Discrete Geometry and Convexity in Honour of Imre Bárány

Discrete Geometry and Convexity
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Preface

This special volume is contributed by the speakers of the Discrete Geometry and Convexity conference, held in Budapest, June 19–23, 2017. The aim of the conference is to celebrate the 70th birthday and the scientific achievements of professor Imre Bárány, a pioneering researcher of discrete and convex geometry, topological methods, and combinatorics. The extended abstracts presented here are written by prominent mathematicians whose work has special connections to that of professor Bárány. Topics that are covered include: discrete and combinatorial geometry, convex geometry and general convexity, topological and combinatorial methods.

The research papers are presented here in two sections. After this preface and a short overview of Imre Bárány's works, the main part consists of 20 short but very high level surveys and/or original results (at least an extended abstract of them) by the invited speakers. Then in the second part there are 13 short summaries of further contributed talks.

We would like to dedicate this volume to Imre, our great teacher, inspiring colleague, and warm-hearted friend.

Budapest, 11. June 2017.

Gergely Ambrus, Károly J. Böröczky and Zoltán Füredi

A few great results of Imre Bárány

Imre Bárány is not only an outstanding mathematician of our age but also an inspiration and a friend for many of us. His smile constantly sends the message, 'don't worry, there is a solution'. Most of his work has geometric flavor but one of his strength is to combine ideas from various fields in mathematics like algebraic topology, number theory, game theory, theory of algorithms and combinatorics. His 160+ research papers opened up new directions in various branches related to convexity. Not only his research interest is versatile, he is an avid sportsman and traveler, having positions both at University College London and at the Rényi Institute, Budapest.

Imre's oeuvre was awarded by numerous prestigious prizes. He was an invited speaker at ICM 2002, he has been a holder of an ERC grant, recipient of the Széchenyi Prize, and he is a member of the Hungarian Academy of Sciences.

He started his carrier by providing a short and elegant proof [1] concerning the chromatic number of Kneser graphs by using Gale transform and the theory of neighborly polytopes about the same time as Lovász' solution. A little bit later Imre [3] proved the colorful version of the Charathéodory theorem. Together with Katchalski and Pach [4], they initiated the study of the quantitative versions of the Helly theorem. If one restricts attention to lattice points in the intersection of d-dimensional convex sets, Bell and Scarf showed earlier that the optimal Helly constant is 2^d, not $d + 1$. However, Imre and Matoušek [15] managed to verify a weighted (fractional) version of the Helly theorem for lattice points where only subfamilies of size $d + 1$ had to be considered. All these beautiful theorems are not only classical 'textbook' results of convexity by now but subjects of intensive current research.

Imre was one of the pioneers making algebraic topology an everyday tool in convexity and discrete geometry. The paper with Shlossmann and Szűcs [2] extended the Borsuk-Ulam theorem leading to a topological version of the Tverberg theorem. Their result and approach has been applied intensively ever since when Borsuk-Ulam type theorems can be used in combinatorics.

Concerning geometrically motivated algorithms, Imre with Füredi [7] showed that for given d and any polynomial algorithm calculating the volume of d-dimensional convex bodies there is a polytope whose volume is calculated only up to a factor essentially d^d. This result, improving on Elekes' earlier estimate, partially motivated Dyer, Frieze and Kannan to invent their famous randomized algorithm. Discrete programming is another area where Imre had fundamental contributions. Together with Howe and Scarf [11], they have proved that the so called Scarf complex is homeomorphic to the corresponding Euclidean space. Imre has even played around with game theory, as well [9], which paper was preceded by his work with Füredi [5].

A great achievements by Imre is to help to understand the family of lattice polytopes. Together with Vershik [10], he answered a long standing conjecture on the number of combinatorial types of lattice polytopes by Arnold motivated by work on singularities of complex manifolds. One of Imre's landmark results is to describe the typical lattice polygon contained in a given convex domain K. He showed [12] that among the lattice polygons with respect to rescaled integer lattice, the typical one is asymptotically close to the convex domain of maximal affine perimeter contained in K.

Speaking about approximating a d-dimensional convex body by random polytopes, it was Imre who first provided comprehensive estimates for all important aspects of approximation [8]. Solving a problem going back to Sylvester, Imre and Larman [13] showed that a random polytope in a ball is very close to the corresponding floating body. Imre even managed to answer the celebrated question of Sylvester on the probability that a certain number of points are in a convex position [14] .

However, not all conjectures are true. Call a system $\mathcal{C}$ of open disjoint planar discs a 6-neighboured circle packing if every member is tangent to at least 6 other elements of $\mathcal{C}$. Imre with coauthors [6] verified L. Fejes Tóth's conjecture that either $\mathcal{C}$ is the regular hexagonal packing or $\inf_{C\in\mathcal{C}} r(C) = 0$. However the rate of convergence was very slow, so they conjectured that more was true, $\mathcal{C}$ must have a limit point. The counterexample below was constructed by Kenneth Stephenson.

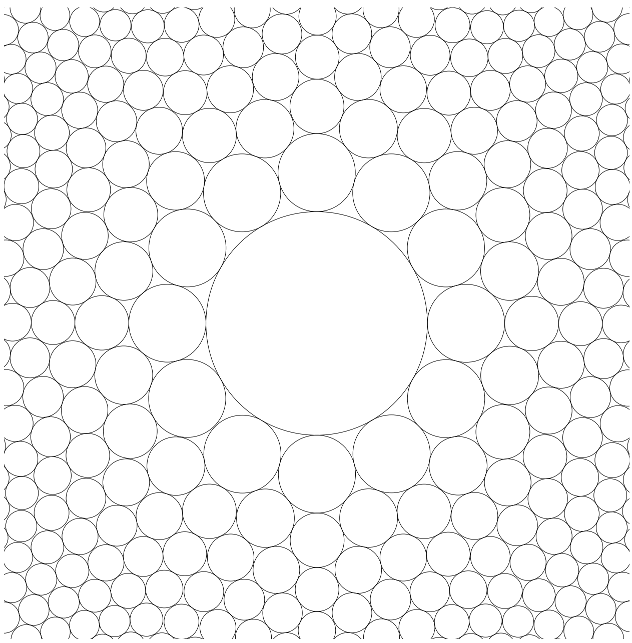

Figure 1: A 6-neighboured circle packing without a limit point

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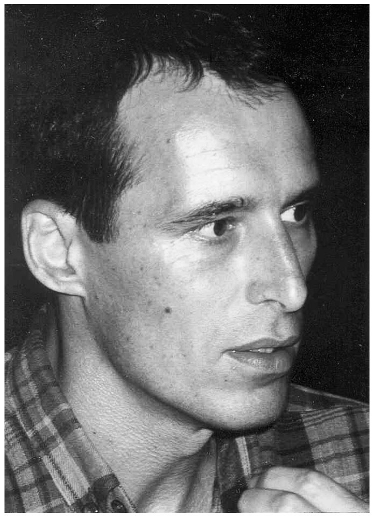
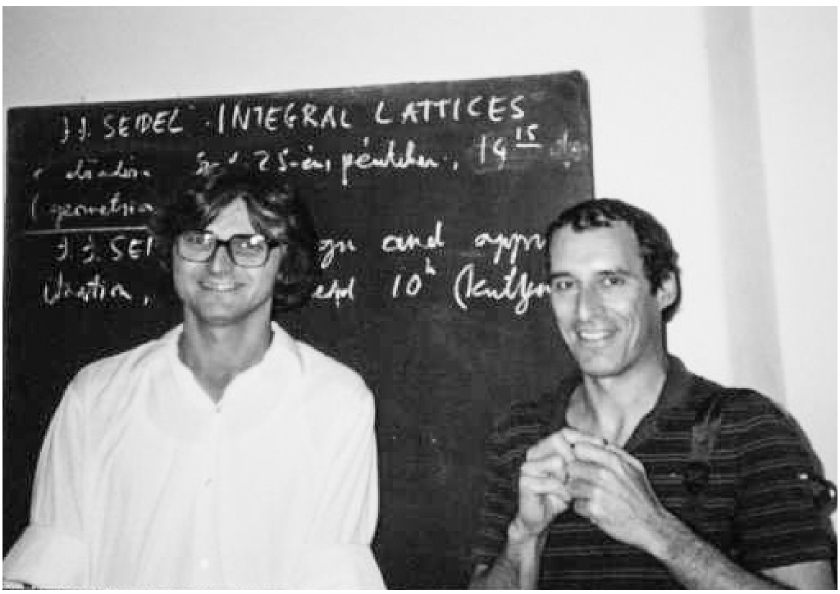
A FEW SNAPSHOTS OF IMRE BÁRÁNY

1965 and 1988. No changes at all! Photos: KöMaL, K. Jacobs



1987, with Z. Füredi in the Institute. Photo: J. J. Seidel

2014, an evening in Oberwolfach. Photo: G. Ambrus

2014, becoming regular member of the Academy. Photo: MTA

Invited articles

The beauty and the mystery of the symmetric moment curve

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1. The moment curves

The *symmetric moment curve* is a closed curve in $\mathbb{R}^{2k}$ with parameterization

$$S_k(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots,\ \cos(2k-1)t,\ \sin(2k-1)t) \quad \text{for} \quad 0 \le t \le 2\pi.$$

We view S_k as an embedding of the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ in $\mathbb{R}^{2k}$. The curve is called *symmetric* since

$$S_k(t+\pi) = -S_k(t), \tag{1}$$

so that a pair of antipodal points in $\mathbb{S}^1$ is mapped into a pair of antipodal points of $\mathbb{R}^{2k}$. The words *moment curve* relate to the *trigonometric moment curve* in $\mathbb{R}^{2k}$, the study of which goes back to Carathéodory [5],

$$T_k(t) = (\cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \ldots,\ \cos kt,\ \sin kt) \quad \text{for} \quad 0 \le t \le 2\pi,$$

which, in turn, is a closed version of the ordinary *moment curve* in $\mathbb{R}^d$,

$$M_d(t) = \left(t, t^2, \ldots, t^d\right) \quad \text{for} \quad 0 \le t \le 1.$$

The curve M_d has the remarkable property that every affine hyperplane in $\mathbb{R}^d$ intersects the image of M_d in at most d points (an easy exercise). Such curves are called *convex*. If $d = 2k$ is even, then T_k provides an example of a closed convex curve, while if d is odd no closed convex curve exists (a little harder exercise). It follows that the convex hull of any finite set of points on $M_d(t)$ or $T_k(t)$ is a *neighborly polytope*, that is, a polytope $P \subset \mathbb{R}^d$ such that every $\lfloor d/2 \rfloor$ vertices of P span a face of P [6]. It also follows that any k distinct points $T_k(t_1), \ldots, T_k(t_k)$ span a $(k-1)$-dimensional simplicial face of the convex hull of $T_k(t)$ and that the k-parameter family of those $(k-1)$-dimensional faces sweep the whole boundary of the convex hull, which topologically is the $(2k-1)$-dimensional sphere $\mathbb{S}^{2k-1}$.

It is not hard to see that any affine hyperplane in $\mathbb{R}^{2k}$ intersects the image of $S_k(t)$ in at most $4k-2$ points and that one cannot replace $4k-2$ by a smaller number

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for any curve satisfying the symmetry condition (1). Curiously, the symmetric curve in $\mathbb{R}^d$ parameterized by

$$t \longmapsto \Big(t,\ t^3,\ \ldots, t^{2d-1}\Big) \quad \text{for} \quad -1 \leq t \leq 1$$

is not optimal in that respect, since there are hyperplanes intersecting it in $2d-1$ points. Hence it was suggested in [4] to use $S_k(t)$ to construct *symmetric* polytopes with the largest possible number of faces. Such polytopes are needed for efficient reconstruction of sparse signals, cf. [10].

2. Faces of the convex hull

The facial structure of the convex hull of $S_2(t)$ in $\mathbb{R}^4$ was completely described by Smilansky [12]. There are 0-dimensional faces $S_2(t)$ for $t \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, there are 1-dimensional faces spanned by the pairs of points $S_2(a)$ and $S_2(b)$ such that the length of the arc spanned by a and b in $\mathbb{S}^1$ is shorter than $2\pi/3$ and there are 3-dimensional faces spanned by the triples of points $S_2(a)$, $S_2(b)$ and $S_2(c)$ such that a, b and c are the vertices of an equilateral triangle in $\mathbb{S}^1$. Thus there is a 2-parameter family of 1-dimensional faces which therefore sweeps a 3-dimensional piece of the boundary of the convex hull, and there is a 1-parameter family of 2-dimensional faces which sweeps a complementary 3-dimensional piece of the boundary. Topologically, the boundary of the convex hull is the 3-dimensional sphere $\mathbb{S}^3$ and a moment's thought (no pun intended) convinces us that the two pieces provide the Hopf decomposition of $\mathbb{S}^3$ into the union of two solid tori.

But even the relatively simple case of $S_2(t)$ has its mysteries. The convex hull of $S_2(t_1), \ldots, S_2(t_N)$, where N is even and $t_1, \ldots, t_N \in \mathbb{S}^1$ are equally spaced points, is a 4-dimensional symmetric polytope with roughly $N^2/3$ edges. However, if we pick $N \equiv 0 \mod 4$ points $t_1, \ldots, t_N \in \mathbb{S}^1$ clustering around the vertices of a square, we get a 4-dimensional symmetric polytope with at least $3N^2/8$ edges, the current record (it is shown in [4] that a 4-dimensional symmetric polytope with N vertices cannot have more than $15N^2/32$ edges). It is not clear whether one can get any improvement by carefully positioning the points in the clusters around the vertices of the square. There is a vague feeling, reinforced somewhat by the iteration construction sketched below, that the optimal symmetric choice of $t_1, \ldots, t_N \in \mathbb{S}^1$ maximizing the number of edges of the convex hull should follow some "fractal pattern".

