

ON THE DENSITY OF PLANAR SETS WITHOUT UNIT DISTANCES

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ABSTRACT. We improve the upper bound of the density of a planar, measurable set containing no two points at distance 1 to 0.25688 by involving higher order convolutions of the autocorrelation function of the set.

1. INTRODUCTION

What is the maximal upper density of a measurable planar set with no two points at distance 1? This 60-year-old question has attracted some attention recently, and this paper provides a new upper estimate for $m_1(\mathbb{R}^2)$. Our argument builds on the proof method of [7]. We are going to use the notation and results therein extensively. Apart from polishing the previous techniques to near perfection, the improvement mainly stems from applying estimates involving higher order convolutions, and thus establishing new linear algebraic conditions for the autocorrelation function f corresponding to the set.

Let A be a *1-avoiding set* in \mathbb{R}^2 , that is, a subset of the plane containing no two points at distance 1. Let $m_1(\mathbb{R}^2)$ denote the supremum of possible upper densities of a Lebesgue measurable, 1-avoiding planar sets. Erdős conjectured in the 1970's [4] that $m_1(\mathbb{R}^2)$ is less than $1/4$, a conjecture that has been open ever since.

One of the easiest non-trivial upper bounds for $m_1(\mathbb{R}^2)$ is $1/3$, shown by the fact that A can contain at most one of the vertices of any regular triangle of edge length 1. This idea was further strengthened by Moser [8] using a special graph, the Moser spindle (see Section 4), implying that $m_1(\mathbb{R}^2) \leq 2/7 \approx 0.285$. Székely [10] improved the upper bound to ≈ 0.279 . Applying the linear programming bounds generated by carefully selected regular triangles, Oliveira and Vallentin [9] proved that $m_1(\mathbb{R}^2) \leq 0.268$. Including further constraints, Keleti, Matolcsi, Oliveira Filho and Ruzsa [7] was able to obtain the currently strongest upper bound of ≈ 0.259 .

Improving on the previous upper bounds, we prove the following theorem.

Theorem 1. *Any Lebesgue measurable, 1-avoiding planar set has upper density at most 0.25688.*

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All these upper bounds are considerably far from the largest lower bound for $m_1(\mathbb{R}^2)$. This is given by a construction of Croft [2], which has density approximately 0.22936.

The question may be formulated in higher dimensions as well. The articles of Bachoc, A. Passuello, and A. Thiery [1] and of DeCorte, Oliveira Filho and Vallentin [3] contain detailed historical accounts and a complete overview of recent results in that direction.

Perhaps the most famous related question is the Hadwiger-Nelson problem about the chromatic number $\chi(\mathbb{R}^2)$ of the plane: how many colours are needed to colour the points of the plane so that no two points at distance 1 receive the same colour? Obviously, $\chi(\mathbb{R}^2) \geq 1/m_1(\mathbb{R}^2)$. Recently, A. de Grey [5] proved that $\chi(\mathbb{R}^2) \geq 5$, a result which stirred up interest in this area.

2. GEOMETRIC CONSTRAINTS

Our proof follows the technique of [7]. Let $A \subset \mathbb{R}^2$ be a measurable, 1-avoiding set. For technical reasons, we may assume (due to a trivial argument taking limits) that A is periodic with respect to a lattice $L \subset \mathbb{R}^2$, i.e. $A = A + L$. The *autocorrelation function* $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of A is defined by

$$(1) \quad f(x) = \delta(A \cap (A - x)).$$

Then $\delta(A) = f(0)$, and the fact that A is 1-avoiding translates to the condition that $f(x) = 0$ for all unit vectors x . The estimate for $m_1(\mathbb{R}^2)$ of Keleti et al. [7] is based on the following lemma. A graph is called a *unit distance graph* if its vertex set is a subset of \mathbb{R}^2 , and its edges are given by the pairs of points being at distance 1. The independence number (i.e. the maximal number of independent vertices) of a graph G is denoted by $\alpha(G)$. For simplicity, if not specified otherwise, we denote the vertex set of a graph G by the same letter G .

Lemma 1 ([7], [9],[10]). *Let f be the autocorrelation function of a measurable, periodic, 1-avoiding set $A \subset \mathbb{R}^2$, as defined in (1). Then:*

- (C0) $f(x) = 0$ for every $x \in \mathbb{R}^2$ with $|x| = 1$;
- (C1) If G is a finite unit distance graph, then

$$\sum_{x \in G} f(x) \leq f(0)\alpha(G);$$

- (C2) If $C \subset \mathbb{R}^2$ is a finite set of points, then

$$\sum_{\{x,y\} \in \binom{C}{2}} f(x-y) \geq |C|f(0) - 1.$$

Constraint (C1) was first used by Oliveira and Vallentin [9], while Székely applied (C2) in his argument [10].

