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A method for solving LSM problems of small size in the AHP*

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Abstract The Analytic Hierarchy Process (AHP) is one of the most popular methods used in Multi-Attribute Decision Making. It provides with ratio-scale measurements of the priorities of elements on the various levels of a hierarchy. These priorities are obtained through the pairwise comparisons of elements on one level with reference to each element on the immediate higher level.

The Eigenvector Method (EM) and some distance minimizing methods such as the Least Squares Method (LSM), Logarithmic Least Squares Method (LLSM), Weighted Least Squares Method (WLSM) and Chi Squares Method (X^2M) are of the tools for computing the priorities of the alternatives. This paper studies a method for generating all the solutions of the LSM problem for 3×3 matrices. We observe non-uniqueness and rank reversals by presenting numerical results.

1 Introduction

The Analytic Hierarchy Process was developed by Thomas L. Saaty [9]. It is a systematic procedure for representing the elements of any problem, hierarchically. It organizes the basic rationality by breaking down a problem into its smaller and smaller parts and then guides decision makers through a series of pairwise comparison judgments to express the relative strength or intensity of the impact of the elements in the hierarchy. These judgments are translated into numbers.

We will study only one part of the decision problem, i.e. when one matrix is obtained from pairwise comparisons.

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First, let us suppose that we have n alternatives and we know all the alternatives with respect to a criterion, denoting them by $w_1, w_2, w_3, \ldots, w_n$. Define the matrix of weight ratios as $W = [w_{ij}]_{n\times n} = [\frac{w_i}{w_j}]_{n\times n}$:

$$\begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \ldots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \ldots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & 1 & \ldots & \frac{w_3}{w_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \ldots & 1 \end{pmatrix}.$$

Note that for any i, j, k indices

$$w_{ij} > 0, \tag{1}$$

$$w_{ij} = \frac{1}{w_{ji}}, \tag{2}$$

$$w_{ij} = w_{ik} w_{kj}. \tag{3}$$

A matrix is called consistent if its components satisfy (3) for any $i, j, k = 1, \ldots, n$.

In real life applications, the decision maker's responses are not perfect, human judgments may contain some inconsistency, and we have an $n \times n$ positive reciprocal matrix in the form

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & 1 & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & 1 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & 1 \end{pmatrix},$$

where for any $i, j = 1, \ldots, n$,

$$a_{ij} > 0,$$

$$a_{ij} = \frac{1}{a_{ji}}.$$

Once we have matrices from pairwise comparisons, we want to find a weight vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_+$ representing the priorities, where $\mathbb{R}^n_+$ is the positive orthant.

1.1 Eigenvector Method (EM)

Saaty calculated the weights by the Eigenvector Method (EM) which computes the largest real eigenvalue λ_{max} of A, and w is the right-hand eigenvector of A corresponding to λ_{max}.

Saaty's Consistency Index (CI) is defined as

$$CI = \frac{\lambda_{max} - n}{n - 1}.$$

By computing this ratio for random matrices of the same size and taking the expected value, we obtain $MRCI_n$. The Consistency Ratio (CR) is defined as

$$CR = \frac{CI}{MRCI_n}.$$

Saaty suggested that a Consistency Ratio of about 10% or less should be usually considered acceptable.

Some distance minimizing methods are known for estimating the priorities. The Least Squares Method (LSM) and Weighted Least Squares Method (WLSM) was proposed by Chu, Kalaba and Spingarn [2].

1.2 Least Squares Method (LSM)

$$\begin{aligned} \min \sum_{i=1}^{n} \sum_{j=1}^{n} & \left(a_{ij} - \frac{w_i}{w_j} \right)^2 \\ \sum_{i=1}^{n} & w_i = 1, \\ & w_i > 0, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

1.3 Weighted Least Squares Method (WLSM)

$$\begin{aligned} \min \sum_{i=1}^{n} \sum_{j=1}^{n} & (w_j a_{ij} - w_i)^2 \\ \sum_{i=1}^{n} & w_i = 1, \\ & w_i > 0, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

Logarithmic Least Squares Method (LLSM) was introduced by DeJong [4] and Crawford and Williams [3].

1.4 Logarithmic Least Squares Method (LLSM)

$$\min \sum_{i=1}^{n} \sum_{i<j} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right) \right]^2$$
$$\prod_{i=1}^{n} w_i = 1,$$
$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

Chi Square Method (X^2M) was defined by Jensen [6].

1.5 Chi Square Method (X^2M)

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\left(a_{ij} - \frac{w_i}{w_j}\right)^2}{\frac{w_i}{w_j}}$$
$$\sum_{i=1}^{n} w_i = 1,$$
$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

In the case of consistent matrices, all these methods — including EM — give the same solution. In inconsistent cases, the priorities depend on the applied method. All of them, except for LSM, has unique solution which can be quite easily computed.

An explicit formula exists, e.g., for the LLSM's solution [3].

Theorem 1 *The optimal solution of LLSM can be obtained from the geometric mean of the elements of rows:*

$$w_i^{LLSM} = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \qquad i = 1, 2, \ldots, n.$$

One of the proofs can be found in [1]. An advantage of LLSM is the easy way of computing the priorities.