For larger k, we have only some fragmentary information regarding the facial structure of the convex hull of $S_k(t)$. First, we know that any two distinct points $S_k(a)$ and $S_k(b)$ such that the length of the arc spanned by $a, b \in \mathbb{S}^1$ is smaller than $\pi(2k-2)/(2k-1)$ span a 1-dimensional face (edge) of the convex hull of $S_k(t)$ [4] and that there are no other edges [13]. Second, we know that the convex hull of $S_k(t)$ is "at least quarter-neighborly": there is $\pi/2 < \alpha_k < \pi$ such that for any k distinct $t_1, \ldots, t_k \in \mathbb{S}^1$ that lie in an arc of length α_k, the points $S_k(t_1), \ldots, S_k(t_k)$

span a $(k-1)$-dimensional simplicial face of the convex hull of $S_k(t)$. Moreover, α_k converges to $\pi/2$ as k grows [1]. This k-parameter family of $(k-1)$-dimensional simplices sweeps a "solid" $(2k-1)$-dimensional part of the boundary of the convex hull (which is topologically a $(2k-1)$-dimensional sphere) but there are other big chunks. For example, it is easy to see that there is a 1-parameter family of $(2k-2)$-dimensional simplicial faces, which also sweeps a solid piece of the boundary of the convex hull. In general, no other families of faces appear to be known.

3. The iterated moment curve

Let us consider the following *iterated symmetric moment curve*:

$$t \longmapsto (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t, \ldots, \cos 3^m t,\ \sin 3^m t)\,.$$

It is shown in [2] that if $A_m \subset \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the set of $2(3^m - 1)$ equally spaced points then the convex hull of the corresponding set of points on the curve is a 2-neighborly (that is, every two non-antipodal vertices span an edge) symmetric polytope P of dimension $d = 2(m+1)$ with $2(3^m - 1)$ vertices. In terms of the dimension d, the polytope P has roughly $3^{d/2}$ vertices (it is shown in [8] that a d-dimensional symmetric 2-neighborly polytope cannot have more than 2^d vertices). The fact that P is 2-neighborly follows from the facial description of the convex hull of $S_2(t)$ and the fact that any two distinct non-antipodal points $t_1, t_2 \in A_m$ after at most $m-1$ iterations of the map

$$t \longmapsto 3t \mod 2\pi$$

of the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ end up spanning an arc of length greater than 0 but less than $2\pi/3$.

Similarly, by carefully choosing points on the iterated curve

$$t \longmapsto (S_k(t),\ S_k(3t),\ \ldots,\ S_k(3^m t))$$

one can construct a centrally symmetric k-neighborly d-dimensional polytope with $2^{c_k d}$ vertices for some $c_k > 0$ (one can choose $c_k = 3/20k^2 2^k$) [2]. Also, for any k and arbitrarily large N and d, one can construct a d-dimensional symmetric polytope with N vertices and at least $\left(1 - \delta_k^d\right)\binom{N}{k+1}$ faces of dimension k, for some $0 < \delta_k < 1$ [3].

4. Moment curves and isoperimetric problems

The symmetric moment curve (up to a simple linear transformation) appears in the work of Nudel'man [9], who proved that among all convex curves of length L in

$\mathbb{R}^{2k}$, the largest volume of the convex hull is achieved by the curve

$$t \longmapsto \frac{L}{\pi\sqrt{k}}\left(\cos t,\ \sin t,\ \frac{\cos 3t}{3},\ \frac{\sin 3t}{3},\ \ldots,\ \frac{\cos(2k-1)t}{2k-1},\ \frac{\sin(2k-1)t}{2k-1}\right)$$
$$\text{for} \quad 0 \leq t \leq \pi.$$

This followed a result by Schoenberg [11] that the similarly scaled trigonometric moment curve T_k maximizes the volume of the convex hull in the class of all closed convex curves of a given length in $\mathbb{R}^{2k}$. In the odd dimensions, in the class of all convex curves of a given length, the maximizer was found by Krein and Nudel'man, see [7], in the form of a properly scaled trigonometric moment curve $T_k(t)$ appended by a linear function in the parameter t.

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Monotonicity of functionals of random polytopes

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Dedicated to Imre Bárány on the occasion of his 70th birthday.

1. Introduction

Let n random points $X_1, ..., X_n$ be chosen independently and according to a given density function ϕ in $\mathbb{R}^d$. We call the convex hull $P_n = [X_1, ..., X_n]$ of these points a random polytope. Various properties of these objects have been studied in the last decades, e.g. the number of j-dimensional faces and the intrinscic volumes. Classical papers dealt with the expected values of these functionals, see e.g. Bárány [2], Bárány and Buchta [3], Bárány and Larman [5], Reitzner [25]. More recently, distributional properties have been investigated intensively, e.g. variance estimates, central limit theorems and and large deviation inequalities, see e.g. Bárány, Fodor, and Vigh [4], Bárány and Reitzner [6, 7], Calka, Schreiber and Yukich [10], Calka and Yukich [11, 12, 13] , Pardon [19, 20] Reitzner [23, 24, 25], Schreiber and Yukich [28], and Vu [29].

For all these questions the expectation is a central object. We denote by $\mathbb{E}V_d(P_n)$ the expected volume of the random polytope and by $\mathbb{E}f_j(P_n)$ the expectation of the number of j-dimensional faces.

In this short note we concentrate on monotonicity questions concerning the quantities $\mathbb{E}V_d(P_n)$ and $\mathbb{E}f_j(P_n)$ which have been investigated in the last years. For more information on random polytopes and related questions we refer to the survey articles [15] and [26].

2. Monotonicity of the volume with respect to set inclusion

Let $K \subset \mathbb{R}^d$ be a convex set with nonempty interior and $\phi(\cdot) = V_d(K)^{-1}\mathbb{1}_K(\cdot)$, thus the points X_i are chosen according to the uniform distribution in K and P_n is a random polytope in the interior of K. It seems to be immediate that increasing the convex body K should also increase the random polytope and thus its volume.

More precisely, assume that K, L are two d-dimensional convex sets. Choose independent uniform random points $X_1, \ldots, X_n$ in K and $Y_1, \ldots, Y_n$ in L, and denote by P_n the convex hull $[X_1, \ldots, X_n]$, and by Q_n the convex hull $[Y_1, \ldots, Y_n]$. Is it true that $K \subset L$ implies

$$\mathbb{E}V_d(P_n) \leq \mathbb{E}V_d(Q_n)? \tag{1}$$

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The starting point for the investigation should be a check for the first nontrivial case $n = d + 1$ where a random simplex is chosen in K, resp. L. In this form, the question was first raised by Meckes [17] in the context of high-dimensional convex geometry.

In dimension one the monotonicity is immediate. In 2012 in a groundbreaking paper by Rademacher [21] proved that this is also true in dimension two, but that there are counterexamples for dimensions $d \geq 4$ and $n = d + 1$. Only recently the three-dimensional case could be settled by Kunis, Reichenwallner and Reitzner [16] where monotonicity of $\mathbb{E}V_3(P_4)$ also fails. The question of monotonicity of higher moments $\mathbb{E}V_d(P_n)^k$ was investigated in [22].

It remains an open problem whether there is a number N, maybe depending on K or only on the dimension d, such that monotonicity holds for $n \geq N$.

3. Monotonicity of the number of faces with respect to n

Choose $X_1, ..., X_n$ according to a given density function ϕ in $\mathbb{R}^d$. A natural guess is that the expected number $\mathbb{E}f_j(P_n)$ of j-dimensional faces behaves monotone if the number of generating points increases. The asymptotic results suggest that at least for random points chosen uniformly in a smooth convex set and or a polytope (see [2], [3], [5], [25]) the expectation $\mathbb{E}f_j(P_n)$ should be increasing in n,

$$\mathbb{E}f_j(P_n) \leq \mathbb{E}f_j(P_{n+1}) \quad \forall n \in \mathbb{N}.$$

On the other hand Bárány [1] showed that the behaviour for generic convex sets is extremely complicated and thus monotonicity is not obvious.

The first results concerning this issue have been gained by Devillers et al. [14]. They considered convex hulls of uniformly distributed random points in a convex body K. It is proven that for planar convex sets the expected number of vertices $\mathbb{E}f_0(P_n)$ (and thus also edges) is increasing in n. Furthermore they showed that for $d \geq 3$ the number of facets $\mathbb{E}f_{d-1}(P_n)$ is increasing for n large enough if $\lim_{n\to\infty} \frac{\mathbb{E}f_{d-1}(P_n}{An^c} = 1$ for some constants A and $c > 0$, e.g. for K being a smooth convex body. In the PhD thesis of Beermann [8] the cases of ϕ being the Gaussian distribution or the uniform distribution in a ball are settled. We sketch the proof in the Appendix. The method used for these results was extented by Bonnet et al. [9] who settled the cases of random points on the sphere, on a halfsphere, random points chosen according to a certain heavy-tailed distribution, and beta-type distributions.

It should be noted that these results carry over to monotonicity results for convex hulls of random points chosen from a Poisson point process with a suitable density.

All these results only deal with the number of facets. Only in the Gaussian case it seems to be possible to extend this monotonicity results to general j-dimensional faces. This is the content of a recent preprint by Kabluchko. But even for other 'most simple cases' like uniform points in a ball the general question is widely open.

4. Appendix: Facet numbers of random polytopes

Let Φ be a probability measure in $\mathbb{R}^d$ with density φ. Choose n random points $X_1, \ldots, X_n$ independently according to Φ, and let P_n be the convex hull of these random points. We start by developing a well known formula for $\mathbb{E} f_{d-1}$. Each $(d-1)$-dimensional face of P_n is the convex hull of exactly d random points with probability one. Since $X_1, ..., X_n$ are chosen independently and identically it holds

$$\mathbb{E} f_{d-1}(P_n) \quad = \quad \binom{n}{d} \mathbb{P}([X_1, \ldots, X_d] \text{ is a facet }).$$

We denote by $H_{1,\ldots,d}$ the affine hull of the $(d-1)$-dimensional simplex $P_d = [x_1, \ldots, x_d]$ which divides $\mathbb{R}^d$ into the two halfspaces $H^+_{1,\ldots,d}$ and $H^-_{1,\ldots,d}$. If P_d is a facet then the other points $X_{d+1}, \ldots, X_n$ are either all located in $H^+_{1,\ldots,d}$, or in $H^-_{1,\ldots,d}$. This happens with probability $\Phi(H^+_{1,\ldots,d})^{n-d}$, resp. $\Phi(H^-_{1,\ldots,d})^{n-d}$, hence

$$\mathbb{E} f_{d-1}(P_n) \quad = \quad \binom{n}{d} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \left(\Phi(H^+_{1,\ldots,d})^{n-d} + \Phi(H^-_{1,\ldots,d})^{n-d}\right) \prod_{i=1}^d \phi(x_i) \mathrm{d}x_i. \quad (2)$$

We use the classical affine Blaschke-Petkantschin formula (see [27], Theorem 7.2.1.) to conclude

$$\begin{aligned}
\mathbb{E} f_{d-1}(P_n) \quad &= \quad (d-1)! \binom{n}{d} \int\limits_{S^{d-1}} \int\limits_0^\infty \left(\Phi(H(p,\omega)^+)^{n-d} + \Phi(H(p,\omega)^-)^{n-d}\right) \\
&\qquad \times \int\limits_{H(p,\omega)} \cdots \int\limits_{H(p,\omega)} \Delta_{d-1}(x_1, ..., x_d) \prod_{i=1}^d \phi(x_i) \mathrm{d}x_i \; \mathrm{d}p \mathrm{d}\omega \\
&= \quad (d-1)! \binom{n}{d} \int\limits_{S^{d-1}} \int\limits_{-\infty}^\infty \Phi(H(p,\omega)^+)^{n-d} \\
&\qquad \int\limits_{H(p,\omega)} \cdots \int\limits_{H(p,\omega)} \Delta_{d-1}(x_1, ..., x_d) \prod_{i=1}^d \phi(x_i) \mathrm{d}x_i \; \mathrm{d}p \mathrm{d}\omega
\end{aligned}$$

where we parametrize the hyperplane by $H(p,\omega) = \{x \colon \langle x, \omega \rangle = p\}$, and the half-spaces by $H(p,\omega)^- = \{x \colon \langle x, \omega \rangle \leq p\}$ and $H(p,\omega)^+ = \{x \colon \langle x, \omega \rangle \geq p\}$. In the inner integral $\Delta_{d-1}(x_1, ..., x_d)$ is the $(d-1)$-dimensional volume of the convex hull of $x_1, ..., x_d$. We fix the direction ω and want to prove the montonicity in n of

$$\mathcal{I}(n) = \binom{n}{d} \int\limits_{\mathbb{R}} \Phi(H(p,\omega)^+)^{n-d} \int\limits_{H(p,\omega)} \dots \int\limits_{H(p,\omega)} \Delta_{d-1}(x_1, ..., x_d) \prod_{i=1}^{d} \phi(x_i) \mathrm{d}x_i \, \mathrm{d}p$$