We start off with a straightforward strengthening of (C1).

Lemma 2. *Let f be as in Lemma 1. Then:*

(C1') If G is a finite unit distance graph with vertex set V and $\eta : V \rightarrow \mathbb{R}$ is a weight function so that

$$\sum_{x \in W} \eta(x) \leq \alpha$$

for every independent subset W of V with some constant α , then

$$\sum_{x \in V} \eta(x) f(x) \leq f(0) \alpha.$$

Setting $\eta(x) \equiv 1$ we arrive back at constraint (C1).

Proof. Consider the translated copies $A - x$ for $x \in V$. Since A is 1-avoiding, for any point $z \in \mathbb{R}^2$, the set of vertices $\{x_i\} \subset V$ for which $z \in A - x_i$ is independent. Therefore, $\sum_{z \in A - x_i} \eta(x_i) \leq \alpha$. Thus,

$$f(0) = \delta(A) \geq \delta \left(\bigcup_{x \in V} (A \cap (A - x)) \right) \geq \frac{1}{\alpha} \sum_{x \in V} \eta(x) \delta(A \cap (A - x)). \quad \square$$

Constraint (C1') provides stronger conditions for f than (C1), and may prove useful in the future. It is natural to try to apply it to graphs with high fractional chromatic number. Unfortunately, we have not been able to obtain any numerical improvement on $m_1(\mathbb{R}^2)$ along these lines so far.

We continue with further improvements on the constraints, which will eventually lead to the improved upper bound. The proof of (C2) (cited from [7]) is based on the inclusion-exclusion principle:

$$\begin{aligned} 1 &\geq \delta \left(\bigcup_{x \in C} (A - x) \right) \\ &\geq \sum_{x \in C} \delta(A - x) - \sum_{\{x,y\} \in \binom{C}{2}} \delta((A - x) \cap (A - y)) \\ &= |C| \delta(A) - \sum_{\{x,y\} \in \binom{C}{2}} \delta(A \cap (A - (x - y))), \end{aligned}$$

which implies (C2). Note that when applying the inclusion-exclusion principle above, intersections of at least three sets are omitted. Including these may improve the related bounds for f . These estimates also involve higher order convolutions of f , which account for the density of prescribed polygons in A (in place of the density of prescribed segments.)

The main technical result of the paper is the following statement.

Lemma 3. *Let f be as in Lemma 1. Then the following hold:*

(C3) *For every finite set of points V in \mathbb{R}^2 ,*

$$\sum_{x \in V} f(x) - \sum_{\{x,y\} \in \binom{V}{2}} f(x - y) \leq f(0);$$

(C4) If G is a finite unit distance graph with vertex set V , satisfying $\alpha(G) \leq 4$, then

$$\sum_{x \in V} f(x) - \sum_{\{x,y\} \in \binom{V}{2}} f(x-y) \leq 1 + (2 - |V|)f(0).$$

Proof. Let us introduce the notations

$$\Sigma_i = \sum_{\{x_1, \dots, x_i\} \in \binom{V}{i}} \delta((A - x_1) \cap \dots \cap (A - x_i))$$

and

$$\Sigma_i^\circ = \sum_{\{x_1, \dots, x_i\} \in \binom{V}{i}} \delta(A \cap (A - x_1) \cap \dots \cap (A - x_i)).$$

Thus, $\Sigma_0 = 1$, $\Sigma_1 = |V|f(0)$, $\Sigma_0^\circ = f(0)$, and $\Sigma_1^\circ = \sum_{x \in V} f(x)$. Using these notations, the quantity to estimate is

$$\sum_{x \in V} f(x) - \sum_{\{x,y\} \in \binom{V}{2}} f(x-y) = \Sigma_1^\circ - \Sigma_2.$$

Obviously,

$$(2) \quad \Sigma_i^\circ \leq \Sigma_i$$

holds for every i .

By Bonferroni's inequality and (2), one has

$$(3) \quad \begin{aligned} f(0) = \delta(A) &\geq \delta\left(\bigcup_{x \in V} (A \cap (A - x))\right) \\ &\geq \Sigma_1^\circ - \Sigma_2^\circ \\ &\geq \Sigma_1^\circ - \Sigma_2, \end{aligned}$$

which is constraint (C3).