LSM, however, is rather difficult to solve because the objective function is nonlinear and usually nonconvex, moreover, no unique solution exists

[6,7] and the solutions are not easily computable. As Jensen wrote in [7, p.319.], “there is no closed-form solution to this problem, but a solution to any desired degree of accuracy may be obtained iteratively.” Farkas [5] applied Newton’s method of successive approximation.

2 Resultant method

In this section, a general method [8] is presented for solving systems of nonlinear equations of small size, then it will be used for solving the LSM problem.

Definition 1 *A monomial in $x_1, x_2, \dots, x_n$ is a product of the form*

$$x_1{}^{\alpha_1} x_2{}^{\alpha_2} \cdots x_n{}^{\alpha_n},$$

where all of the exponents $\alpha_1, \alpha_2, \dots, \alpha_n$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1 + \alpha_2 + \dots + \alpha_n$.

Definition 2 *Let $\mathbb{K}$ be a field. A polynomial f in $x_1, x_2, \dots, x_n$ with coefficients in $\mathbb{K}$ is a finite linear combination of monomials, e.g., f can be written in the form*

$$f = \sum_{(\alpha_1,\alpha_2,\dots,\alpha_n)} c_{\alpha_1,\alpha_2,\dots,\alpha_n} x_1{}^{\alpha_1} x_2{}^{\alpha_2} \cdots x_n{}^{\alpha_n},$$

where the sum is over a finite number of n-tuples $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $c_{\alpha_1,\alpha_2,\dots,\alpha_n} \in \mathbb{K}$. The set of all polynomials in $x_1, x_2, \dots, x_n$ with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}[x_1, x_2, \dots, x_n]$.

In this section $\mathbb{R}[x]$, $\mathbb{C}[x]$ and $\mathbb{R}[x, y]$ are used. First, recall one of Gauss’ main results.

Theorem 2 *(The Fundamental Theorem of Algebra) Every nonconstant polynomial $f \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.*

Now, let $f, g \in \mathbb{R}[x]$ be polynomials in one variable with real coefficients:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n,$$
$$g(x) = b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + a_m,$$

where $a_0 \neq 0$, $b_0 \neq 0$. As it is known from the Fundamental Theorem of Algebra f and g can be written as

$$f(x) = a_0 \prod_{i=1}^{n} (x - \alpha_i), \tag{4}$$

$$g(x) = b_0 \prod_{j=1}^{m} (x - \beta_j), \tag{5}$$

where $\alpha_i, \beta_j \in \mathbb{C}$, $\quad i = 1, \dots, n$, $j = 1, \dots, m$.

Definition 3 *The resultant of f and g denoted by $R(f,g)$ is*

$$R(f,g) = a_0^m b_0^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j). \tag{6}$$

From (5),

$$g(\alpha_i) = b_0 \prod_{j=1}^{m} (\alpha_i - \beta_j),$$

and an equivalent form of $R(f,g)$ is

$$R(f,g) = a_0^m \prod_{i=1}^{n} g(\alpha_i).$$

It is easy to see that f and g have common roots in $\mathbb{C}$ if and only if $R(f,g) = 0$. Observe that the definition of the resultant is not symmetric in f and g, however,

$$R(g,f) = b_0^n a_0^m \prod_{j=1}^{m} \prod_{i=1}^{n} (\beta_j - \alpha_i) = (-1)^{nm} R(f,g).$$

An equivalent formula for $R(g,f)$ can be given as follows:

$$R(g,f) = b_0^n \prod_{j=1}^{m} f(\beta_j).$$

By the following theorem, $R(f,g)$ can be computed not only from the roots of f and g but also from the coefficients of f and g.

Theorem 3 *[8] Let D denote the determinant of the Sylvester matrix of f and g given by*

$$D = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n & & & \\ & a_0 & a_1 & \dots & a_{n-1} & a_n & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_m & & & \\ & b_0 & b_1 & \dots & b_{m-1} & b_m & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_0 & b_1 & b_2 & \dots & b_m \end{vmatrix}_{(n+m)\times(n+m)},$$

where the empty spaces are filled by zeros. Then,

$$D = R(f,g).$$

Example 1 Let f and g be

$$f(x) = x^2 - 4x - 5,$$
$$g(x) = x^2 - 7x + 10.$$

By Theorem 3,

$$R(f,g) = \begin{vmatrix} 1 & -4 & -5 & 0 \\ 0 & 1 & -4 & -5 \\ 1 & -7 & 10 & 0 \\ 0 & 1 & -7 & 10 \end{vmatrix} = 0,$$

so f and g have a common root. This result is obtained without computing the roots of f and g. ($x = 5$ is a common root.)