For a given direction ω we put $\psi(t) = \int_{H(t,\omega)} \phi(x)\mathrm{d}x$ and $\Psi(p) = \int_{-\infty}^{p} \psi(t)\mathrm{d}t$ which defines the push forward measure of Φ under the projection onto the line $\{t\omega \colon t \in \mathbb{R}\}$. Observe that on the support of ψ, the mass of halfspaces $\Psi(p) = \Phi(H(p,\omega)^-)$ is an increasing function in p and thus there is an inverse function $\Psi^{-1}(s)$, also increasing, with

$$\frac{d}{ds}\Psi^{-1}(s) = \Big(\frac{d}{dp}\Phi(H(p,\omega)^-)|_{p=\Psi^{-1}(s)}\Big)^{-1} = \Big(\psi(p)|_{p=\Psi^{-1}(s)}\Big)^{-1} = \Big(\psi(\Psi^{-1}(s))\Big)^{-1}$$

for $s \in (0,1)$, and thus $\mathrm{d}p = (\psi(\Psi^{-1}(s)))^{-1}\mathrm{d}s$. Substituting by $s = \Psi(p) = \Phi(H(p,\omega)^-)$ we end up with

$$\mathcal{I}(n) = \binom{n}{d} \int\limits_0^1 (1-s)^{n-d} \psi(\Psi^{-1}(s))^{d-1} \, \mathbb{E}_{H(\Psi^{-1}(s),\omega)} \Delta_{d-1}(X_1, \dots X_d) \, \mathrm{d}s$$

where $\mathbb{E}_{H(p,\omega)}\Delta_{d-1}(X_1, \dots X_d)$ is the volume of a random simplex where the points $X_1, \dots, X_d$ are chosen independently according to the normalized density φ in $H(p,\omega)$. Thus to prove monotonicity we have to show that $\triangle_n \mathcal{I} = \mathcal{I}(n) - \mathcal{I}(n-1)$ is positive,

$$\triangle_n \mathcal{I} = \frac{1}{n}\binom{n}{d} \int\limits_0^1 (1-s)^{n-d-1}(d-ns) L(s)^{d-1} \, \mathrm{d}s$$

with

$$L(s) = \psi(\Psi^{-1}(s)) \Big(\mathbb{E}_{H(\Psi^{-1}(s),\omega)} \Delta_{d-1}(X_1, \dots X_d)\Big)^{\frac{1}{d-1}}.$$

In the next two sections we will show that in both cases we are interested in, the function $L(s)$ is concave. This is sufficient, because then the graph of $L(s)$ starts at the origin, is above the line $l(s) = L(\frac{d}{n})\frac{ns}{d}$ in $(0, \frac{d}{n})$, meets the line for $s = \frac{d}{n}$, and is below the line for $s > \frac{d}{n}$. This yields

$$\begin{aligned} \triangle_n \mathcal{I} &\geq \frac{1}{n}\binom{n}{d} \int\limits_0^1 (1-s)^{n-d-1}(d-ns) l(s)^{d-1} \, \mathrm{d}s \\ &= \frac{n^{d-2}}{d^{d-1}} \binom{n}{d} L\Big(\frac{d}{n}\Big)^{d-1} \underbrace{\int\limits_0^1 (1-s)^{n-d-1}(d-ns) s^{d-1} \, \mathrm{d}s}_{=d\mathbf{B}(n-d,d) - n\mathbf{B}(n-d,d+1)} = 0 \, , \end{aligned}$$

and hence $\mathbb{E} f_{d-1}(P_n)$ is increasing.

4.1. The Case of Gaussian Polytopes

In this case we have $\phi(x) = \frac{1}{(2\pi)^{d/2}} \exp\{-\sum x_i^2/2\}$. By the rotation invariance it sufficies to consider the case $\omega = (1, 0, \ldots, 0)$ where it is easy to see that $\psi(t) = \frac{1}{\sqrt{2\pi}} \exp\{-t^2/2\}$ and that $\mathbb{E}_{H(p,\omega)} \Delta_{d-1}(X_1, \ldots X_d)$ is independent of p. Thus $L(s) = c_d \psi(\Psi^{-1}(s))$.

The continously differentiable function $L(s)$ is concave if and only if its derivative is decreasing. Since $\psi'(t) = -t\psi(t)$, it follows that

$$L'(s) = c_d \frac{d}{ds} \psi(\Psi^{-1}(s)) = -c_d \Psi^{-1}(s) \psi(\Psi^{-1}(s)) \big(\psi(\Psi^{-1}(s))\big)^{-1} = -c_d \Psi^{-1}(s).$$

Clearly, $\Psi(s)$ is increasing in s, and therefore $\Phi^{-1}(s)$ too. This implies that $-\Phi^{-1}(s)$ is decreasing and $L(s)$ is concave on $[0, 1]$.

4.2. The Case of Random Polytopes in a Ball

Assume that B^d is the unit ball of volume κ_d. In this case $\phi(x) = \kappa_d^{-1} \mathbb{1}(x \in B^d)$. By the rotation invariance it sufficies to consider the case $\omega = (1, 0, \ldots, 0)$ where $\psi(t) = \kappa_d^{-1} \kappa_{d-1} (1 - t^2)^{(d-1)/2}$. The intersection of $H(t, \omega) \cap B^d$ is always a ball of radius $(1 - t^2)^{1/2}$ and the expected volume of a random simplex in $H(t, \omega) \cap B^d$ is a constant times $V_{d-1}(H(t, \omega) \cap B^d) = \psi(t)$. The constant is determined explicitly in a paper of Miles [18]. Thus $L(s) = c_d \psi(\Psi^{-1}(s))^{\frac{d}{d-1}}$. We have

$$L'(s) = c_d \frac{d}{ds} \psi(\Psi^{-1}(s))^{\frac{d}{d-1}} = c_d \, \frac{d}{d-1} \, \frac{\psi'(\Psi^{-1}(s))}{\psi(\Psi^{-1}(s))^{\frac{d-2}{d-1}}} = c_d \, d \Big(\frac{d}{dp} \psi(p)^{\frac{1}{d-1}} \Big) \Big|_{p = \Psi^{-1}(s)}$$

Because ψ is a concave function, its derivative is decreasing in p. Noting that Ψ^{-1} is increasing shows that L' is decreasing und thus the function $L(s)$ is concave.

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Thieves dividing birthday presents and birthday cakes

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Dedicated to Imre Bárány on the occasion of his 70th birthday

It was the Fall of 2003, my son was just born and I was adjusting to parenthood. At the same time the Mathematical Sciences Research Institute in Berkeley was hosting a semester program *Discrete and Computational Geometry*. Many of the stars of my postgraduate studies were listed as organizers, Jesús A. De Loera, Herbert Edelsbrunner, Jacob E. Goodman, János Pach, Micha Sharir, Emo Welzl, and Günter M. Ziegler. For an unknown PhD with no support and no relevant publications it was almost impossible to attend such an event. Nevertheless on 16 November of 2003 I arrived to attend the last workshop. In the following days I was mesmerized my the lectures of László Lovász, Louis Billera, Bernd Sturmfels, Jiří Matoušek, and Alexander Bobenko.

In a break between the lectures a miracle occurred, though not yet a mathematical one. Imre Bárány approached me and started a conversation. His gentle, familiar and direct way of communicating overwhelmed me, made me feel that I am at the right place, talking to the right person, having a conversation that will make a significant impact on my professional life. While we were talking, the Golden Gate casted an unreal light onto the blackboard filled with the problems that are going to run my life for decades to come. The beauty of Imre's mathematics with clear and right to the heart presentation was stunning. On that day Imre Bárány opened the door of the mathematical research for me and gently and firmly guided me over the first obstacles. Since then he shared with me his knowledge, ideas, insight and most importunely his support and friendship. I am honored to have Imre Bárány as my teacher, collaborator, and a family friend.

The following results are dedicated to a man who gain and has admiration of all of his colleagues, that inspired only the best in us and made our mathematics beautiful, fun and worth doing.

In Berlin on 30 May 2017,

Pavle V. M. Blagojević

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1. Partition problems

Measure partition problems are classical, significant and challenging questions of Discrete Geometry secretly attracted to Algebraic Topology.

We are going to consider convex partitions of the Euclidean space $\mathbb{R}^d$. More precisely, an ordered collection of n closed subsets $\mathcal{K} = (K_1, \dots, K_n)$ of $\mathbb{R}^d$ is a *partition of* $\mathbb{R}^d$ if it is a covering $\mathbb{R}^d = K_1 \cup \dots \cup K_n$, all the interiors $\mathrm{int}(K_1), \dots, \mathrm{int}(K_n)$ are non-empty, and $\mathrm{int}(K_i) \cap \mathrm{int}(K_j) = \emptyset$ for all $1 \le i < j \le n$. A partition $\mathcal{K} = (K_1, \dots, K_n)$ is said to be a *convex partition of* $\mathbb{R}^d$ if each element of the partition is convex. Furthermore, for an integer $r \ge 1$ an *r-labeled (convex) partition of* $\mathbb{R}^d$ is an ordered pair $(\mathcal{K}, \ell)$ where $\mathcal{K} = (K_1, \dots, K_n)$ is a (convex) partition of $\mathbb{R}^d$, and $\ell \colon [n] \longrightarrow [r]$ is an arbitrary function. We use the notation $[n] := \{1, \dots, n\}$.

We assume that all measures in $\mathbb{R}^d$ are probability measures that are absolutely continuous with respect to the Lebesgue measure. In particular this means that the overlapping boundary $\bigcup_{1 \le i < j \le n} K_i \cap K_j$ of the elements of a partition always has measure zero.

One of the widely known measure partitioning results is the Ham Sandwich theorem, which was conjectured by Steinhaus and proved subsequently by Banach in 1938.

Theorem (Ham Sandwich theorem)**.** *Let $d \ge 1$ be an integer. For any collection of d measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$, there exists an affine hyperplane H that simultaneously splits them into halves. Namely, we have*

$$\mu_i(H^+) = \mu_i(H^-)$$

for all $1 \le i \le d$, where H^+ and H^- denote closed half-spaces determined by H.

The reason for its name is an illustration where each of the measures is thought of as a different ingredient floating in $\mathbb{R}^d$ (that is out birthday cake). The goal is to make two sandwiches (or pieces of the cake) with equal amount of each ingredient by cutting $\mathbb{R}^d$ with a single hyperplane slice. On the other hand, if more people (are attending and) want their sandwich (cake pieces) and fancy convex shapes, it has been shown by different groups of authors [10] [8] [4] that *for any collection of d measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$ and any integer $r \ge 1$ there is a convex partition $\mathcal{K} = (K_1, \dots, K_r)$ of $\mathbb{R}^d$ into r parts that simultaneously split each measure into r parts of equal size. Namely,*

$$\mu_i(K_1) = \dots = \mu_i(K_r)$$

for all $1 \le i \le d$.

The second classic and appealing measure partition result is the following "Necklace Splitting" theorem of Hobby and Rice [7].

Theorem (Necklace Splitting theorem). *Let $m \geq 1$ be an integer. Then for any collection of m measures $\mu_1, \dots, \mu_m$ in $\mathbb{R}$, there exists a 2-labeled convex partition $(\mathcal{K}, \ell)$ of $\mathbb{R}$ into a collection of $m+1$ intervals $\mathcal{K} = (K_1, \dots, K_{m+1})$ with $\ell\colon [m+1] \longrightarrow [2]$ such that for every $1 \leq j \leq m$:*

$$\mu_j\Big(\bigcup_{i \in \ell^{-1}(1)} K_i\Big) = \mu_j\Big(\bigcup_{i \in \ell^{-1}(2)} K_i\Big) = \tfrac{1}{2}.$$

The discrete version of this theorem, where the measures are the counting measures of finite sets of points, was first proved by Goldberg and West [6], and then by Alon and West [3]. The common illustration of this theorem is as follows. Two thieves steal an open necklace (that is a birthday gift) with m types of pearls, knowing that there is an even number of each kind of pearls. They will cut the necklace into pieces and distribute those among themselves, so that each receives half of each kind of pearls. The result above shows that this can always be achieved with m cuts, regardless of the order of the pearls. The function ℓ is simply telling us who gets each part. The version with arbitrary number r of thieves was given by Alon in [2], where $(r-1)m$ cuts are shown to be sufficient. The number of cuts cannot be improved. Extensions of this result with additional combinatorial conditions on the distribution of the necklace appear in [1].

Our main goal is to present a common generalization of the Ham Sandwich theorem and the Necklace Splitting theorem. In other words, given more than d ingredients in $\mathbb{R}^d$, we should be able to find a fair distribution among r hungry persons if we are willing to split $\mathbb{R}^d$ into more than r parts. Alternatively, if r thieves steal a high-dimensional necklace, they should be able to distribute it among themselves by splitting it into very few convex parts.

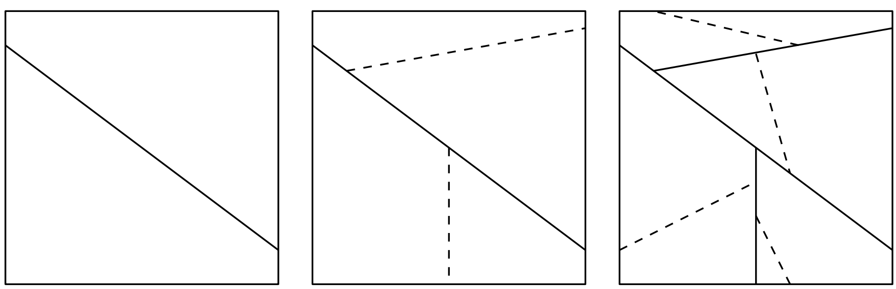

Figure 1: An iterated partition of the plane by successive hyperplane cuts

High-dimensional versions of the necklace splitting problem were given by de Longueville and Živaljević [5], and by Karasev, Roldán-Pensado and Soberón [9]. In [5], the authors proved an analogous result for r thieves and m measures in $\mathbb{R}^d$, where the partitions are made using $(r-1)m$ hyperplanes each of whose directions is fixed in advance and must be orthogonal to a vector of the canonical basis of $\mathbb{R}^d$.

The downside of this type of partitions is that there may be an extremely large number of pieces to distribute.