For the estimate (C4), we first consider the case $\alpha(G) \leq 3$. Then, $\Sigma_i = \Sigma_i^\circ = 0$ for every $i \geq 4$. Since every point of the set $\bigcup_{x \in V} (A \cap (A - x))$ is covered by at most three sets of the form $(A \cap (A - x))$, one obtains that

$$\begin{aligned} f(0) = \delta(A) &\geq \delta\left(\bigcup_{x \in V} (A \cap (A - x))\right) \\ &\geq \frac{1}{2}\Sigma_1^\circ - \frac{1}{2}\Sigma_3^\circ \\ &\geq \frac{1}{2}\Sigma_1^\circ - \frac{1}{2}\Sigma_3, \end{aligned}$$

thus,

$$(4) \quad \Sigma_3 \geq \Sigma_1^\circ - 2f(0).$$

On the other hand, by the inclusion-exclusion principle,

$$\begin{aligned} 1 &\geq \delta\left(\bigcup_{x \in V} A - x\right) = \Sigma_1 - \Sigma_2 + \Sigma_3 \\ &= |V|f(0) - \Sigma_2 + \Sigma_3, \end{aligned}$$

and therefore

$$(5) \quad \Sigma_3 \leq 1 - |V|f(0) + \Sigma_2.$$

Comparing this with (4), we arrive at

$$\Sigma_1^\circ - \Sigma_2 \leq 1 + (2 - |V|)f(0),$$

that is, (C4).

Finally, we handle the case $\alpha(G) = 4$. First, we show that

$$(6) \quad \Sigma_3^\circ - 2\Sigma_4^\circ \leq \Sigma_3 - 2\Sigma_4.$$

Indeed, since $\Sigma_i = \Sigma_i^\circ = 0$ for every $i \geq 5$, the inclusion-exclusion formula yields that for any fixed $\{x, y\} \in \binom{V}{2}$,

$$\begin{aligned} & \delta \left(\bigcup_{z \in V \setminus \{x, y\}} A \cap (A - x) \cap (A - y) \cap (A - z) \right) \\ &= \sum_{z \in V \setminus \{x, y\}} \delta(A \cap (A - x) \cap (A - y) \cap (A - z)) \\ & \quad - \sum_{\{z, w\} \in \binom{V \setminus \{x, y\}}{2}} \delta(A \cap (A - x) \cap (A - y) \cap (A - z) \cap (A - w)). \end{aligned}$$

Therefore,

$$\begin{aligned} 3\Sigma_3^\circ - 6\Sigma_4^\circ &= \sum_{\{x, y\} \in \binom{V}{2}} \delta \left(\bigcup_{z \in V \setminus \{x, y\}} A \cap (A - x) \cap (A - y) \cap (A - z) \right) \\ &\leq \sum_{\{x, y\} \in \binom{V}{2}} \delta \left(\bigcup_{z \in V \setminus \{x, y\}} (A - x) \cap (A - y) \cap (A - z) \right) \\ &= 3\Sigma_3 - 6\Sigma_4, \end{aligned}$$

which shows (6).

Counting the set of points covered k times as before, and using (6),

$$\begin{aligned} f(0) = \delta(A) &\geq \delta \left(\bigcup_{x \in V} (A \cap (A - x)) \right) \\ &\geq \frac{1}{2}\Sigma_1^\circ - \frac{1}{2}\Sigma_3^\circ + \Sigma_4^\circ \\ &= \frac{1}{2}\Sigma_1^\circ - \frac{1}{2}(\Sigma_3^\circ - 2\Sigma_4^\circ) \\ &\geq \frac{1}{2}\Sigma_1^\circ - \frac{1}{2}(\Sigma_3 - 2\Sigma_4), \end{aligned}$$

thus

$$\Sigma_3 - 2\Sigma_4 \geq \Sigma_1^\circ - 2f(0).$$

Meanwhile, the inclusion-exclusion principle implies that

$$\begin{aligned} 1 \geq \delta \left(\bigcup_{x \in V} (A - x) \right) &= \Sigma_1 - \Sigma_2 + \Sigma_3 - \Sigma_4 \\ &\geq \Sigma_1 - \Sigma_2 + \Sigma_3 - 2\Sigma_4 \\ &= |V|f(0) - \Sigma_2 + (\Sigma_3 - 2\Sigma_4) \\ &\geq (|V| - 2)f(0) + \Sigma_1^\circ - \Sigma_2, \end{aligned}$$

which yields (C4) for $\alpha(G) = 4$. \square

With the current best upper estimate for $f(0)$, constraint (C3) is stronger than (C4) iff $|V| \leq 4$.

It is easy to check that by setting $G = C \cup \{0\}$, constraint (C4) for G simplifies to constraint (C2) for C . Thus, (C2) applied for C with $\alpha(C) \leq 3$ is a special case of (C4). This also shows that (C4) may only be applied to graphs not containing the origin in order to obtain new constraints.