Now, suppose that we have the following system:

$$f(x,y) = 0 \quad (7)$$
$$g(x,y) = 0, \quad (8)$$

where $f, g \in \mathbb{R}[x,y]$ are polynomials in two variables x, y with real coefficients. If we consider only x as a variable, then the elements of f and g can be sorted by the powers of x in decreasing order:

$$f(x,y) = a_0(y)x^k + a_1(y)x^{k-1} + \ldots + a_{k-1}(y)x + a_k(y), \quad (9)$$
$$g(x,y) = b_0(y)x^l + b_1(y)x^{l-1} + \ldots + b_{l-1}(y)x + a_l(y), \quad (10)$$

where $a_0(y), a_1(y), \ldots, a_k(y), b_0(y), b_1(y), \ldots, b_l(y) \in \mathbb{R}[y]$ are polynomials in y with real coefficients. We can write $R_x(f,g)$ as the resultant of two polynomials f and g in one variable. Then,

$$R_x(f,g) = \begin{vmatrix} a_0(y) & a_1(y) & a_2(y) & \ldots & a_k(y) & & & \\ & a_0(y) & a_1(y) & \ldots & a_{k-1}(y) & a_k(y) & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_0(y) & a_1(y) & a_2(y) & \ldots & a_k(y) \\ b_0(y) & b_1(y) & b_2(y) & \ldots & b_l(y) & & & \\ & b_0(y) & b_1(y) & \ldots & b_{l-1}(y) & b_l(y) & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_0(y) & b_1(y) & b_2(y) & \ldots & b_l(y) \end{vmatrix} = P(y),$$

where $P(y) \in \mathbb{R}[y]$ is a polynomial in y with real coefficients.

Suppose that system (7) – (8) has a solution $x = \alpha, y = \beta$. Substituting β for y in (9) – (10), we have two polynomials in one variable $f(x,\beta)$ and $g(x,\beta)$. These polynomials have the common root α. If the leading coefficients $a_0(\beta)$ and $b_0(\beta)$ are not zeros, then the resultant of $f(x,\beta)$ and

$g(x,\beta)$ can be written in the form

$$R(f(x,\beta),g(x,\beta)) = \begin{vmatrix} a_0(\beta) & a_1(\beta) & a_2(\beta) & \dots & a_k(\beta) & & & & \\ & a_0(\beta) & a_1(\beta) & \dots & a_{k-1}(\beta) & a_k(\beta) & & & \\ & & \ddots & \ddots & & & \ddots & & \\ & & & & a_0(\beta) & a_1(\beta) & a_2(\beta) & \dots & a_k(\beta) \\ b_0(\beta) & b_1(\beta) & b_2(\beta) & \dots & b_l(\beta) & & & & \\ & b_0(\beta) & b_1(\beta) & \dots & b_{l-1}(\beta) & b_l(\beta) & & & \\ & & \ddots & \ddots & & & \ddots & & \\ & & & & b_0(\beta) & b_1(\beta) & b_2(\beta) & \dots & b_l(\beta) \end{vmatrix},$$

which is equal to $P(\beta)$. Since α is a common root of $f(x,\beta)$ and $g(x,\beta)$, we got that β should be a root of P.

On the other hand, suppose that $P(y) = R_x(f,g)$ has a root β. We have to check the leading coefficients. If $a_0(\beta)$ and $b_0(\beta)$ are not zero, then $P(\beta)$ equals $R(f(x,\beta),g(x,\beta))$. $P(\beta)=0$, thus $f(x,\beta)$ and $g(x,\beta)$ have a common root.

We can conclude that the resultant method is a possible tool for solving nonlinear systems of small size.

3 Solving the LSM problem

Now, we derive a method for generating all the solutions of the LSM minimization problem for 3×3 matrices. Suppose that A is a 3×3 matrix obtained from pairwise comparisons in the form

$$A = \begin{pmatrix} 1 & a & b \\ \frac{1}{a} & 1 & c \\ \frac{1}{b} & \frac{1}{c} & 1 \end{pmatrix}.$$

The aim is to find a positive reciprocal consistent matrix X in the form

$$X = \begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & 1 \end{pmatrix},$$

which minimizes the Frobenius norm

$$\| A - X \|_F^2 = \left(a - \frac{w_1}{w_2}\right)^2 + \left(b - \frac{w_1}{w_3}\right)^2 + \left(\frac{1}{a} - \frac{w_2}{w_1}\right)^2 + \left(c - \frac{w_2}{w_3}\right)^2 + \left(\frac{1}{b} - \frac{w_3}{w_1}\right)^2 + \left(\frac{1}{c} - \frac{w_3}{w_2}\right)^2,$$

where $w_1, w_2, w_3 > 0$. Introducing new variables x, y

$$x = \frac{w_1}{w_2} \tag{11}$$

$$y = \frac{w_2}{w_3}, \tag{12}$$

we get the matrix

$$X = \begin{pmatrix} 1 & x & xy \\ \frac{1}{x} & 1 & y \\ \frac{1}{xy} & \frac{1}{y} & 1 \end{pmatrix},$$

with two variables where $x, y > 0$. This matrix has two variables instead of three. If $f : \mathbb{R}^2_+ \to \mathbb{R}$ is given by

$$f(x,y) = \| A - X \|_F^2 = (a-x)^2 + (b-xy)^2 + \left(\frac{1}{a} - \frac{1}{x}\right)^2 + (c-y)^2 + \left(\frac{1}{b} - \frac{1}{xy}\right)^2 + \left(\frac{1}{c} - \frac{1}{y}\right)^2,$$

then the optimization problem is as follows:

$$\min \quad f(x,y) \tag{13}$$

$$x, y > 0. \tag{14}$$

It follows from the first-order necessary condition of optimality that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. The partial derivatives of f can be computed as

$$\frac{\partial f}{\partial x} = 2\left(-a + x + \frac{1}{ax^2} - \frac{1}{x^3} - by + xy^2 + \frac{1}{bx^2y} - \frac{1}{x^3y^2}\right),$$

$$\frac{\partial f}{\partial y} = 2\left(-c + y + \frac{1}{cy^2} - \frac{1}{y^3} - bx + x^2y + \frac{1}{bxy^2} - \frac{1}{x^2y^3}\right).$$