One way to address this issue in the case when $r = 2$ is to consider iterated hyperplane partitions. A partition of $\mathbb{R}^d$ into convex parts is an *iterated hyperplane partition* if it can be made out of $\mathbb{R}^d$ by successively partitioning each convex part, from a previous partition, with a hyperplane (that means each new hyperplane only cuts one of the existing convex parts, see Figure 1). In [9] the authors showed that for r thieves and m measures in $\mathbb{R}^d$, there is a fair distribution of each measure among the thieves using an iterated hyperplane partition that has $(r-1)m$ hyperplane cuts, whose directions are fixed in advance, as long as r is a prime power. This has the advantage that the total number of parts is $(r-1)m+1$.

In both results, there is little to gain from the increasing dimension. This is a consequence of fixing the directions of the cutting hyperplanes. Thus, it is natural to wonder what can be gained if the fixed directions restriction is disregarded. In this situation we distinguish two different types of labeled partitions. The first type of partitions are labeled partitions of $\mathbb{R}^d$ into n convex parts without any additional requirements. For the second type of partitions we consider iterated convex partitions of $\mathbb{R}^d$ that in the case when $r = 2$ coincide with iterated hyperplane partitions.

Definition. *Let $n \geq 1$, $r \geq 1$ and $d \geq 1$ be integers.*

(1) Let $M = M(n,r,d)$ be the largest integer such that for any collection of M measures $\mu_1, \dots, \mu_M$ in $\mathbb{R}^d$ there exists an r-labeled convex partition $(\mathcal{K}, \ell)$ of $\mathbb{R}^d$ into n parts $\mathcal{K} = (K_1, \dots, K_n)$ with $\ell\colon [n] \longrightarrow [r]$ with the property that for all $1 \leq j \leq M$ and all $1 \leq s \leq r$ we have

$$\mu_j\Big(\bigcup_{i \in \ell^{-1}(s)} K_i\Big) = \tfrac{1}{r}.$$

Every such labeled convex partition into n parts is called a fair distribution between the thieves.

(2) Let $M' = M'(n,r,d)$ be the largest integer such that for any collection of M' measures $\mu_1, \dots, \mu_{M'}$ in $\mathbb{R}^d$ there exists an r-labeled iterated partition $(\mathcal{K}, \ell)$ of $\mathbb{R}^d$ into n convex parts $\mathcal{K} = (K_1, \dots, K_n)$ with $\ell\colon [n] \longrightarrow [r]$ so for all $1 \leq j \leq M'$ and all $1 \leq s \leq r$ we have

$$\mu_j\Big(\bigcup_{i \in \ell^{-1}(s)} K_i\Big) = \tfrac{1}{r}.$$

Every such r-labeled convex partition into n parts is called a fair iterated distribution between the thieves.

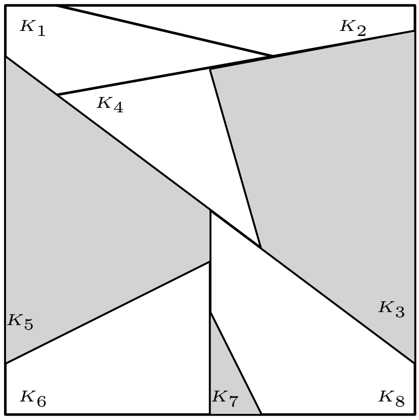

Figure 2: 2-labeled partition by iterated hyperplane cuts with $\ell = \binom{12345678}{11212121}$

Some of the convex parts K_i in the partition $\mathcal{K}$ can be empty. For an example of a 2-labeled convex partition formed by iterated hyperplane cuts see Figure 2.

For all integers $n \geq 1$, $r \geq 1$ and $d \geq 1$ it is clear that $M'(n,r,d) \leq M(n,r,d)$. The Ham Sandwich theorem is equivalent to the statement that $M'(2,2,d) = M(2,2,d) = d$, while the Necklace Splitting theorem for two thieves is equivalent to $M'(n+1,2,1) = M(n+1,2,1) = n$. Their respective extensions for distributions among more persons simply state that $M(r,r,d) = d$ and $M(n,r,1) = \lfloor \frac{n-1}{r-1} \rfloor$.

It is possible to prove the following bound on the function $M'(n,r,d)$.

Theorem 1. *Let $d \geq 1$ and $t \geq 1$ be integers, and let $r \geq 2$ be a prime. Then,*

$$M'(rt,r,d) \geq \lceil \tfrac{td(r-1)+t}{r-1} - 1 \rceil.$$

Moreover, this result is optimal for $M(r,r,d)$ and $M(n,2,1)$.

The labeled partitions we use to prove this theorem have additional property: From the rt convex parts, every thief receives exactly t of them. The result above implies that $M(r,r,d) = d$ for any r using a standard factorization argument. This factorization argument only works well if $t = 1$ or $d = 1$.

For the case $r = 2$, our results actually give iterated hyperplane partitions. Since those also include results for the case when we use an odd number of parts, we state them separately.

Theorem 2. *Let $d \geq 1$ be an integer. Then,*

$$\begin{array}{lcll} M'(2t,2,d) & \geq & t(d+1)-1, & \textit{for } t \geq 1 \\ M'(2t+1,2,d) & \geq & t(d+1), & \textit{for } t \geq 0. \end{array}$$

Moreover, this result is optimal for $M'(n,2,1), M'(3,2,d)$ and $M'(2,2,d)$.

For $r = 2$ the previous theorem says that $M'(n, 2, d) \sim \left\lfloor \frac{n}{2} \right\rfloor (d+1)$, which can be seen as a common extension of the Ham Sandwich theorem and the Necklace Splitting theorem clearer. For larger values of r, the lower bounds we obtain for $M(n, d, r)$ are roughly $\frac{nd}{r}$.

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Polynomials in finite geometry and combinatorics

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In 1993 I gave a talk at the 14th BCC in Keele: Polynomials in finite geometry and combinatorics [5]. My talk will be about (some of) my favourite results that happened after that. We start with introducing the main ingredients: $\mathrm{GF}(q) = \mathbf{F}_q$ is the finite field of order $q = p^h$, p prime.

$\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$ denote the n-dimensional projective and affine space.

Affine space has coordinates $(x_1, \ldots, x_n) \in \mathbf{F}_q^n$. Projective space has homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$.

In particular $\mathrm{PG}(2, q)$ has $q^2 + q + 1$ points and lines, $q + 1$ points on every line, and $q + 1$ lines through every point.

1. Maximal arcs, 1996 and 1998

A (k, n)-arc in $\mathrm{PG}(2, q)$ is a set B of k points, at most n on a line. We see that $k \le 1 + (q + 1)(n - 1)$ with equality if lines intersect the arc in 0 or n points, and then $n \mid q$.
In this case B is called a *maximal arc*.
In 1996 Simeon Ball, Franco Mazzocca and I proved [3, 4]:

Theorem 1 (Ball, B., Mazzocca 1997)**.** *Maximal arcs in planes of odd order do not exist.*

A key ingredient in the proof is the maximal arcs polynomial

$$F(t, x) = \prod_{b \in B} (1 - (1 - bx)^{q-1} t)$$

or reversed

$$G(T, X) = \prod_{b \in B} (T - (X - b)^{q-1})$$

The proof can be completed along the following steps:

1. Look at $B \subset \mathrm{AG}(2, q) \simeq \mathbf{F}_{q^2}$.
2. $(x - b)^{q-1}$ gives the direction of the line $\langle x, b \rangle$.
3. From $x \in B$ you see every direction $n - 1$ times:

$$F(T, x \in B) = (T^{q+1} - 1)^{n-1}$$

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4. From $x \notin B$ you see every direction 0 or n times:

$$F(T, x \notin B) = H(T^n)$$

2. The direction problem, 1999 and 2003

Theorem 2 (Ball, B., Brouwer, Storme, Szőnyi 1999 and Ball 2003)**.** *If the number m of directions determined by $f : \mathbf{F}_q \to \mathbf{F}_q$ is less than $(q+3)/2$, and $f(0) = 0$ then f is linear over a subfield (of $\mathbf{F}_q$). The possible values of m can be determined (in principle, not in practice) [1, 6].*

Geometric view and the case q is prime. If $q = r^e$ then $\mathrm{AG}(2, q) \simeq V(2, \mathbf{F}_q) \simeq V(2e, \mathbf{F}_r)$ and the graph of f is an e-dimensional subspace.

Much more can be said if q is prime. In this case Rédei and Megyesi [14] in 1970 proved $m \geq (q+3)/2$, Lovász and Schrijver [13] in 1981 characterized the case of equality. For larger m we have

Theorem 3 (Gács 2003 [12])**.** *If the number of directions determined by $f : \mathbf{F}_p \to \mathbf{F}_p$ is larger than $(p+3)/2$, then it is at least $2(p-1)/3$.*

3. Directions in space

The strong cylinder conjecture, due to Ball [2], reads as follows:

If C is a set of p^2 points in $AG(3, p)$, such that every plane intersects C in a multiple of p points, then C must be a cylinder, the union of p parallel lines.

He proved the following.

Theorem 4 (Ball, 2008)**.** *Let $S \subset AG(3, q)$ of size q^2 with at least q* not *determined directions. Then every plane intersects S in 0 mod p points.*

4. The finite field Kakeya problem, 2009

The finite field Kakeya problem asks for the (minimal) number of points covered by a set of lines in $\mathrm{AG}(n, q)$, one in every direction.

One of the mathematical gems of the past ten years is the beautiful result by Dvir [11], showing that the order of magnitude is q^n.

Theorem 5 (Z. Dvir, 2010)**.** *The number of points that is covered by such a Kakeya-set is at least $\binom{q+n-1}{n}$.*

Almost 30 years ago Aiden Bruen and I proved the following result [8]:

Theorem 6 (A. B., A. A. Bruen, 1989)**.** *If $q \geq 7$ is odd, then a set of $q+2$ lines in $PG(2, q)$ covers at least $\frac{1}{2}(q+1)(q+2) + \frac{1}{3}(q+2)$ points.*

Of course, if q is even, the $q+2$ lines of a dual hyperoval cover only $\frac{1}{2}(q+1)(q+2)$ points. The result is sharp for $q = 7$ (in two ways), but one would like to prove, for *all* odd q:

$$\frac{1}{2}(q+1)(q+2) + \frac{1}{2}(q-1).$$

This result comes close to solving the Kakeya problem in the plane. This we managed to do much later [9]:

So the problem is: How many points are covered by a set of $q+1$ lines, one in every direction. If q is even, the lines can be taken 'in general position', as we have seen.

If q is odd, one takes q lines in general position, and adds a suitable line in the last direction.

Theorem 7 (A.B., F. Mazzocca, 2009)**.** *The number of points that is covered by a Kakeya-set is at least* $\binom{q+1}{2}$ *if* q *is even and* $\binom{q+1}{2} + \frac{1}{2}(q-1)$ *if* q *is odd.*

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Families of vectors without antipodal pairs

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Abstract

Some Erdős-Ko-Rado type extremal properties of families of vectors from $\{-1, 0, 1\}^n$ are considered.

1. Introduction

The standard n-cube is formed by all vectors $v = (v_1, \dots, v_n)$ with $v_i \in \{0, 1\}$. Setting $F(v) := \{i : v_i = 1\}$ is a natural way to associate a subset of $[n] := \{1, \dots, n\}$ with a vertex of the n-cube. This association has proved very useful in tackling various problems in discrete geometry. In particular, intersection theorems concerning finite sets were the main tool in proving exponential lower bounds for the chromatic number of $\mathbb{R}^n$ and disproving Borsuk's conjecture in high dimensions (cf. [6], [7]).

In this short note we consider $(0, \pm 1)$-vectors, that is, vectors $v = (v_1, \dots, v_n)$, where each v_i is $0, 1$, or -1. Probably the first non-trivial extremal result concerning these objects was a result of Deza and the first author [2] showing that in a certain situation one can prove the same best possible upper bound for $(0, \pm 1)$-vectors as for the restricted case of $(0, 1)$-vectors.

Raigorodskii [15] and others (cf, e.g., [13], [11]) have used a similar approach to improve the bounds for the above-mentioned and related discrete geometry problems, obtained via $(0, 1)$-vectors, by considering $(0, \pm 1)$-vectors.

Motivated by such results we propose to investigate the following problem. Let $k \ge l \ge 1$ be integers and let $V(n, k, l)$ denote the set of all $(0, \pm 1)$-vectors of length n and having exactly k coordinates equal to $+1$ and l coordinates equal to -1. Note that

$$|V(n, k, l)| = \binom{n}{k}\binom{n-k}{l} = \binom{n}{k+l}\binom{k+l}{l}.$$

For two vectors let $\langle v, w \rangle$ denote their scalar product: $\langle v, w \rangle = \sum_{i=1}^n v_i w_i$.

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If $v, w \in V(n,k,l)$, then they possess altogether $2l$ coordinates equal to -1. Thus

$$\langle v, w\rangle \geq -2l.$$

If $\langle v, w\rangle = -2l$, then we call these two vectors antipodal. Note that for a fixed $v \in V$ there are $\binom{k}{l}\binom{n-k-l}{k-2l}$ antipodal vectors $w \in V$. To avoid trivialities, we assume in what follows that $n \geq 2k$.

Example 1. Let $\mathcal{G} \subset V(n,k,l)$ consist of those vectors whose last non-zero coordinate is a -1. Then $|\mathcal{G}| = \binom{n}{k+l}\binom{k+l-1}{l-1}$, and it is easy to see that $\mathcal{G}$ contains no two antipodal vectors.

The purpose of this note is to prove the following two theorems.

Theorem 1. *Suppose that $\mathcal{F} \subset V(n,k,l)$ does not contain two antipodal vectors. Then*

$$|\mathcal{F}| \leq \binom{n}{k+l}\binom{k+l-1}{l-1} + \binom{n}{2k}\binom{2l}{l}\binom{n-2l-1}{k-l-1}. \tag{1}$$

Note that the last term of (1) is $O(n^{k+l-1})$. We also put $\binom{n-2l-1}{k-l-1} := 0$ for $k = l$. Thus (1) shows that Example 1 is asymptotically best possible.