Remark. Using estimates for the density of suitable triangles in A could be useful to obtain stronger estimates for $\delta(A)$. Note that whenever condition (C2) is sharp for a given point-set C , we necessarily have $\Sigma_3 = 0$, that is, all triangles with vertices from C must have density 0 in A . Of course, this is automatically guaranteed for triangles containing an edge of length 1, but the optimal configurations C (see Figure 2) also include triangles containing no unit distance. Lower estimates for the densities of triangles may be obtained by applying the previously seen bound (3)

$$f(0) = \delta(A) \geq \Sigma_1^\circ - \Sigma_2^\circ$$

to a vertex set V . Here, Σ_2° is the sum of triangle densities of triangles with one vertex at the origin and two vertices from V . Specifically, when $V = \{x, y, z\}$ with $|x - z| = |y - z| = 1$, one gets a direct estimate for the density of the triangle $\triangle(0, x, y)$. Unfortunately, our efforts in finding contradictory upper and lower bounds for the densities of specific triangles have been unsuccessful so far.

3. LINEAR PROGRAMMING FORMULATION

We need to find the maximal possible value of $f(0)$ among functions that are the autocorrelation functions of a measurable, periodic, 1-avoiding set. These functions are guaranteed to satisfy constraints (C0)–(C4), which we will only invoke for a finite family of suitably chosen graphs and point-sets. We transform this maximization problem to a finite linear programming problem by using a discrete approximation of the Fourier series of f . This technique has become somewhat an industry standard by now. Elaborate technical details of it may be found in [6] and [7], therefore we present here only the outline. Assume that L is a lattice in the plane with fundamental parallelogram Λ , and $f, g : \mathbb{R}^2 \rightarrow \mathbb{C}$ are L -periodic functions. We define the inner product of f and g by

$$\langle f, g \rangle = \frac{1}{|\Lambda|} \int_{\Lambda} f(x) \overline{g(x)} dx.$$

If L^* denotes the dual lattice of L , i.e. $L = \{x \in \mathbb{R}^2 : \langle x, y \rangle \in \mathbb{Z} \forall y \in L\}$, then the set of functions

$$\{\chi_u(x) = e^{iu \cdot x}, u \in 2\pi L^*\}$$

is a complete orthonormal system among square-integrable, L -periodic functions in the plane. Thus, they provide a Fourier decomposition of the function f , with Fourier coefficients $\widehat{f}(u) = \langle f, \chi_u \rangle$. By the Fourier inversion formula,

$$f(x) = \sum_{u \in 2\pi L^*} \widehat{f}(u) e^{iu \cdot x}$$

holds with L^2 convergence. By Parseval's identity, we also have

$$(7) \quad \langle f, g \rangle = \sum_{u \in 2\pi L^*} \widehat{f}(u) \overline{\widehat{g}(u)}.$$

If f is the autocorrelation function of the L -periodic measurable set A , then

$$f(x) = \langle \mathbf{1}_A, \mathbf{1}_{A-x} \rangle,$$

where $\mathbf{1}_A$ is the indicator function of A . Since $\widehat{\mathbf{1}_{A-x}}(u) = \widehat{\mathbf{1}_A}(u) e^{iu \cdot x}$, by Parseval's identity (7), we have that

$$f(x) = \sum_{u \in 2\pi L^*} |\widehat{\mathbf{1}_A}(u)|^2 e^{-iu \cdot x} = \sum_{u \in 2\pi L^*} |\widehat{\mathbf{1}_A}(-u)|^2 e^{iu \cdot x}$$

with convergence for all x . Therefore, the inversion formula shows that $\widehat{f}(u) = |\widehat{\mathbf{1}_A}(-u)|^2$, in particular, all the Fourier coefficients of f are non-negative. In other words, f is positive definite.

Let

$$\Omega_2(|x|) = \frac{1}{2\pi} \int_{S^1} e^{ix \cdot \xi} d\omega(\xi) = J_0(|x|),$$

where ω is the perimeter measure on S^1 , and J_0 is the Bessel function of the first kind with parameter 0.

As usual, we radialize f by taking its radial average \mathring{f} :

$$\mathring{f}(x) = \frac{1}{2\pi} \int_{S^1} f(\xi|x|) d\omega(\xi).$$

Then, by Fourier inversion,

$$\mathring{f}(x) = \frac{1}{2\pi} \int_{S^1} \sum_{u \in 2\pi L^*} \widehat{f}(u) e^{iu \cdot \xi|x|} d\omega(\xi) = \sum_{u \in 2\pi L^*} \widehat{f}(u) \Omega_2(|u||x|).$$

Introducing the notation

$$\kappa(t) = \sum_{u \in 2\pi L^*, |u|=t} \widehat{f}(u),$$

where $t \geq 0$, the previous equation simplifies to

$$(8) \quad \mathring{f}(x) = \sum_{t \geq 0} \kappa(t) \Omega_2(t|x|).$$

We have the following properties of the function $\kappa(t)$:

$$(9) \quad \kappa(t) \geq 0 \text{ for every } t \geq 0,$$

$$(10) \quad \kappa(0) = \widehat{f}(0) = \delta^2(A),$$

$$(11) \quad \sum_{t \geq 0} \kappa(t) = \sum_{u \in 2\pi L^*} \widehat{f}(u) = f(0) = \delta(A).$$

Since the function Ω_2 is bounded, the above equations show that the sum in (8) is absolutely convergent. Also note that (C0) implies $\mathring{f}(1) = 0$.