Dividing them by 2 and multiplying $\frac{\partial f}{\partial x}$ by x^3y^2, and $\frac{\partial f}{\partial y}$ by x^2y^3, we obtain the p and q polynomials in variables x, y :

$$p(x,y) = x^4y^4 + x^4y^2 - bx^3y^3 - ax^3y^2 + \frac{1}{a}xy^2 + \frac{1}{b}xy - y^2 - 1,$$

$$q(x,y) = x^4y^4 + x^2y^4 - bx^3y^3 - cx^2y^3 + \frac{1}{c}x^2y + \frac{1}{b}xy - x^2 - 1.$$

The aim is to find the $(x,y) \in \mathbb{R}^2_+$ solution(s) of the system

$$\begin{aligned} p(x,y) &= 0, \\ q(x,y) &= 0, \\ x, y &> 0. \end{aligned} \tag{15}$$

By using the resultant method and collecting p and q as polynomials in x,

$$p(x,y) = \left(y^2 + y^4\right)x^4 + \left(-y^2a - y^3b\right)x^3 + \left(\frac{y^2}{a} + \frac{y}{b}\right)x - y^2 - 1,$$

$$q(x,y) = y^4x^4 - y^3bx^3 + \left(-y^3c + y^4 + \frac{y}{c} - 1\right)x^2 + \frac{y}{b}x - 1.$$

$R_x(p,q)$ can be computed as the determinant of the following matrix:

$$\left|\begin{matrix} y^2+y^4 & -y^2a-y^3b & 0 & \frac{y(a+yb)}{ab} & -y^2-1 & 0 & 0 & 0 \\ 0 & y^2+y^4 & -y^2a-y^3b & 0 & \frac{y(a+yb)}{ab} & -y^2-1 & 0 & 0 \\ 0 & 0 & y^2+y^4 & -y^2a-y^3b & 0 & \frac{y(a+yb)}{ab} & -y^2-1 & 0 \\ 0 & 0 & 0 & y^2+y^4 & -y^2a-y^3b & 0 & \frac{y(a+yb)}{ab} & -y^2-1 \\ y^4 & -y^3b & \frac{-y^3c^2+y^4c+y-c}{c} & \frac{y}{b} & -1 & 0 & 0 & 0 \\ 0 & y^4 & -y^3b & \frac{-y^3c^2+y^4c+y-c}{c} & \frac{y}{b} & -1 & 0 & 0 \\ 0 & 0 & y^4 & -y^3b & \frac{-y^3c^2+y^4c+y-c}{c} & \frac{y}{b} & -1 & 0 \\ 0 & 0 & 0 & y^4 & -y^3b & \frac{-y^3c^2+y^4c+y-c}{c} & \frac{y}{b} & -1 \end{matrix}\right| .$$

Computing $R_x(p,q)$, we get a polynomial P in y of degree 28. Moreover, its last nonzero element is y^4, so we have in fact a polynomial of degree 24.

Applying a polynomial-solver algorithm to find all the positive real roots of P, we have the solutions $y_1, y_2, \ldots, y_t$ where $1 \leq t \leq 24$.

Substituting these solutions y_i, $i = 1, \ldots, t$, back in p and q, we get polynomials in x of degree 4. The leading coefficients never become zero because we consider the positive real roots y_i, $i = 1, \ldots, t$ only. Solving these polynomials in x, we have to check whether p and q really have common positive real roots.

Suppose that we have an $(x^\star, y^\star)$ solution of (15), where $y^\star \in \{y_i, 1 \leq i \leq t\}$. Then, we have to check the Hessian matrix of f to be sure that it is a local minimum point. The Hessian matrix of f can be computed as

$$\begin{pmatrix} 2+\frac{6}{x^4}-4\frac{1}{x^3a}+2\,y^2+6\,\frac{1}{x^4y^2}-4\,\frac{1}{x^3yb} & 4\,xy-2\,b+4\,\frac{1}{x^3y^3}-2\,\frac{1}{x^2y^2b} \\ 4\,xy-2\,b+4\,\frac{1}{x^3y^3}-2\,\frac{1}{x^2y^2b} & 2\,x^2+6\,\frac{1}{x^2y^4}-4\,\frac{1}{xy^3b}+2+\frac{6}{y^4}-4\,\frac{1}{y^3c} \end{pmatrix}.$$

If the Hessian matrix is positive definite at $(x^\star, y^\star)$, we have a strict local minimum point.