Example 2. Let $\mathcal{E} \subset V(n,k,l)$ consist of those vectors whose first coordinate is a 1. Then $|\mathcal{E}| = \binom{n-1}{k+l-1}\binom{k+l-1}{k-1} = \frac{k}{n}|V(n,k,l)|$, and $\mathcal{E}$ contains no two antipodal vectors.

Theorem 2. *Suppose that $\mathcal{F} \subset V(n,k,l)$ does not contain two antipodal vectors. If $2k \leq n \leq 3k-l$, then*

$$|\mathcal{F}| \leq \frac{k}{n}|V(n,k,l)|. \tag{2}$$

Theorem 2 shows that Example 2 is best possible for $2k \leq n \leq 3k-l$. We note that the case $n \leq 2k$ can be easily reduced to sets setting and the Erdős-Ko-Rado theorem (see below). Let us also mention that in [4] we gave the complete solution for the case $l = 1$:

Theorem (Frankl, Kupavskii [4]). *Suppose that $\mathcal{F} \subset V(n,k,1)$ does not contain two antipodal vectors. Then one has*

$$|\mathcal{F}| \leq k\binom{n-1}{k} \qquad \text{for } 2k \leq n \leq k^2,$$

$$|\mathcal{F}| \leq k\binom{k^2-1}{k} + \binom{k^2}{k} + \binom{k^2+1}{k} + \ldots + \binom{n-1}{k} \qquad \text{for } n > k^2.$$

Both inequalities are best possible.

The proof of Theorem 1 is rather short, but it relies on some classical results in extremal set theory.

Definition 1. *Two families $\mathcal{A}, \mathcal{B}$ of finite sets are called* cross-intersecting, *if for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ one has $A \cap B \neq \emptyset$. For the case $\mathcal{A} = \mathcal{B}$ we use the term* intersecting.

For $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the collection of all k-subsets of $\{1, \dots, n\}$.

Theorem (Erdős-Ko-Rado [3]). *Suppose that $n \geq 2k > 0$, and the family $\mathcal{A} \subset \binom{[n]}{k}$ is intersecting. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}. \tag{3}$$

As Daykin [1] observed, (3) can be deduced from the Kruskal-Katona Theorem ([10], [8]). the same approach yields the following version of (3) for cross-intersecting families.

Proposition 3. *Let a, b, m be integers, $m \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[m]}{a}$ and $\mathcal{B} \subset \binom{[m]}{b}$ are cross-intersecting. Then either $|\mathcal{A}| \leq \binom{m-1}{a-1}$ or $|\mathcal{B}| \leq \binom{m-1}{b-1}$ hold.*

Note that stronger versions of this proposition were proved by Pyber [14], Matsumoto and Tokushige [12], and the authors of this note [5].

2. The proof of Theorem 1

Let $\mathcal{F}$ be our family of vectors. For a $(0, \pm 1)$-vector $v = (v_1, \dots, v_n)$ let $S(v)$ denote its support, i.e.,

$$S(v) := \{i \in [n] : v_i \neq 0\}.$$

Define also

$$S_+(v) := \{i : v_i = +1\},$$
$$S_-(v) := \{i : v_i = -1\}.$$

Obviously, $S(v) = S_+(v) \sqcup S_-(v)$.

Also, for $k \geq l$ two vectors v, w satisfy $(v, w) = -2l$ if and only if $S_-(v) \subset S_+(w), S_-(w) \subset S_+(v)$ and $S_+(v) \cap S_+(w) = \emptyset$ hold simultaneously.

Our assumption is that no such pair v, w exist in $\mathcal{F}$. For a pair A, B of disjoint l-element sets we define $\mathcal{F}(A, B)$ to be the family of those $(k-l)$-element sets C that the vector u defined by $S_+(u) = A \sqcup C$, $S_-(u) = B$ is in $\mathcal{F}$.

Lemma 4. *For disjoint l-subsets $A, B \subset [n]$ the two families $\mathcal{F}(A, B)$ and $\mathcal{F}(B, A)$ are cross-intersecting.*

Proof. Suppose the contrary and let $C \in \mathcal{F}(A, B), D \in \mathcal{F}(B, A)$ be disjoint $(k-l)$-sets. Then the vectors v, w determined by $S_+(v) = A \sqcup C$, $S_-(v) = B$, $S_+(w) = B \sqcup D$, $S_-(w) = A$ are both in $\mathcal{F}$. However, $\langle v, w \rangle = -2l$, a contradiction. □

This lemma and Proposition 3 motivate the following procedure. For all $\binom{n}{2l}\binom{2l}{l}$ choices of a pair of disjoint l-sets A and B, if $|\mathcal{F}(A,B)| \leq \binom{n-2l-1}{k-l-1}$, then delete from $\mathcal{F}$ *all* vectors v with $S_-(v) = B$, $A \subset S_+(v)$.

Let $\mathcal{F}'$ be the collection of remaining vectors and note:

$$|\mathcal{F}'| \geq |\mathcal{F}| - \binom{n}{2l}\binom{2l}{l}\binom{n-2l-1}{k-l-1}. \tag{4}$$

Let us fix now a $(k+l)$-element set $T \subset [n]$ and consider the family $\mathcal{B} \subset \binom{T}{l}$ defined as follows:

$$\mathcal{B} := \left\{B \in \binom{T}{l} : \exists v \in \mathcal{F}', S(v) = T, S_-(v) = B\right\}.$$

Lemma 5. *The family $\mathcal{B}$ is intersecting.*

Proof. Suppose for contradiction that $A, B \in \mathcal{B}$ are disjoint. By the definition of $\mathcal{B}$ there are $u, v \in \mathcal{F}'$ satisfying $S_-(u) = A$, $S_-(v) = B$, $S(u) = S(v) = T$. This implies $A \subset S_+(v), B \subset S_+(u)$.

Since both u and v survived the deletion process, we have

$$|\mathcal{F}(A,B)| > \binom{n-2l-1}{k-l-1},$$
$$|\mathcal{F}(B,A)| > \binom{n-2l-1}{k-l-1}.$$

However, Proposition 3 shows that $\mathcal{F}(A,B)$ and $\mathcal{F}(B,A)$ are *not* cross-intersecting. This contradicts Lemma 4. □

Since $k \geq l$, the Erdős-Ko-Rado Theorem implies

$$|\mathcal{B}| \leq \binom{k+l-1}{l-1}.$$

Consequently,

$$|\mathcal{F}'| \leq \binom{n}{k+l}\binom{k+l-1}{l-1}.$$

Combining with (4), the inequality (1) follows.

3. The proof of Theorem 2

The proof is based on the application of the general Katona's circle method [9] to $\mathcal{V}(n,k,l)$. Consider the following subfamily $\mathcal{H}$ of $\mathcal{V}(n,k,l)$:

$$\mathcal{H} := \{\mathbf{v} = (v_1, \ldots, v_n) : \exists i \in [n] : \ v_i = \ldots = v_{i+k-1} = 1,$$
$$v_{i-k} = \ldots = v_{i-k+l-1} = -1\}$$

We remark that all indices are written modulo n. Note that $|\mathcal{H}| = n$. For any permutation σ of $[n]$ we define $\mathcal{H}(\sigma) := \{\sigma(H) : H \in \mathcal{H}\}$.

Take a family $\mathcal{F}$ with no two antipodal vectors.

Lemma 6. *For any permutation σ we have $|\mathcal{H}(\sigma) \cap \mathcal{F}| \le k$.*

Proof. Denote by $\mathcal{F}' \subset \binom{[n]}{k}$ the family $\{S_+(F) : F \in \mathcal{F}\}$, and, similarly, $\mathcal{H}' := \{S_+(H) : H \in \mathcal{H}(\sigma)\}$.

We claim that $\mathcal{H}' \cap \mathcal{F}'$ is an intersecting family. Assume that there are two sets $F_1', F_2' \in \mathcal{H}' \cap \mathcal{F}'$, that are disjoint. W.l.o.g., $F_2' = [k+1, 2k]$. Then F_1' is obliged to contain $[1, l]$, since any cyclic interval of length k in $[n] \setminus [k+1, 2k]$ contains $[1, l]$, provided that $n \le 3k - l$.

We conclude that the corresponding vector $v_1 \in \mathcal{F} \cap \mathcal{H}$ satisfies $S_+(v) \supset [1, l]$. At the same time, by the definition of $\mathcal{H}$, the vector v_2 corresponding to F_2' satisfies $S_-(v_2) = [1, l]$. That is, $S_-(v_2) \subset S_+(v_1)$. Interchanging the roles of F_1, F_2, we get that $S_-(v_1) \subset S_+(v_2)$. Moreover, $S_+(v_1) \cap S_+(v_2) = \emptyset$. This means that v_1 and v_2 are antipodal, a contradiction.

Therefore, the family $\mathcal{H}' \cap \mathcal{F}'$ is intersecting. It is proven in [9] that in this case $|\mathcal{H}' \cap \mathcal{F}'| \le k$, but we sketch the proof of this simple fact here for completeness. Take a set $H \in \mathcal{H}' \cap \mathcal{F}'$. Then the $2k-2$ sets from $\mathcal{H}'$ that intersect H can be split into pairs of disjoint sets. We can take only one set from each pair. $\square$

The rest of the argument is a standard averaging argument. Let us count in two ways the number of pairs (permutation σ, a vector from $\mathcal{H}(\sigma) \cap \mathcal{F}$). On the one hand, each vector from $\mathcal{F}$ is counted $nk!l!(n-k-l)!$ times. On the other hand, for each permutation, there are at most k pairs by Lemma 6. Therefore,

$$|\mathcal{F}|\, n\, k!\, l!\, (n-k-l)! \le kn! \quad \Leftrightarrow \quad |\mathcal{F}| \le \frac{k}{n} \frac{n!}{k!\, l!\, (n-k-l)!} = \frac{k}{n} |V(n,k,l)|.$$

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On a Helly-type question for central symmetry

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Abstract

We study a certain Helly-type question by Konrad Swanepoel. Assume that X is a set of points such that every k-subset of X is in centrally symmetric convex position, is it true that X must also be in centrally symmetric convex position? It is easy to see that this is false if $k \leq 5$, but it may be true for sufficiently large k. We investigate this question and give some partial results.

Dedicated to Imre Bárány on his 70th birthday.

1. Introduction

The classical Carathéodory theorem in dimension 2 can be stated in the following equivalent way: Let X be a set of points in the plane, if any 4 points from X are in convex positions then X is in convex position. In 2010, Konrad Swanepoel [5] asked the following Helly-type question which was inspired by this formulation of Carathéodory's theorem.

For brevity, we say that a set of points is in *c.s.c. position* (short for centrally symmetric convex position) if it is contained in the boundary of a centrally symmetric convex body.

Question. *Does there exist a number k such that for any planar set X the following holds: If any k points from X are in c.s.c position, then the whole set X is in c.s.c. position.*

It is clear from Carathéodory's theorem that X should be in convex position. One can also see that $k \geq 6$ since any 5 points are in c.s.c. position. This follows from the fact that any 5 points pass through a quadric curve. Since the points must be in convex position, the points lie on an ellipse, parabola or a branch of a hyperbola, and in each of these cases there is a centrally symmetric convex body containing these points on its boundary.

It is not clear that such a k exists although we suspect that it does. In this short note, we prove the following two results in Sections 3 and 4, respectively.

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Theorem 1. *There is a set X consisting of 9 points that is not in c.s.c. position such that any 8 of its points are in c.s.c. position. This implies that, if k exists, then $k \geq 9$.*

Theorem 2. *Let Γ be a continuous closed curve such that any 7 points of Γ are in c.s.c. position, then Γ bounds a centrally symmetric convex region.*

Before proving these theorems, we describe a way to decide whether a finite set X is in c.s.c. position or not. For more information on Carathéodory's theorem and Helly-type theorems we recommend [2] and [3].

2. Centrally symmetric convex position

We start with a useful definition.

Definition. *Let X be a point set and O be a point. The set X_O denotes the reflexion of X with respect to O, i.e., $X_O = 2O - X$. If $X \cup X_O$ is in convex position then we say that O is an* admissible center for X*, the set of all admissible centers is denoted by $\mathcal{M}_X$.*

Swanepoel's question can be reformulated in terms of admissible centers, since X is in c.s.c. position if and only if $\mathcal{M}_X$ is non-empty. The main goal of this section is to give a simple way of constructing $\mathcal{M}_X$. We start with the simplest possible case. The description of the set of admissible center for a finite set X can be obtained from the following simple lemmas.

Lemma 3. *Let $\triangle = \{a, b, c\}$ be three non-collinear points. The three lines passing through the midpoints of the sides of conv$(\triangle)$ divide the plane into 7 regions. The set $\mathcal{M}_\triangle$, shown in Figure 1, is the union of the closed components of this division that do not intersect $\triangle$.*

The set $\mathcal{M}_\triangle$ is naturally represented as the union of 4 convex subsets. We call these subsets the *center-part*, *a-part*, *b-part* and *c-part* as in Figure 1.

Lemma 4. *For a given set X in convex position we have that*

$$\mathcal{M}_X = \bigcap \{\mathcal{M}_Y : Y \subset X, \#(Y) = 3\}.$$

This last Lemma provides us with a way to construct the set of admissible centers of a set with n points in convex position as the intersection of $\binom{n}{3}$ sets. We see below how we can achieve the same thing using fewer sets.

Definition 5. *Assume X is a finite set of points in convex position such that X is not contained in a line. Let ab be a side of conv(X) and let $c \in X$ be a farthest point from the line ab. We call the triangle abc a* tallest triangle of X with respect to side ab.