We introduce yet one more transformation: we normalize the coefficients $\kappa(t)$ by defining

$$(12) \quad \widetilde{\kappa}(t) = \frac{\kappa(t)}{\delta(A)}.$$

Notice that all the constraints of Lemma 1, Lemma 2 and Lemma 3 are rotation-invariant in the sense that they hold for all rotated copies of a given graph or point-set. Therefore, equations (8),(9),(11),(12) and constraints (C0)–(C4) imply that for any 1-avoiding, planar, measurable set A with density $\delta(A)$, the sequence of coefficients $\widetilde{\kappa}(t)$ corresponding to the autocorrelation function f of A must satisfy the following linear constraints with $\delta = \delta(A)$:

$$(\widehat{\text{CP}}) \quad \widetilde{\kappa}(t) \geq 0 \text{ for every } t \geq 0,$$

$$(\widehat{\text{CS}}) \quad \sum_{t \geq 0} \widetilde{\kappa}(t) = 1,$$

$$(\widehat{\text{C0}}) \quad \sum_{t \geq 0} \widetilde{\kappa}(t) \Omega_2(t) = 0,$$

$$(\widehat{\text{C1}}) \quad \text{For every finite unit distance graph } G,$$

$$\sum_{t \geq 0} \widetilde{\kappa}(t) \sum_{x \in G} \Omega_2(t|x|) \leq \alpha(G)$$

$$(\widehat{\text{C2}}) \quad \text{For every finite set } C \text{ of points in the plane,}$$

$$\sum_{t \geq 0} \widetilde{\kappa}(t) \sum_{\{x,y\} \in \binom{C}{2}} \Omega_2(t|x-y|) \geq |C| - \frac{1}{\delta}.$$

$$(\widehat{\text{C3}}) \quad \text{For every finite unit distance graph } T,$$

$$\sum_{t \geq 0} \widetilde{\kappa}(t) \left(\sum_{x \in T} \Omega_2(t|x|) - \sum_{\{x,y\} \in \binom{T}{2}} \Omega_2(t|x-y|) \right) \leq 1.$$

$$(\widehat{\text{C4}}) \quad \text{For every finite unit distance graph } D \text{ with independence number } \alpha(D) \leq 4,$$

$$\sum_{t \geq 0} \widetilde{\kappa}(t) \left(\sum_{x \in D} \Omega_2(t|x|) - \sum_{\{x,y\} \in \binom{D}{2}} \Omega_2(t|x-y|) \right) \leq 2 - |D| + \frac{1}{\delta}.$$

Since $\widetilde{\kappa}(0) = \delta(A)$, considering the coefficients $\widetilde{\kappa}(t)$ with $t \geq 0$ as variables, the solution of the continuous linear program

$$(13) \quad \begin{aligned} & \text{maximize } \tilde{\kappa}(0) \\ & \text{subject to } (\widehat{\text{CP}}), (\widehat{\text{CS}}), (\widehat{\text{C1}}) - (\widehat{\text{C4}}) \end{aligned}$$

must be at least $\delta(A)$. In other words, if for some collection of graphs and point-sets and for some value of δ the above linear program is feasible, and its solution μ is at least δ , then $\delta(A) \leq \mu$ must hold for any 1-avoiding, L -periodic set. (In fact, at the optimal estimate for a given set of constraint graphs and points sets, we have $\delta = \mu$). By linear programming duality, the value of μ is provided by a witness function:

Proposition 1. *Let \mathcal{C} be a finite collection of finite sets of points in \mathbb{R}^2 , \mathcal{G} , \mathcal{T} and \mathcal{D} be finite families of finite unit distance graphs, and assume that the independence number of graphs in \mathcal{D} is at most 4. Suppose that for the non-negative numbers v_0, v_1, w_G for $G \in \mathcal{G}$, w_C for $C \in \mathcal{C}$, w_T for $T \in \mathcal{T}$, and w_D for $D \in \mathcal{D}$, the function $W(t)$ defined by*

$$(14) \quad \begin{aligned} W(t) &= v_0 + v_1 \Omega_2(t) \\ &+ \sum_{G \in \mathcal{G}} w_G \sum_{x \in G} \Omega_2(t|x|) \\ &- \sum_{C \in \mathcal{C}} w_C \sum_{\{x,y\} \in \binom{C}{2}} \Omega_2(t|x-y|) \\ &+ \sum_{T \in \mathcal{T}} w_T \left(\sum_{x \in T} \Omega_2(t|x|) - \sum_{\{x,y\} \in \binom{T}{2}} \Omega_2(t|x-y|) \right) \\ &+ \sum_{D \in \mathcal{D}} w_D \left(\sum_{x \in D} \Omega_2(t|x|) - \sum_{\{x,y\} \in \binom{D}{2}} \Omega_2(t|x-y|) \right) \end{aligned}$$

satisfies $W(0) \geq 1$ and $W(t) \geq 0$ for $t > 0$.