4 Numerical results

Now, some numerical results are shown which were computed by the method studied in the previous section. The first important question is the uniqueness of solution. We ask what conditions are needed to guarantee that the LSM solution is unique. Consider the matrix

$$A = \begin{pmatrix} 1 & a & \lambda \\ \frac{1}{a} & 1 & c \\ \frac{1}{\lambda} & \frac{1}{c} & 1 \end{pmatrix}, \tag{16}$$

where a and c are fixed, λ is a parameter. Remember that $\lambda = ac$ provides the consistent case with a unique solution. We examine the parameter λ for which the LSM approximation of (16) is unique. Tables 1a and 1b show that

Table 1a. The first 7 columns of the table of non-uniqueness intervals.

$a \backslash c$	$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$
9	7.911 0.085 11.75 0.126	7.426 0.080 9.197 0.129	6.846 0.075 7.243 0.132	0.070 0.136	0.065 0.141	0.060 0.146	0.055 0.152
8	7.726 0.108 12.44 0.134	7.265 0.103 9.656 0.137	6.725 0.097 7.484 0.141	0.091 0.145	0.085 0.149	0.078 0.152	0.072 0.161
7	7.524 0.138 13.24 0.146	7.086 0.133 10.22 0.148	6.581 0.127 7.816 0.151	5.952 0.121 6.064 0.155	0.113 0.160	0.105 0.166	0.096 0.172
6	7.305 14.19	6.888 10.9	6.414 0.164 8.255 0.167	5.844 0.160 6.249 0.171	0.152 0.175	0.143 0.180	0.132 0.187
5	7.069 15.3	6.672 11.73	6.224 8.817	5.700 6.547		0.195 0.203	0.186 0.208
4	6.815 16.59	6.593 12.72	6.014 9.518	5.527 6.969	4.924 5.109		
3	6.548 18.07	6.184 13.88	5.783 10.36	5.328 7.524	4.785 5.367		
2	6.271 19.73	5.924 15.21	5.543 11.36	5.115 8.211	4.616 5.752		
1	5.995 21.58	5.664 16.70	5.303 12.5	4.900 9.013	4.439 6.235	3.871 4.202	
$\frac{1}{2}$	5.848 22.63	5.525 17.56	5.174 13.17	4.786 9.496	4.343 6.537	3.810 4.328	
$\frac{1}{3}$	5.781 23.06	5.461 17.92	5.114 13.47	4.731 9.725	4.299 6.689	3.788 4.388	
$\frac{1}{4}$	5.731 23.34	5.412 18.17	5.067 13.68	4.690 9.896	4.272 6.812	0.415 4.393	3.788 4.388
$\frac{1}{5}$	5.686 23.56	5.368 18.36	5.025 13.86	0.683 10.05	0.202 6.928	4.272 6.812	4.299 6.689
$\frac{1}{6}$	0.985 23.74	0.436 18.54	0.253 14.02	0.128 10.20	0.683 10.05	4.690 9.896	4.731 9.725
$\frac{1}{7}$	0.211 23.92	0.152 18.71	0.089 14.19	0.253 14.02	5.025 13.86	5.067 13.68	5.114 13.47
$\frac{1}{8}$	0.105 24.08	0.067 18.88	0.152 18.71	0.436 18.54	5.368 18.36	5.412 18.17	5.461 17.92
$\frac{1}{9}$	0.052 24.25	0.105 24.08	0.211 23.92	0.985 23.74	5.686 23.56	5.731 23.34	5.781 23.06
	$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$

Table 1b. The second 7 columns of the table of non-uniqueness intervals.

$a\backslash b$	$\frac{1}{2}$	1	2	3	4	5	6
9	0.050 0.159	0.046 0.166	0.044 0.170	0.043 0.172	0.042 0.174	0.042 0.175	0.042 1.015
8	0.065 0.168	0.059 0.176	0.056 0.180	0.055 0.183	0.055 0.184	0.054 0.186	0.053 2.289
7	0.087 0.180	0.079 0.188	0.075 0.193	0.074 0.195	0.073 0.197	0.072 0.198	0.071 3.938
6	0.121 0.195	0.110 0.204	0.105 0.208	0.102 0.211	0.101 0.213	0.099 1.464	0.098 7.803
5	0.173 0.216	0.160 0.225	0.152 0.230	0.149 0.232	0.146 0.234	0.144 4.940	0.099 1.464
4		0.237 0.258	0.231 0.262	0.227 0.263	0.227 2.405	0.146 0.234	0.101 0.213
3					0.227 0.263	0.149 0.232	0.102 0.211
2					0.231 0.262	0.152 0.230	0.105 0.208
1					0.237 0.258	0.160 0.225	0.110 0.204
$\frac{1}{2}$						0.173 0.216	0.121 0.195
$\frac{1}{3}$						0.186 0.208	0.132 0.187
$\frac{1}{4}$	3.810 4.328	3.871 4.202				0.195 0.203	0.143 0.180
$\frac{1}{5}$	4.343 6.537	4.439 6.235	4.616 5.752	4.785 5.367	4.924 5.109		0.152 0.175
$\frac{1}{6}$	4.786 9.496	4.900 9.013	5.115 8.211	5.328 7.524	5.527 6.969	5.700 6.547	5.844 0.160 6.249 0.171
$\frac{1}{7}$	5.174 13.17	5.303 12.5	5.543 11.36	5.783 10.36	6.014 9.518	6.224 8.817	6.414 0.164 8.255 0.167
$\frac{1}{8}$	5.525 17.56	5.664 16.7	5.924 15.21	6.184 13.88	6.593 12.72	6.672 11.73	6.888 10.9
$\frac{1}{9}$	5.848 22.63	5.995 21.58	6.271 19.73	6.548 18.07	6.815 16.59	7.069 15.3	7.305 14.19
	$\frac{1}{2}$	1	2	3	4	5	6