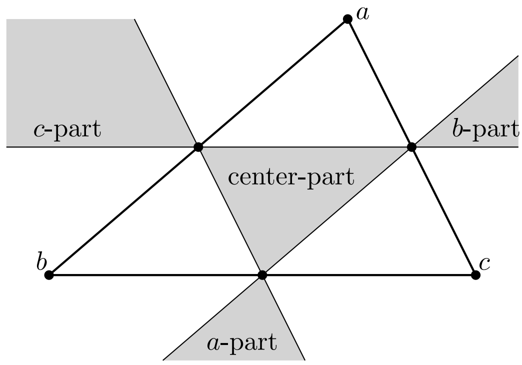

Figure 1: Set of admissible centers for a triangle.

The tallest triangle has appeared before, at least as source of interesting questions for mathematical Olympiads (see e.g. [4] or [1]).

Theorem 6. *If X is a finite set of points in convex position, then the set of admissible centers for X is the intersection of the sets of admissible centers of the tallest triangles of X, i.e.,*

$$\mathcal{M}_X = \bigcap \left\{ \mathcal{M}_{\{a,b,c\}} : abc \text{ is a tallest triangle of } X \right\}.$$

Proof. The set $\mathcal{M}_X$ is included in the intersection on the right-hand side of the formula, so we only need to prove that any point from the intersection is in $\mathcal{M}_X$.

Let O be any point from the intersection and let a be any point from X. We will show that it is possible to find a supporting line of $\operatorname{conv}(X \cup X_O)$ at a.

Let b be one of the neighbors of a on the boundary of $\operatorname{conv}(X)$, say in the clockwise direction. Let abc be a tallest triangle of X with respect to ab. If O lies in a-part or b-part of the admissible set for triangle abc, then ab is a supporting line for $\operatorname{conv}(X \cup X_O)$. Therefore O lies in the c-part or in the central part of $\mathcal{M}_{\{a,b,c\}}$.

Similarly, if d is the other neighbor of a on the boundary of X (in the counter-clockwise direction), and ade is a tallest triangle of X with respect to ad, then O lies in the central part or in the e-part of $\mathcal{M}_{\{a,d,e\}}$, otherwise we are done.

There are two possibilities for the positions of c and e. Either they coincide, or e is in counter-clockwise from c (that is, we meet e before a in this direction from c). In the case $c = e$, the only admissible point from $\mathcal{M}_{\{a,b,c\}}$ is the midpoint of ac, which also belongs to the b-part of $\mathcal{M}_{\{a,b,c\}}$. So, the line ab is a supporting line of $\operatorname{conv}(X \cup X_O)$ as we have shown before.

In the latter case, the point $O \in abc \cap ade$, and c and e are connected by a sequence of sides of X. Then there is a side pq of X in the angle $\angle cae$ such that O is inside triangle apq. It is not difficult to see that apq is a tallest triangle of X. Since O is inside apq and in $\mathcal{M}_{\{a,p,q\}}$, it is in the central part of this set of admissible centers. It follows that the line parallel to pq through a is also a supporting line of $\operatorname{conv}(X \cup X_O)$. □

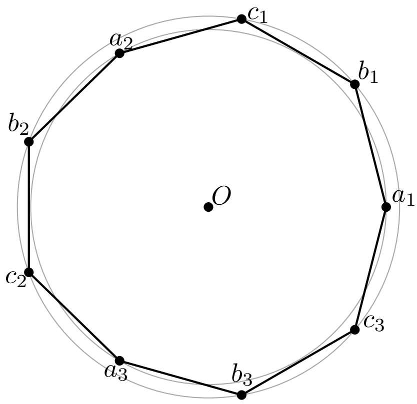

Figure 2: The 9-gon for Theorem 1.

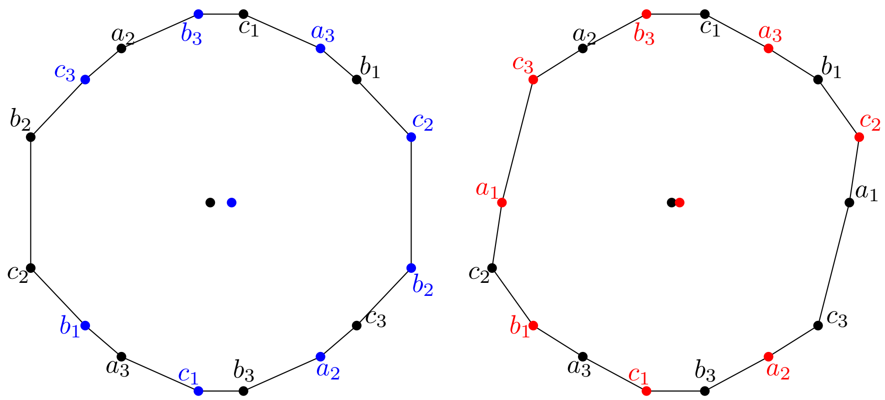

Figure 3: The original and reflected 8-gons with their respective centers.

3. Example showing $k \geq 9$

In this section we prove Theorem 1 by giving an explicit example of a set X with 9 points such that $\mathcal{M}_X = \emptyset$, but $\mathcal{M}_Y \neq \emptyset$ for every $Y \subset X$ with 8 points.

Proof of Theorem 1. Start with a regular 9-gon with center O and label its vertices as a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 in counter-clockwise order. Now, take the triangle $a_1a_2a_3$ and, with center O, scale it down by a factor of 0.93. Then we are left with an almost regular 9-gon such as the one shown in Figure 2. This will be the set X.

A subset Y of X with 8 points can be of two types, depending on whether or not it is missing a point a_i from X. For each of these, a point of $\mathcal{M}_Y$ close to O will serve as an admissible center. If we choose coordinates so that $O = (0, 0)$ and $b_1 = (1, 0)$, then points in $\mathcal{M}_Y$ corresponding to $Y = X \setminus \{a_1\}$ and $Y = X \setminus \{b_2\}$ are $(0.04, 0)$ and $(0.02, 0)$, respectively (see Figure 3).

All that is left is to show that $\mathcal{M}_X = \emptyset$. By Lemma 4, we only need to consider

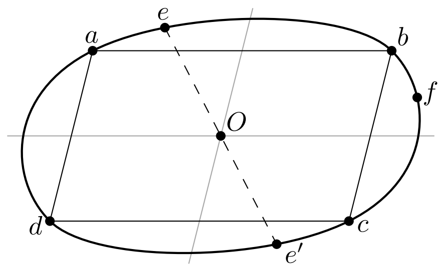

Figure 4: An inscribed parallelogram and auxiliary points.

the triangles determined by X. Let us consider first the triangle $a_1b_2c_2$, it is not hard to see that $\mathcal{M}_X$ must be a subset of the center part of $\mathcal{M}_{\{a_1,b_2,c_2\}}$. By the threefold symmetry of X, the same is true for the triangles $a_2b_3c_3$ and $a_3b_1c_1$. However, the center parts of these sets are triangles that do not intersect, so $\mathcal{M}_X$ must be empty. □

4. The case of convex curves

In this section we will show that for a convex curve Γ, Swanepoel's question has a positive answer. In fact, it is enough to take $k = 7$ in order to force Γ to be centrally symmetric. This result is based on the following simple fact, which can be proved easily using Lemma 4.

Lemma 7. *The set of admissible centers for the vertex-set of a parallelogram P is the union the two lines passing through the center of P and each parallel to a side of P.*

Proof of Theorem 2. The convexity of Γ is trivial. Take distinct points $a, b \in \Gamma$ such that ab is not an affine diameter of Γ, then there are $c, d \in \Gamma$ such that $abcd$ is a non-degenerate parallelogram with its vertices labeled in cyclic order. Moreover, we may chose this parallelogram so that its boundary intersects Γ at precisely $\{a, b, c, d\}$ (see Figure 4).

Let O be the center of this parallelogram and choose $e \in \Gamma \setminus \{a, b, c, d\}$. Let e' be the other point of intersection of the line eO with Γ. We may assume without loss of generality that e is on the arc of Γ between a and b. Let $f \in \Gamma$ be a point between b and c. The set $Y = \{a, e, b, f, c, e', d\}$ has 7 points so, by hypothesis, $\mathcal{M}_Y \neq \emptyset$. Since Y contains the vertices of a parallelogram, Lemma 7 tells us that $\mathcal{M}_Y$ is contained in the union of two lines passing through O. However, by considering the points e and f, the only possibility is $\mathcal{M}_Y = \{O\}$. This implies that O is the midpoint of ee'. Since e is arbitrary, we have shown that O is the center of symmetry of Γ. □

5. Acknowledgments

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An old method with two new geometrical applications

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The semi-random method was introduced for graphs in [1]. Later it was extended to 3-uniform hypergraphs in [8]. The method was further extended in [2] and [4].

A hypergraph $\mathcal{H}$ on the vertex set V is a subset of $\mathcal{P}(V)$, the power set of V. I.e. $\mathcal{H}$ is a collection of certain subsets of V, called edges. If the edges have a common size, say k, then we say that $\mathcal{H}$ is k-uniform. In a hypergraph $\mathcal{H}$ a vertex set $I \subset V$ is called an independent set iff it doesn't contain any edge as a subset. The maximum size of the independent sets of $\mathcal{H}$ is denoted by $\alpha(\mathcal{H})$. There are several results concerning independent sets in 3-uniform uncrowded hypergraphs. From hypergraph theory we recall that the degree of a vertex x ($deg(x)$) is the number of edges, containing x. Also a k-cycle ($k \geq 2$) in $\mathcal{H}$ is a sequence of k different vertices: $x_1, \ldots, x_{k-1}, x_k = x_0$ and a sequence of k different edges: $E_1, \ldots, E_k$ such that $x_{i-1}, x_i \in E_i$ for $i = 1, 2 \ldots, k$. The cycle above is called a simple cycle iff $E_i \cap \left(\cup_{j:j\neq i} E_j\right) = \{x_{i-1}, x_i\}$ for $i = 1, 2 \ldots, k$. We quote the earliest result on hypergraphs using the semi-random method.

Theorem 1 ([8], Lemma 1)**.** *Let $\mathcal{H}$ be a 3-uniform hypergraph on v vertices. Let $\overline{d}$ denote the average degree of $\mathcal{H}$. Assume that $\overline{d} \leq t^2$ and $1 \ll t \ll v^{1/10}$.*

If $\mathcal{H}$ doesn't contain simple cycles of length at most 4, then

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}\sqrt{\log t}\right).$$

In our applications we might have many simple cycles of length 3 and 4. We need the following strengthening of the basic bound:

Theorem 2 ([4], Theorem 2)**.** *Let $\mathcal{H}$ be a k-uniform hypergraph on v vertices. Let Δ be the maximum degree of $\mathcal{H}$. Assume that $\Delta \leq t^{k-1}$ and $1 \ll t$. If $\mathcal{H}$ doesn't contain a 2-cycle (two edges with at least two common vertices), then*

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}(\log t)^{\frac{1}{k-1}}\right).$$

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In [6] we gave two new geometrical applications of the above bound.

In the first application we consider a question asked by Gowers [5]. Given a planar point set $\mathcal{P}$, what is the minimal size of $\mathcal{P}$ that guarantees that one can find n points on a line or n independent points (no three on a line) in it? He noted that the grid shows that $\Omega(n^2)$ many points are necessary, and in the case of $2n^3$ many points without n points on a line a simple greedy algorithm finds n independent points. Payne and Wood [9] improved the upper bound to $\mathcal{O}(n^2 \log n)$. They also considered an arbitrary point set with much fewer points than n^3 and without n points on a line. But instead of the greedy algorithm they used Spencer's lemma, which is based on a simple probabilistic sparsification.

We improved the previous upper bound methods. We also start with a random sparsification. After some additional preparation (we get rid of 2-cycles) we are able to use a semi-random method (see [4]) to find a large independent set.

Theorem 3. *Let $\mathcal{P}$ be an arbitrary planar point set of size $\Omega\left(\frac{n^2 \log n}{\log \log n}\right)$. Then we can find n points in $\mathcal{P}$, that are incident to a line or independent.*

Our second application is closely related to Heilbronn's triangle problem [10], [15], [11], [12], [13], [14], [7]. Take a "nice" unit area domain D (usually a square, disc or a regular triangle). Place n points into D and find the smallest area among the triangles determined by the chosen points. Let $H_\triangle(n)$ denote the maximum of this parameter over all possible choices of n points.

Instead of triangles we can take k-tuples of our point set and consider the area of the convex hull of the k chosen points. We denote the corresponding parameter by $H_k(n)$ (so $H_3(n) = H_\triangle(n)$). The best lower bound on $H_\triangle(n)$ [8], and some trivial observations are summarized in the next line:

$$\Omega\left(\frac{\sqrt{\log n}}{n^2}\right) = H_\triangle(n) \le H_4(n) \le H_5(n) \le \ldots = \mathcal{O}\left(\frac{1}{n}\right).$$

We mention two major open problems: Is it true that $H_\triangle(n) = O(1/n^{2-o(1)})$ and $H_4(n) = o(1/n)$?

Our interest is in the lower bound on $H_4(n)$. Schmidt [15] proved that $H_4(n) = \Omega(n^{-3/2})$. The proof is a construction of a point set by a simple greedy algorithm. In [3] the authors provide a new proof, and extensions of this result. They also proposed an open question, which they have not yet been able to resolve: that is whether Schmidt's bound can be improved by a logarithmic factor. With the help of the semi-random method we were able to improve Schmidt's bound and settle the problem of [3].