Then $m_1(\mathbb{R}^2) \leq \delta$, where δ is the positive solution of the equation

$$(15) \quad \begin{aligned} \delta^2 &= \delta \left(v_0 + \sum_{G \in \mathcal{G}} w_G \alpha(G) - \sum_{C \in \mathcal{C}} w_C |C| + \sum_{T \in \mathcal{T}} w_T \right. \\ &\left. + \sum_{D \in \mathcal{D}} w_D (2 - |D|) \right) + \sum_{C \in \mathcal{C}} w_C + \sum_{D \in \mathcal{D}} w_D. \end{aligned}$$

Proof. For any function $W(t)$ satisfying $W(0) \geq 1$ and $W(t) \geq 0$ for $t > 0$ we have, by (10) and (12),

$$(16) \quad \delta = \tilde{\kappa}(0) \leq \sum_{t \geq 0} \tilde{\kappa}(t) W(t).$$

If $W(t)$ is in the form (14), then inequalities $(\widehat{\text{CP}})$, $(\widehat{\text{CS}})$, $(\widehat{\text{C1}}) - (\widehat{\text{C4}})$, and (16) imply

$$\begin{aligned} \delta \leq & v_0 + \sum_{G \in \mathcal{G}} w_G \alpha(G) - \sum_{C \in \mathcal{C}} w_C |C| + \sum_{T \in \mathcal{T}} w_T \\ & + \sum_{D \in \mathcal{D}} w_D (2 - |D|) + \frac{1}{\delta} \left(\sum_{C \in \mathcal{C}} w_C + \sum_{D \in \mathcal{D}} w_D \right). \quad \square \end{aligned}$$

4. NUMERICAL BOUNDS

Finding the optimal configurations of points for which to invoke constraints $(\widehat{\text{C1}}) - (\widehat{\text{C4}})$ is a tedious task, where we utilized a bootstrap algorithm. Once some constraints are fixed, and the corresponding linear program is solved, one has to find numerically a configuration of points for which some of the constraints $(\widehat{\text{C1}}) - (\widehat{\text{C4}})$ is violated. Adding this to the list of constraints, one again has to execute the above procedure, until no significant improvement is possible to be made this way. At the final step, the large number of graphs and point-sets can be pruned by dropping the non-binding constraints. Our construction in its polished form has 26 linear constraints.

In order to handle the linear program numerically, we discretize it. Based on the previous results, we will only search for the coefficients $\tilde{\kappa}(t_i)$, where $t_i = i\varepsilon_0$, with $\varepsilon_0 = 0.05$ and $i \leq 12000$, thus, $t_i \in [0, 600]$. For all other values of $t \geq 0$, we set $\tilde{\kappa}(t) = 0$.

We are going to solve the discretized linear program defined by the following graphs and point-sets.

Constraint $(\widehat{\text{C1}}) - \mathcal{G}$ family. As in [7], we also found that the optimal graphs to apply $(\widehat{\text{C1}})$ for are suitably positioned copies of the *Moser spindle* M , see Figure 1. This unit distance graph has 7 vertices and 11 edges. The optimal locations are so that the origin lies on the axis of symmetry of M , outside the convex hull of M , see Figure 1. The distance of the vertex of M with degree 4 (called the *apex* of M) and the origin ranges between 0.55 and 0.85. We apply constraint $(\widehat{\text{C1}})$ to a family \mathcal{G} of 8 Moser spindles, as described in Table 1.

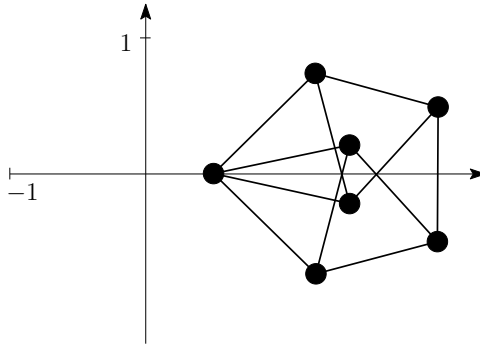


FIGURE 1. The Moser spindle. All the indicated edges are of unit length.