for every given pair (a, c), there are 0,1 or 2 intervals. If λ is in this/these interval/s then the LSM solution is not unique.

Let, e.g., $a = \frac{1}{4}, c = \frac{1}{2}$. We have one interval in the table corresponding to $(\frac{1}{4}, \frac{1}{2})$, which is $[3.810, 4.328]$. This means that if λ is less than 3.810 or greater than 4.328, the solution is unique, else we have two or more solutions, which are local minimum points of (13)-(14). For some pairs (a, c) (for example $a = \frac{1}{9}, c = 7$), we have two intervals and for some pairs (a, c) (for example $a = 3, c = \frac{1}{2}$), we have no intervals, which means that for any λ, the LSM solution is unique.

In Figure 1., the area of (a, c) can be observed, for which the LSM solution is always unique (the white area in the middle.) The area lined vertically and the other area lined horizontally mean that the corresponding (a, c) pairs have one interval I, for which if $\lambda \in I$, the solution regarding λ is not unique. The intersection of these two areas (in two corners) have two intervals.

Note that we choose a special scale of the axes in a similar way as it is in Table 1. For numbers less than 1, we choose reciprocal scale, for numbers larger than or equal to 1, we choose the traditional scale. This multiplicative symmetry on 1 can be easily understood when we consider the function $f(x, y)$ in (13)-(14) which has also this symmetry on 1. Both the variables and their reciprocals are present in every formula.

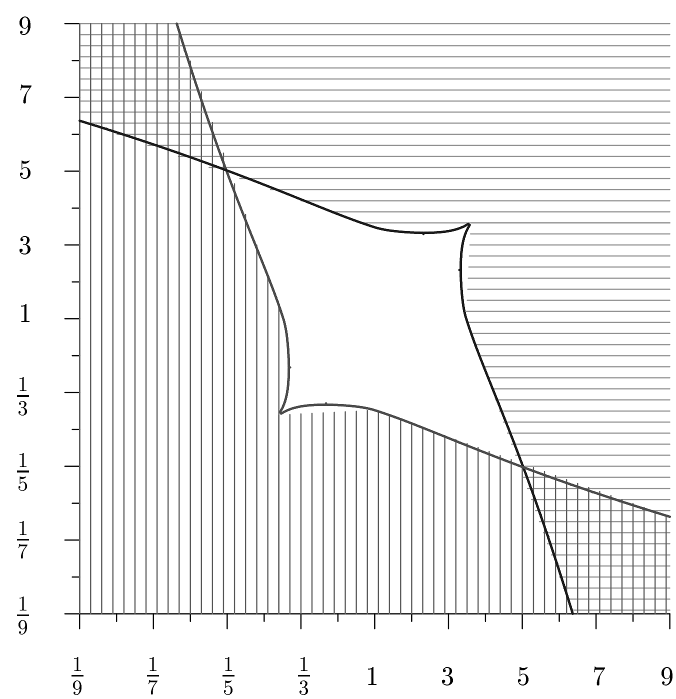

Figure 1. LSM solution is always unique in the area in the middle.

We want to demonstrate more of Table 1. Consider the row of Table 1. corresponding to $\frac{1}{5}$. Figure 2. shows the lower and upper bounds of the intervals while c changes from $\frac{1}{9}$ to 5.