Theorem 4. *There exists a point set of size n in the unit square that doesn't contain four points with convex hull of area $\mathcal{O}(n^{-3/2}(\log n)^{1/2})$.*

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Helly theorems for connected sets in the plane

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Dedicated to Imre Bárány on the occasion of his seventieth birthday.

1. Colorful interactions with Imre

The first time I heard of Imre was from my advisor, Helge Tverberg: *"You know my generalization of Radon's theorem... Bárány showed that it holds for any continuous map!"* Whether or not he left out that there should be a prime number of parts doesn't really matter. I was in awe.

At the time I was a fresh graduate student at the University of Bergen and Helge had suggested that I visit Imre at UCL. During our first meeting I told him the following problem (due to Katchalski):

Let B_1, B_2, and B_3 be pairwise disjoint unit balls in $\mathbb{R}^3$. Show that the set of directions for which there exists a directed line which intersects them in the order B_1, B_2, B_3 is convex.

I was somewhat surprised to hear that Imre already knew of this problem, and even more surprised to hear that he had a counter-example! That is, he had a counter-example written in a notebook in Budapest, and he promised to send me a copy when he had a chance. Sure enough, a few months after returning to Bergen, I received a letter in the mail with xeroxed copies of Imre's hand-written notes.[2]

More importantly, it was during this visit to UCL that I first learned about "colorful" and "fractional" versions of various classical theorems in combinatorial convexity, such as "colorful Carathéodory", "fractional Helly", and "fractional Erdős–Szekeres". Imre took the time to explain several of these results to me, giving me problems and exercises related to them as well. For instance he explained an open problem concerning the fractional Helly number for convex lattice sets in $\mathbb{R}^d$. Their Helly number is 2^d, but what is their fractional Helly number? Is it $d+1$, just as for standard convex sets? This is indeed the case, which was shown by Imre together with Matoušek [5].

Although I found these generalizations fascinating, it took a long time before I was able to appreciate the depth of these results and their impact on problems in discrete geometry. The colorful Carathéodory theorem [4] and the fractional Helly theorem [13] are crucial tools in the construction of weak ε-nets for convex sets [1] and play important roles in the proof of the celebrated (p, q)-theorem due to Alon

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[2]I must admit that I never read the notes carefully, because shortly before receiving Imre's letter I succeeded in proving the convexity of the set of directions!

and Kleitman [3]. The fractional Erdős–Szekeres theorem [6, 15] plays a prominent part in the recent breakthrough on the Erdős–Szekeres convex polygon problem due to Suk [16].

Apart from their applications, the "colorful" and "fractional" theorems are often examples of the beautiful interaction between topology, combinatorics, and geometry. Imre's seminal results in this area have been a great inspiration to many of us, and in what follows we present some recent "colorful" and "fractional" topological Helly theorems obtained together with Minki Kim and Seunghun Lee.

2. Connected families in the plane

Let $\mathcal{F} = \{X_1, \ldots, X_n\}$ be a family of open sets in the plane. (We assume that all families are finite throughout.) We call $\mathcal{F}$ a ***connected family*** provided that any non-empty intersection of members in $\mathcal{F}$ is connected. (In particular, each member of $\mathcal{F}$ is connected.) The following is a well-known Helly theorem for connected families in the plane.

Theorem 1. *Let $\mathcal{F}$ be a connected family of open sets in the plane. If every four members of $\mathcal{F}$ have a point common, then there is a point that intersects every member of $\mathcal{F}$.*

Here is a "folklore" proof of Theorem 1: Suppose $|\mathcal{F}| = n \geq 5$ and that every proper subfamily of $\mathcal{F}$ has a point in common. For every member X in $\mathcal{F}$, choose a point $p_X \in \bigcap_{Y \in \mathcal{F} \setminus \{X\}} Y$, and for every pair of members $X \neq Y$ in $\mathcal{F}$ draw a path $\pi_{X,Y}$ connecting p_X to p_Y in such a way that $\pi_{X,Y}$ is contained in $\bigcap_{Z \in \mathcal{F} \setminus \{X,Y\}} Z$. Notice that we have now made a drawing in the plane of the complete graph K_n, and since $n \geq 5$, there must be a pair of vertex disjoint edges that intersect in a point contained in every member of $\mathcal{F}$.

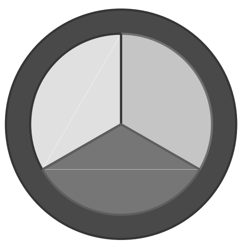

Figure 1: A connected family which shows that the Helly number *four* in Theorem 1 is best possible: Any three members intersect, but not all four intersect.

The proof above generalizes to higher dimensions by applying the van Kampen–Flores theorem, but requires a suitable adjustment of the notion of "connected family" which takes into account higher topological connectedness. Even further generalizations were obtained by Matoušek [14] which allow the sets to have several connected components, and more recently by Goaoc et al. [8] which allow for even more general sets. In both cases, the Helly numbers they obtain are bounded by repeated applications of Ramsey's theorem and are enormous! Unfortunately, the methods developed in [14] and [8] do not seem flexible enough to obtain more robust Helly type theorems such as “colorful" and “fractional" versions.

Our goal here is to present such generalizations of Theorem 1. In particular we have the following “colorful” version.

Theorem 2. *Let $\mathcal{F}$ be a connected family of open sets in the plane and let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be subfamilies of $\mathcal{F}$. If $X_1 \cap X_2 \cap X_3 \cap X_4 \neq \varnothing$ for every choice $X_i \in \mathcal{F}_i$, then $\bigcap_{X\in\mathcal{F}_i} X \neq \varnothing$ for some $1 \le i \le 4$.*

Although the (colorful) Helly number for connected families is four, it turns out that the “fractional” Helly number for such families is only three.

Theorem 3. *For every $\alpha \in (0,1]$ there exists a $\beta > 0$ such that the following holds. Let $\mathcal{F}$ be a connected family of open sets in the plane. If at least $\alpha\binom{|\mathcal{F}|}{3}$ of the triples in $\mathcal{F}$ have non-empty intersection, then there is a point contained in at least $\beta|\mathcal{F}|$ members of $\mathcal{F}$.*

Using the results of Alon et al. [2] we obtain, as a consequence of Theorem 3, a generalization of the planar (p,q) theorem.

Theorem 4. *For any integers $p \ge q \ge 3$ there exists an integer $C = C(p,q)$ such that the following holds. Let $\mathcal{F}$ be a connected family of open sets in the plane. If among any p members of $\mathcal{F}$ there are some q that have a point in common, then there exists a set of at most C points that intersects every member of $\mathcal{F}$.*

Let $\mathcal{F} = \{S_1, \dots, S_n\}$ be a family of sets. The intersection patterns of $\mathcal{F}$ can be encoded by its nerve $N(\mathcal{F})$ which is an abstract simplicial complex defined as

$$N(\mathcal{F}) = \{\sigma \subset [n] : \textstyle\bigcap_{i\in\sigma} S_i \neq \varnothing\}.$$

In the case when $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, a basic consequence of the nerve theorem from algebraic topology is that the homology of $N(\mathcal{F})$ vanishes in all dimensions greater or equal to d. Roughly speaking, this means that the “complexity” of the nerve is bounded, and in many cases this bound on the complexity is sufficient to obtain “colorful” and “fractional” versions of Helly's theorem, as has been show in a series of works by Kalai and Meshulam [9, 10, 11, 12].

Our goal is therefore to obtain similar bounds on the “complexity” on $N(\mathcal{F})$ when $\mathcal{F}$ is a connected family of open sets in the plane.

3. Towards a nerve theorem for graphs

Rather than working with connected families of open sets in the plane it turns out that it is more natural to deal with families of connected graphs. Let $\mathcal{F} =$

$\{G_1, \ldots, G_n\}$ be a family of induced subgraphs of a fixed graph G. We say that $\mathcal{F}$ is a ***connected family*** in G provided that $\bigcap_{i\in\sigma} G_i$ is connected for every $\sigma \in N(\mathcal{F})$. What we are looking for is a "nerve theorem" which relates the "complexity" of the nerve $N(\mathcal{F})$ to the "structure" of the underlying graph G.

A simple parameter which measures the "complexity" of the nerve $N(\mathcal{F})$ is the greatest dimension for which the homology of $N(\mathcal{F})$ is non-vanishing. (Throughout we use homology with rational coefficients.) By taking the maximum over all connected families $\mathcal{F}$ in a fixed graph G, we get a measure of the complexity of G.

More precisely, for a graph G we define the ***homological dimension*** of G to be the greatest integer d such that $H_d(N(\mathcal{F})) \neq 0$ for some connected family $\mathcal{F}$ in G. We denote this parameter by $\gamma(G)$, and for the single vertex graph K_1 we define $\gamma(K_1) = -1$.

As for the "structure" of the graph G it turns out that it is natural to use the well-known minor relation from graph theory. Recall that a graph H is a minor of G if there exists pairwise disjoint connected subgraphs in G, one for each vertex in H, such that for any pair of adjacent vertices in H there exists an edge between the corresponding subgraphs in G. We denote this by writing $H \prec G$.

It is not difficult to see that the homological dimension of a graph is minor-monotone. By this we mean that if $H \prec G$, then $\gamma(H) \leq \gamma(G)$. It is also easy to see that for K_{d+2}, the complete graph on $d+2$ vertices, we have $\gamma(K_{d+2}) = d$. And so if $K_{d+2} \prec G$, then $\gamma(G) \geq d$. We conjecture that the converse also holds, and we can prove this for small values of d.

Theorem 5. *For any graph G we have the following.*

1. $\gamma(G) \geq 1 \iff K_3 \prec G.$
2. $\gamma(G) \geq 2 \iff K_4 \prec G.$
3. $\gamma(G) \geq 3 \iff K_5 \prec G.$

It is easy to see that if $\mathcal{F} = \{X_1, \ldots, X_n\}$ is a connected family of open sets in the plane (as discussed in section 2), then there exists a connected family of graphs $\mathcal{F}' = \{G_1, \ldots, G_n\}$ in a *planar* graph G such that the nerve complexes $N(\mathcal{F})$ and $N(\mathcal{F}')$ are isomorphic. As a consequence we get the following.

Corollary 6. *For any connected family $\mathcal{F}$ of open sets in the plane, $H_i(N(\mathcal{F})) = 0$ holds for all $i \geq 3$.*

Now Theorem 2 follows from the colorful topological Helly theorem due to Kalai and Meshulam [11]. Applying Kalai's "upper bound theorem" for d-Leray complexes [9, 10] we get the following.

Corollary 7. *Let $\mathcal{F}$ be a connected family of open sets in the plane. If at least $\alpha\binom{|\mathcal{F}|}{4}$ of the 4-membered subfamilies of $\mathcal{F}$ have non-empty intersection, then there is a point in common to at least $\beta|\mathcal{F}|$ members of $\mathcal{F}$, where $\beta = 1 - (1-\alpha)^{1/4}$.*

In order to establish Theorem 3 we need some additional combinatorial arguments. The first is the following "weak colorful Helly theorem".

Lemma 8. *Let $\mathcal{F}$ be a connected family of open sets in the plane with a partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ where $|\mathcal{F}_i| \geq 10$. If every colorful triple intersects, then there are some four members of $\mathcal{F}$ that intersect.*

By a standard application of a theorem of Erdős–Simonovits [7] together with Lemma 8, one can show that if a constant fraction of the triples in $\mathcal{F}$ intersect, then also a constant fraction of the 4-membered subfamilies of $\mathcal{F}$ intersect. And so Theorem 3 follows from Corollary 7.

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Problems for Imre Bárány's Birthday

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I will give several problems in the interface between combinatorics and geometry and mainly around Helly's theorem and Tverberg's theorem. This is one of the areas on which Imre Bárány had immense impact. My lecture in the conference will focus on some of these problems.

1. Around Tverberg's theorem

Tverberg's Theorem states the following: Let $x_1, x_2, \dots, x_m$ be points in $\mathbb{R}^d$ with $m \geq (r-1)(d+1)+1$. Then there is a partition $S_1, S_2, \dots, S_r$ of $\{1, 2, \dots, m\}$ such that $\cap_{j=1}^{r} conv(x_i : i \in S_j) \neq \emptyset$. This was a conjecture by Birch who also proved the planar case. The bound of $(r-1)(d+1)+1$ in the theorem is sharp as can easily be seen from configuration of points in sufficiently general position. The case $r = 2$ is Radon's theorem.

1.1. Prescribing the sizes of parts in Tverberg's theorem and the number of Tverberg's partitions

Problem 1. *Suppose that $a_1, a_2, \dots, a_r$ is a partition of $m = (r-1)(d+1)+1$ such that $1 \leq a_i \leq d+1$ for every i. Is there a configuration of m points in $\mathbb{R}^d$ of which all of Tverberg's partitions are of type $(a_1, a_2, \dots, a_r)$?*

This problem was raised by Micha A. Perles many years ago and a positive answer was recently given by Moshe White.

Conjecture 2 (Sierksma Conjecture)**.** *The number of Tverberg's r-partitions of a set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ is at least $((r-1)!)^d$.*

White's examples provide a rich family of examples for cases of equality in Sierksma's conjecture. An even more general family of constructions for equality cases, based on stairway convexity, was given by Boris Bukh and Gabriel Nivasch.

1.2. Topological Tverberg

Conjecture 3 (Topological Tverberg)**.** *Let f be a continuous function from the m-dimensional simplex σ^m to $\mathbb{R}^d$. If $m \geq (d+1)(r-1)$ then there are r pairwise disjoint faces of σ^m whose images have a point in common.*

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If f is a linear function this conjecture reduces to Tverberg's theorem. The case $r = 2$ was proved by Bajmoczy and Bárány using the Borsuk-Ulam theorem. In this case you can replace the simplex by any other polytope of the same dimension. The case where r is a prime number was proved in a seminal 1978 paper of Bárány, Shlosman and Szucs. The prime power case was proved by Özaydin. For the prime power case, the proofs are quite difficult and are based on computations of certain characteristic classes.