Constraint $(\widehat{C2})$ - \mathcal{C} point-sets. The optimal \mathcal{C} graphs that we found all have 6 vertices. Up to numerical approximations, there are two types of these extremal graphs, see Figure 2. First, take the union of a rhombus of edge length 1 and a segment of length 1 parallel to its longer diagonal, located symmetrically so that the midpoint of the segment lies on the line defined by the shorter diagonal. In the extremal configurations, the distance between the center of the rhombus and the midpoint of the segment is either about 0.9 or about 2.4. The second configuration type is the union of a regular triangle and its translate by a unit vector. The resulting 6-point unit distance graph has 9 edges. Altogether, we use 10 constraints of this type, listed in Table 2.

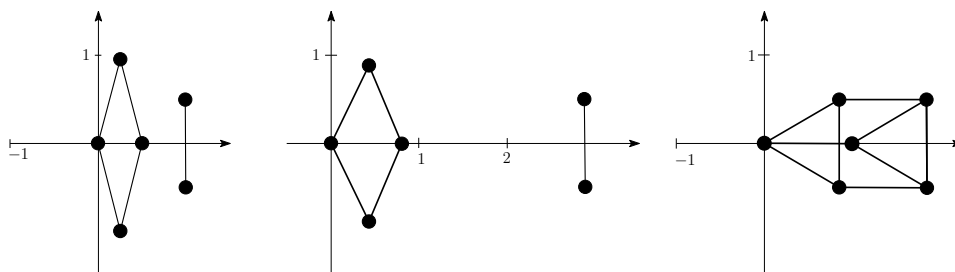


FIGURE 2. The point-sets of family \mathcal{C}

Constraint $(\widehat{C3})$ - \mathcal{T} triangles. The graphs chosen here are isosceles triangles with the origin lying on their axis of symmetry. The side-lengths of the triangles are either $(1, 1, a)$ with $a < 1$ or $(b, b, 1)$ with $b \approx 2$. We invoke the constraint for a family \mathcal{T} of 5 triangles, three smaller and two bigger ones, presented in Table 3.

Constraint $(\widehat{C4})$ - D graph. Finally, we apply the last constraint to only one graph: D is a generalized Moser spindle, that is, the union of two rhombi of unit side length and unit shorter diagonal, placed so that they share a common apex, and the distance between the other apices is about 2.24, see Figure 3. Then, D has 7 vertices with independence number 3; the coordinates of its vertices are given in Table 4.

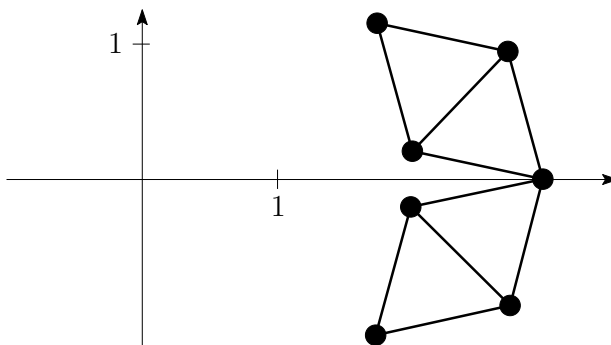


FIGURE 3. The generalized Moser spindle D .

Using these graphs, we construct the witness function $W(t)$ as in Proposition 1 with the coefficients described in Table 5. The coefficients arise as the solution of the dual linear program. Because of discretization, the positivity of the function is only guaranteed at the node points. Complete positivity must be checked numerically. We found that the minimum of the function is about -0.0005567 , attained at $t = 3.82528$. Subtracting this minimum value from the original value of v_0 leads to the set of coefficients described in Table 5 and to the function $W(t)$ which satisfies the condition of Proposition 1. Further technical details about the verification are described in [7].

With this construction of $W(t)$, the quadratic equation (15) takes the form

$$\delta^2 + 8.412286934947 \delta - 2.226873241160 = 0,$$

whose positive solution is $\delta = 0.256873013577$.

The coefficients $\tilde{\kappa}(t)$ obtained as the solution of the linear program (13) also provide the normalized, radialized autocorrelation function $\mathring{f}(x)$ via (8) and (12). This function is plotted on Figure 4.

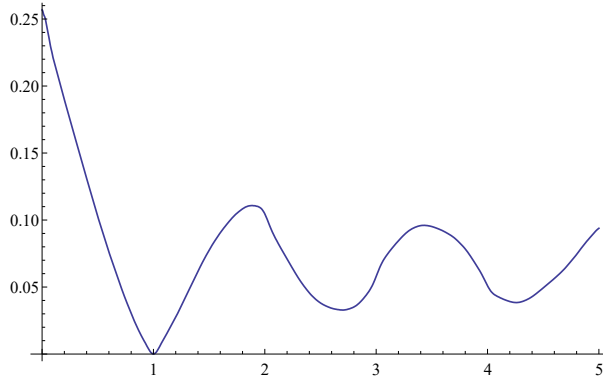


FIGURE 4. The function $\mathring{f}(x)/\delta$.