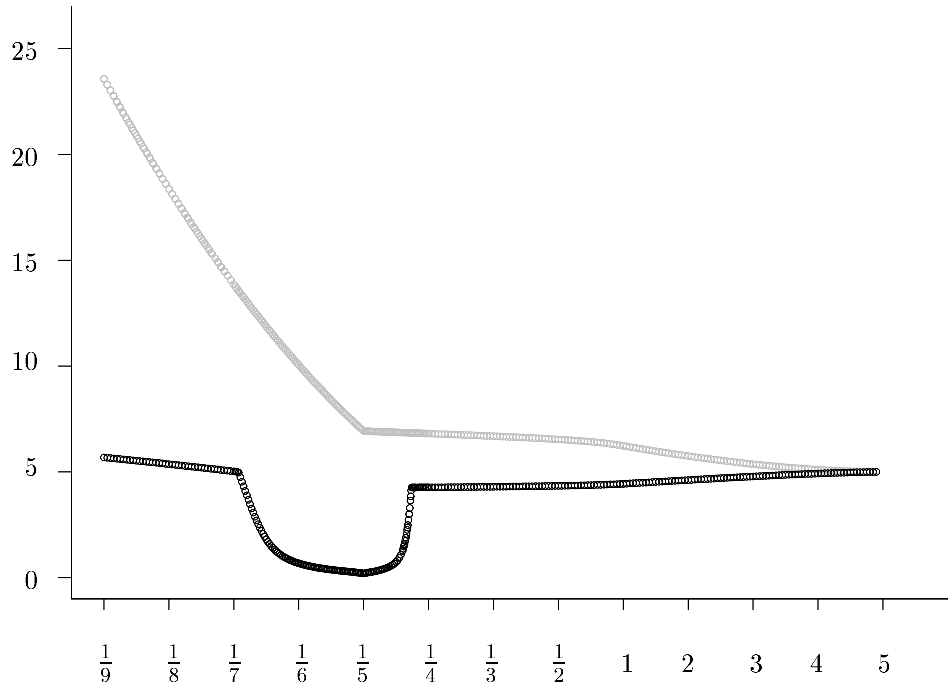

Figure 2. Endpoints of non-uniqueness intervals for matrices $\begin{pmatrix} 1 & \frac{1}{5} & \lambda \\ 5 & 1 & c \\ \frac{1}{\lambda} & \frac{1}{c} & 1 \end{pmatrix}$.

Now, let us see how the weights of alternatives, w_1, w_2, w_3 change while the parameter λ changes. Let $a = 4$, $c = 4$, and λ change from $\frac{1}{6}$ to 5. Figure 3. shows the weights of the alternatives. We can see that if $\lambda < 0.227$ or $\lambda > 2.405$ (which can be read from Table 1.,) the solution is unique, otherwise, we have two or three solutions.

We can observe an interesting behaviour of the solutions. Let the 3×3 pairwise comparison matrix be as follows:

$$\begin{pmatrix} 1 & \lambda & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 1 & \lambda \\ \lambda & \frac{1}{\lambda} & 1 \end{pmatrix},$$

where $\lambda > 0$. Figure 4. shows the speciality of this matrix or LSM problem.

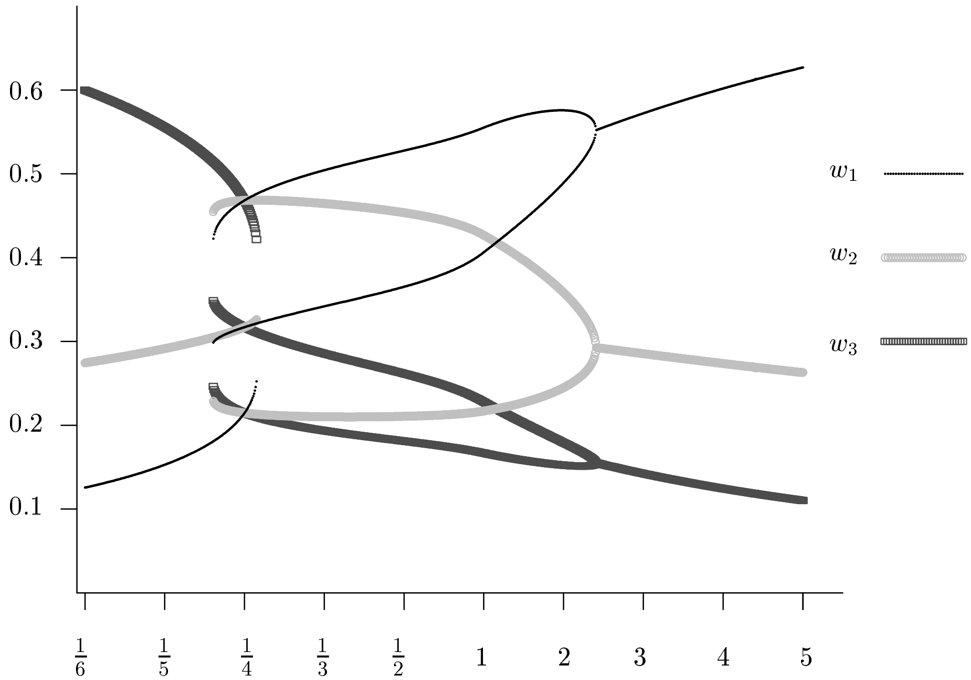

Figure 3. LSM weights for matrices $\begin{pmatrix} 1 & 4 & \lambda \\ \frac{1}{4} & 1 & 4 \\ \frac{1}{\lambda} & \frac{1}{4} & 1 \end{pmatrix}$, where λ changes from $\frac{1}{6}$ to 5.

