy + sy^2)(u + vu^2 + uv + sv^2)y(u + u^2 + uv + sv^2) + svy = \\ &(x + x^2 + xy + sy^2)v + (u + vu^2 + uv + sv^2)y + svy.\end{aligned}$$

□

Therefore, in the group $\mathrm{PGL}(2, q^2)$ of all automorphisms of ℓ, the maps ψ_{c_1,c_2}, with $c_1^{q+1} \neq 1 \neq c_2^{q+1}, \gamma(c_1, c_2) \neq 0$ form an abelian subgroup Ψ and the order of each automorphism in Ψ is divisible by $q + 1$.

A good choice for c_1, c_2 is $c_1 = s$ and $c_2 = is^{-1}$. In this case, $c_1^q c_2(1 + c_1 c_2^q) = i^2$ and $d(c_1, c_2) = i^{q-1}$. Hence $\psi_{c_1,c_2}((1, m, 0)) = (1, i^{q-1}m, 0)$. Since i^{q-1} is a primitive $(q + 1)$-st root of unity, Ψ contains a cyclic subgroup of order $q + 1$. Since Ψ leaves $\mathcal{H} \cap \ell$ invariant, this shows that Ψ acts on $\mathcal{H} \cap \ell$ regularly, and Ψ is a cyclic group of order $q + 1$.

3 Embedding of the polar classical unital in $\mathrm{PG}(2, q^2)$

Let $\mathcal{U}$ be a classical polar unital isomorphic, as design, to a Hermitian unital of $\mathrm{PG}(2, q^2)$. Assume that $\mathcal{U}$ is embedded in $\mathrm{PG}(2, q^2)$. Take any point P is outside $\mathcal{U}$. Since the arguments used in Section 2 only involve points, secants and their incidences, all assertions stated there for a Hermitian unital remains true for $\mathcal{U}$. This together with the results proven in Section 2 show that there is a cyclic automorphism group C_{q+1} of the line ℓ which preserves $\ell \cap \mathcal{U}$. We are not claiming that C_{q+1} extends to a collineation group of $\mathrm{PG}(2, q^2)$. We only use the facts that C_{q+1} consists of automorphisms leaving $\ell \cap \mathcal{U}$ invariant and that C_{q+1} acts on it regularly. By Dickson's classification of subgroups of $\mathrm{PGL}(2, q^2)$, see [12] or [7, Theorem A.8], the automorphism group of ℓ, we have that C_{q+1} is conjugate to the subgroup Σ consisting of all maps $m' = wm$ where $w^{q+1} = 1$. In other words, we can change the projective frame so that $\ell \cap \mathcal{U}$ becomes a (nontrivial) Σ-orbit. Since each nontrivial Σ-orbit is a Baer subline of ℓ, so is $\ell \cap \mathcal{U}$. As the unitary group $\mathrm{PGU}(3, q)$ acts transitively on the block of $\mathcal{U}$, we get that each block is a Baer subline, giving $\mathcal{U}$ is projectively equivalent to a Hermitian unital in $\mathrm{PG}(2, q^2)$, see [4, 9].

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