5. APPENDIX: NUMERICAL VALUES

G_1	0.549394809875	G_2	0.574818438480	G_3	0.589129645929
G_4	0.730924687690	G_5	0.745526562977	G_6	0.810398355949
G_7	0.821514458830	G_8	0.844783781915		

TABLE 1. The family \mathcal{G} of the copies of Moser spindles, which are located so that the origin lies on the axis of symmetry, outside the convex hull. The norm of the apex is listed.

C_1	(0.406817237137, 0) (0.212653201726, 0.979424301472) (0.215416368899, -0.980091365010) (1.076758799342, -0.479441317129) (1.092173797015, 0.514761385947)	C_2	(0.304947737631, 0) (1.011095086193, 0.495273011342) (0.156665823457, 1.003515826428) (0.156665686884, -1.003515829754) (1.011094960857, -0.495273032288)
C_3	(0.722707873034, 0) (2.879596197060, -0.497750114205) (0.383263232831, 0.926963720323) (0.383263417300, -0.926963701532) (2.879595735930, 0.497757370331)	C_4	(0.739999420118, 0) (0.387683003943, -0.925475734398) (0.387683391889, 0.925475626816) (2.873857316371, 0.497906683939) (2.873855991861, -0.497912547251)
C_5	(1.909586793158, 0) (0.964461163843, 0.264077537545) (0.958064561452, -0.293919382104) (0.700166154949, 0.716490286642) (1.665508028159, 0.957346037548)	C_6	(1.966773467400, 0) (0.985154735434, -0.207655298471) (1.541736634458, 0.899612796891) (0.547931107154, 0.835021131823) (0.983603399841, 0.182718007396)
C_7	(1.971873970831, 0) (0.987645642144, 0.175007132703) (1.585316335898, 0.914734021728) (0.598742912958, 0.798018249330) (0.987533227103, -0.186346100707)	C_8	(1.977604739521, 0) (0.990398355678, 0.157171481656) (0.599752268603, 0.799043686844) (0.991555248207, -0.187167815493) (1.586930662216, 0.912324929889)
C_9	(1.995304619556, 0) (1.578711851561, 0.899112691976) (1.000327108557, -0.177090995730) (0.999910611150, 0.136637775888) (0.592124446295, 0.803209372546)	C_{10}	(2.019750708108, 0) (0.556472755654, 0.826305009124) (1.012516246452, -0.142557099959) (1.544457973623, 0.871840304076) (1.012305158444, 0.114063226498)

TABLE 2. Collection \mathcal{C} of point-sets used. Each of the sets also contains the origin $(0, 0)$, so each set has 6 points.

T_1	(-0.726108320018, 0) (0.155933903426, -0.381702431017) (0.155933903426, 0.381702431017)	T_2	(-0.746389938122, 0) (0.151459324640, 0.350598185034) (0.151459324640, -0.350598185034)
T_3	(-0.799108011229, 0) (0.110687746283, 0.328281974673) (0.110687746283, -0.328281974673)	T_4	(-0.122154919695, 0) (1.939660687516, 0.500789551558) (1.939660687516, -0.500789551558)
T_5	(-0.133851861793, 0) (1.934710898446, 0.500150467476) (1.934710898446, -0.500150467476)		

TABLE 3. The family \mathcal{T} of triangles used for the estimates.

	(2.929162925441, 0)
	(2.590194413036, 0.940797718746)
	(1.944923944982, 0.176843516547)
D	(1.605955432577, 1.117641235293)
	(2.590194413036, -0.940797718746)
	(1.944923944982, -0.176843516547)
	(1.605955432577, -1.117641235293)

TABLE 4. The graph D , a generalized Moser spindle.

v_0	1.000556688164	v_1	21.830086484852	w_{G_1}	0.125663001758
w_{G_2}	0.319084834731	w_{G_3}	0.152397370898	w_{G_4}	0.121683128933
w_{G_5}	0.291693251794	w_{G_6}	0.092105000475	w_{G_7}	0.343036540489
w_{G_8}	0.191307438874	w_{C_1}	0.483479288239	w_{C_2}	0.064553075196
w_{C_3}	0.131728260470	w_{C_4}	0.192138472887	w_{C_5}	0.265727605935
w_{C_6}	0.006785989291	w_{C_7}	0.093924572541	w_{C_8}	0.442799912237
w_{C_9}	0.279332895469	$w_{C_{10}}$	0.152185012025	w_{T_1}	0.185804289684
w_{T_2}	0.010281183916	w_{T_3}	0.094109619652	w_{T_4}	0.130215766322
w_{T_5}	0.139825671498	w_D	0.114218156868		

TABLE 5. Coefficients of the witness function $W(t)$ of Proposition 1.

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