When λ is around 1, i.e., not less than $\approx \frac{1}{3.6}$ but less than ≈ 3.6, the optimal LSM solution is:

$$w_1 = w_2 = w_3 = \tfrac{1}{3}.$$

When λ is larger than ≈ 3.73 or less than $\approx \frac{1}{3.73}$, we have three optimal LSM solutions which can be obtained from eachother by a simple permutation of the indices. When, e.g., $\lambda = 4$ the matrix has the form

$$\begin{pmatrix} 1 & 4 & \frac{1}{4} \\ \frac{1}{4} & 1 & 4 \\ 4 & \frac{1}{4} & 1 \end{pmatrix}.$$

The optimal solutions are as follows:

$$\begin{array}{lll} w_1^1 = 0.215 & w_1^2 = 0.468 & w_1^3 = 0.317 \\ w_2^1 = 0.317 & w_2^2 = 0.215 & w_2^3 = 0.468 \\ w_3^1 = 0.468 & w_3^2 = 0.317 & w_3^3 = 0.215 \end{array}$$

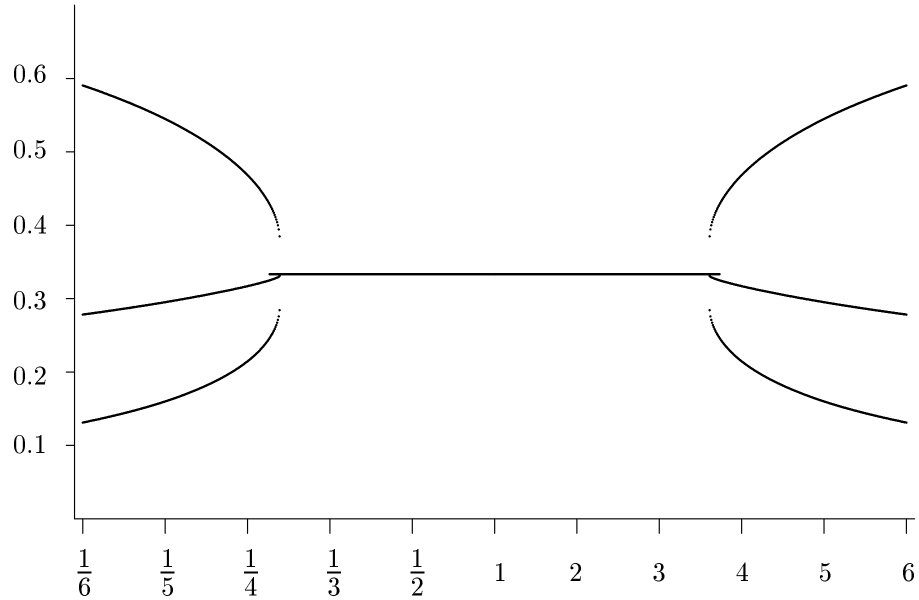

Figure 4. LSM weights for matrices $\begin{pmatrix} 1 & \lambda & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 1 & \lambda \\ \lambda & \frac{1}{\lambda} & 1 \end{pmatrix}$, where λ changes from $\frac{1}{6}$ to 6.

There are two small intervals [3.61,3.73] and $[\frac{1}{3.73}, \frac{1}{3.61}]$, when there are four optimal solutions of LSM: the equal priorities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the three symmetric solutions.

5 Comparison of LSM to other methods

In this section, a few examples are shown to see the difference among LSM and other methods. The LSM solution is not unique, in general, as stated in the previous section. Consider the following 3×3 matrix [6]:

				LSM Solutions			*EM* Solution
	A	B	C	w^{LSM_1}	w^{LSM_2}	w^{LSM_3}	w^{EM}
A	1	9	$\frac{1}{9}$	0.670	0.242	0.088	0.333
B	$\frac{1}{9}$	1	9	0.088	0.670	0.242	0.333
C	9	$\frac{1}{9}$	1	0.242	0.088	0.670	0.333

This matrix is degenerated because the ratios heavily contradict each other. It is impossible to justify the preference between A, B, C from these values. Note that $\lambda_{max} = 10.111$, and $CR = 6.13$, — as it was defined by Saaty, — which is much higher than the 0.10 limit.

LSM non-uniqueness, however, may arise even when there is no degeneracy, i.e., when the EM's eigenvector components are unequal suggesting a basis for differential row importance weighting. Let us see the following example:

	A	B	C	w^{LSM_1}	w^{LSM_2}	w^{EM}
A	1	8	4	0.751	0.450	0.712
B	$\frac{1}{8}$	1	7	0.105	0.467	0.214
C	$\frac{1}{4}$	$\frac{1}{7}$	1	0.144	0.083	0.074

Now, $\lambda_{max} = 3.825$ and $CR = 0.786$, which is still high but not extremely high. A is the best by w^{EM} and w^{LSM_1}, but B is the best by w^{LSM_2}. Moreover, the order by w^{EM} is A, B, C, while w^{LSM_1} yields A, C, B.

In practice, we are dealing with reasonable inconsistency (e.g., $CR \leq 0.10$,) expecting that each method, including LSM, gives unique solution. The final order, however, is not always the same, using different methods. Consider the following example:

	A	B	C	w^{EM}	w^{X^2M}	w^{LSM}
A	1	2	7	0.559	0.500	0.425
B	$\frac{1}{2}$	1	9	0.383	0.442	0.516
C	$\frac{1}{7}$	$\frac{1}{9}$	1	0.058	0.058	0.059

Now, $\lambda_{max} = 3.1$ and $CR = 0.095$, which must be considered consistent. Despite the above, LSM yields a different order.

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