In 2015 the topological Tverberg conjecture was disproved. This involves some early result on vanishing of topological obstructions by Özaydin, a theory developed by Mabillard and Wagner extending Whitney's trick to k-fold intersections, and a fruitful reduction by Gromov and by Blagojević, Frick and Ziegler.

Conjecture 4. *Let f be a linear function from an m-dimensional polytope P to $\mathbb{R}^d$. If $m \geq (d+1)(r-1)$ then there are r pairwise disjoint faces of P whose images have a point in common.*

Problem 5. *Does the conclusion of the topological Tverberg conjecture holds if the images of faces under f form a "good cover?" (Namely, all those images and all non empty intersections are contractible.)?*

1.3. Colorful Tverberg

Let $C_1, \cdots, C_{d+1}$ be disjoint subsets of $\mathbb{R}^d$, called colors, each of cardinality at least t. A $(d+1)$-subset S of $\bigcup_{i=1}^{d+1} C_i$ is said to be multicolored if $S \cap C_i \neq \varnothing$ for $i = 1, \cdots, d+1$. Let r be an integer, and let $T(r,d)$ denote the smallest value t such that for every collection of colors $C_1, \cdots, C_{d+1}$ of size at least t there exist r disjoint multicolored sets $S_1, \cdots, S_r$ such that $\bigcap_{i=1}^{r} \operatorname{conv}(S_i) \neq \varnothing$.

A seminal theorem of Zivaljevic and Vrecica asserts that $T(r,d) \leq 4r-1$ for all r, and $T(r,d) \leq 2r-1$ if r is a prime. The only known proofs for this theorem rely on topological arguments.

Conjecture 6 (Bárány-Larman's colorful Tverberg conjecture)**.**

$$T(r,d) = r.$$

The case where $r+1$ is a prime was proved by Blagojevic, Matschke, and Ziegler.

Colorful Caratheodory and the Rota basis conjecture

Consider $d+1$ sets $A_1, A_2, \cdots, A_{d+1}$ of points in $\mathbb{R}^d$. Assume that each $|A_i| = d+1$ and that the interior of $conv(A_i)$ contains the origin.

Problem 7 (D. H. J. Polymath)**.** *Can we find a partition of all points into $d+1$ rainbow parts such that the interior of the convex hulls of the parts have a point in common. (A rainbow set is a set containing one element from each A_i.)*

This question was raised in Chow's polymath12 dealing with Rota's basis conjecture. Note that Bárány's famous colorful Caratheodory asserts that there is a rainbow set whose convex hull contains the origin. (I don't know what is the maximum guaranteed number of disjoint rainbow sets with this property.) Without the words "the interiors of" this is a special case of the colorful Tverberg conjecture. A positive answer would be a strong variant of Reay's conjecture (below) on the dimension of Tverberg points, and also a strong form of (a somewhat special case) of Rota's basis conjecture.

1.4. Eckhoff's partition conjecture

Let X be a set endowed with an abstract closure operation $X \to cl(X)$. The only requirements of the closure operation are:

(1) $cl(cl(X)) = cl(X)$ and

(2) $A \subset B$ implies $cl(A) \subset cl(B)$.

Define $t_r(X)$ to be the largest size of a (multi)set in X which cannot be partitioned into r parts whose closures have a point in common.

Conjecture 8 (Eckhoff's Partition Conjecture:). *For every closure operation*

$$t_r \leq t_2 \cdot (r-1).$$

If X is the set of subsets of $\mathbb{R}^d$ and $cl(A)$ is the convex hull operation then Radon's theorem asserts that $t_2(X) = d+1$ and Eckhoff's partition conjecture would imply Tverberg's theorem. In 2010 Eckhoff's partition conjecture was refuted by Boris Bukh. Bukh's beautiful paper contains several important ideas and further results. I will mention one ingredient: Let me take for granted the nerve construction for moving from a family of n convex sets to a simplicial complex with n vertices recording their empty and non empty intersections. Bukh studied simplicial complexes whose vertex sets correspond to the power set of a set of size n: Starting with n points in $\mathbb{R}^d$ or some abstract convexity space consider the nerve of convex hulls of all subsets of these points!

1.5. Dimensions of Tverberg's point

A conjecture of Reay

For a set A, denote by $T_r(A)$ those points in $\mathbb{R}^d$ which belong to the convex hull of r pairwise disjoint subsets of A. We call these points Tverberg points of order r.

Conjecture 9 (Reay). *If A is a set of $(d+1)(r-1)+1+k$ points in general position in $\mathbb{R}^d$ then*

$$dim T_t(A) \geq k$$

.

In particular, Reay conjecture asserts that a set of $(d+1)r$ points in general position in $\mathbb{R}^d$ can be partitioned into r sets of size $d+1$ so that the simplices described by these sets have an interior common point.

The cascade conjecture

Conjecture 10. *For every $A \subset \mathbb{R}^d$,*

$$\sum_{r=1}^{|A|} \dim T_r(A) \geq 0.$$

(Note that $\dim \varnothing = -1$.) The conjecture was proved for $d \leq 2$ by Akiva Kadari (unpublished M. Sc thesis in Hebrew).

A special case

A special case of the cascade Conjecture asserts that given $2d+2$ points in $\mathbb{R}^d$ then you can either partition them into two simplices whose interior intersects, or you can find a Tverberg partition into 3 parts. A reformulation based on positive hulls is:

Given $2d$ non zero vectors in $\mathbb{R}^d$ so that the origin is a vertex of the cone spanned by them then either:

- You can divide the points into two sets A and B so that the cones spanned by them have a d-dimensional intersection, or
- You can divide them into three sets A B and C so that the cones spanned by them have a non-trivial intersection.

Another interesting reformulation is obtained when we dualize using the Gale transform, and this have led to the problem we consider next.

A question about directed graphs that can be described as the union of two trees

A very special class of configurations arise from graphs. Start from a directed graph on n vertices and $2n-2$ edges and associate to each directed edge $\{i,j\}$ the vector $e_j - e_i$. This has led to the problem we discuss next.

Problem 11. *Let G be a directed graph with n vertices and $2n-2$ edges. When can you divide your set of edges into two trees T_1 and T_2 (so far we disregard the orientation of edges,) so that when you reverse the directions of all edges in T_2 you get a strongly connected digraph.*

I conjectured that if G can be written as the union of two trees, the only additional obstruction is that there is a cut consisting only of two edges in reversed directions. Maria Chudnovsky and Paul Seymour found an additional necessary condition: There is no induced cycle $c_1 - ... - c_{2k} - c_1$ in G, s.t. each c_i is cubic, the edges of the cycle alternate in direction, and none of $c_1, .., c_{2k}$ are sources or sinks of G.

1.6. Another conjecture by Reay

Problem 12. *What is the smallest integer $R(d, r)$ such that If $x_1, x_2, \dots, x_m$ be points in $\mathbb{R}^d$ with $m \geq R(d, r)$, then there is a partition $S_1, S_2, \dots, S_r$ of $\{1, 2, \dots, m\}$ such that $conv(x_i : i \in S_j) \cap conv(x_i : i \in S_k) \neq \emptyset$, for every $1 \leq j < k \leq r$.*

Reay conjectured that you cannot improve the value given by Tverberg's theorem, namely that

Conjecture 13 (Reay)**.** $R(d, r) = (r - 1)(d + 1) + 1$.

Micha A. Perles conjectures that Reay's conjecture is false even for $r = 3$ for large dimensions, but with Moria Sigron he proved the strongest positive results in the direction of Reay's conjecture.

1.7. Another old problem

Problem 14. *How many points $T(d; s, t)$ in $\mathbb{R}^d$ guarantee that they can be divided into two parts so that every union of s convex sets containing the first part has a non empty intersection with every union of t convex sets containing the second part.*

I would like to explain why $R(d; s, t)$ is finite. This is a fairly general Ramsey-type argument and it gives us an opportunity to mention a few recent important results. The argument has two parts:

1) Prove that $T(d; s, t)$ is finite (with good estimates) when the points are in cyclic position.
2) Use the fact that for every d and m there is $f(d, m)$ so that among every n points in general position in $\mathbb{R}^d$, $n > f(d, n)$ one can find m points in cyclic position.

The finiteness follows (with horrible bounds) from these two ingredients by standard Ramsey-type results.

Recently a fairly good understanding of $f(d, n)$ was achieved in a series of beautiful papers,

Theorem 1.

$$f(d, n) = twr_d(\theta(n).)$$

Here, twr_d is the d-fold tower function. The lower bound is by Suk (improving earlier bounds by Conlon, Fox, Pach, Sudakov and Suk) and the upper bounds are by Bárány, Matousek, and Por.

2. Helly and fractional Helly

2.1. A conjecture by Jie Gao, Michael Langberg, and Leonard Schulman

I will start with a Helly-type conjecture by Gao, Landberg and Schulman:

For a convex set K in $\mathbb{R}^d$ an ϵ enlargement of K is $K + \epsilon(K - K)$. (Where $K - K = \{x - y : x, y \in K\}$.)

Conjecture 15. *For every d, k and ϵ there is some $h = h(d, k, \epsilon)$ with the following property. Let $\mathcal{F}$ be a family of unions of k convex sets. Let $\mathcal{F}^\epsilon$ be the family obtained by enlarging all the involved convex sets by ϵ.*

If every h members of $\mathcal{F}$ have a point in common then all members of $\mathcal{F}^\epsilon$ have a point in common.

2.2. A new exciting topological Helly

Let me mention a recent exciting Helly-type theorem by Xavier Goaoc, Pavel Paták, Zuzana Safernová, Martin Tancer and Uli Wagner.

Theorem 2. *For every $\gamma > 0$ there is $h(\gamma, d)$ with the following property: Let $\mathcal{U}$ be a family of sets in $\mathbb{R}^d$. Suppose that for every intersection L of m members of $\mathcal{U}$ and every $i \leq [d-1/2]$, we have$b_i(L) \leq \gamma$. Then if every $h(\gamma, d)$ members of $\mathcal{U}$ have a point in common there is a point in common to all sets in $\mathcal{U}$.*

2.3. Fractional Helly

We will mention here two conjectures regarding the fractional Helly property and two related theorems. A class of simplicial complexes is hereditary if it is closed under induced subcomplexes. For a simplicial complex K, $f_i(K)$ is the number of i-faces of K, $b(K)$ is the sum of (reduced) Betti numbers of K.

Conjecture 16 (Kalai and Meshulam)**.** *Let $C > 0$ be a positive number. Let $\mathcal{K}$ be the hereditary family of simplicial complexes defined by the property that for every simplicial complex $K \in \mathcal{K}$ with n vertices,*

$$b(K) \leq Cn^d.$$

Then for every $\alpha > 0$ there is $\beta = \beta(d, C) > 0$, with the following property. For $K \in \mathcal{K}$, Then if $f_d(K) \geq \alpha\binom{n}{d+1}$ then $\dim(K) \geq \beta \cdot n$.

The conclusion of the conjecture is referred to as the fractional Helly property of degree d.

Conjecture 17. *Let $\mathcal{U}$ be a family of sets in $\mathbb{R}^d$. Suppose that for every intersection L of m members of $\mathcal{K}$, $b(L) \leq \gamma M^{d+1}$. Then $\mathcal{U}$ satisfies a fractional Helly property of order d.*

Theorem 3 (Bárány and Matoušek)**.** *Families of integral points in convex sets in $\mathbb{R}^d$ satisfies a fractional Helly property of order d.*

Theorem 4 (Matoušek)**.** *Families of sets of bounded VC-dimension in $\mathbb{R}^d$ satisfies a fractional Helly property of order d.*

Problem 18. *Does Radon theorem imply the fractional Helly property?*

3. Two questions by Imre on convex polytopes

Problem 19. *Let P be a d-polytope. Is $f_k(P) \geq \max f_0(P), f_{d-1}(P)$?*

Problem 20. *Is there, for every positive integer d, a constant c_d such that the number of maximal flags of faces for a d-polytope is at most by c_d times the total number of faces (of all dimensions) of P?*

Bárány's first question must be correct (or, so I think, most of the times) but we cannot prove it. It falls into a much more general statement called *the generalized upper bound theorem.*

Conjecture 21 (Generalized upper bound conjecture)**.** *Let K be a polyhedral complex that can be embedded into $\mathbb{R}^d$, let C be the boundary complex of a cyclic d-polytope. Then if $f_i(K) \leq f_i(C)$ for some $i \geq 0$ it follows that $f_j(K) \leq f_j(C)$ for every $j \geq i$.*

Bárány's second question is very mysterious and I don't know what the answer should be even for $d = 5$. I would guess that for polyhedral spheres the answer is negative but this remains open as well.

Happy birthday, dear Imre!

Geometric representations and quantum physics

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As the basic setup in quantum physics, the state of a physical system can be described by a vector of unit norm in a (complex) Hilbert space; seemingly discrete events like whether the spin of an electron points "up" or "down" are also represented by vectors. Labeling the nodes of a graph by vectors has turned out a natural and very useful tool in graph theory: interesting connections between the combinatorial and geometric structures of vector-labeled graphs have been revealed in the theory of rigidity of bar-and-joint structures, orthogonal representations, and other geometric representations.

The fact that quantum physics assigns vectors to discrete objects like "properties" suggests analogies with geometric representations of graphs. As it turns out, this is more than just an analogy; we describe three problems, where quantum physics makes strong use of the theory of geometric representations.

Entanglement. Consider two quantum systems A and B. Separately, their states can be described by unit vectors $\mathbf{x} \in \mathbb{C}^d$ and $\mathbf{y} \in \mathbb{C}^e$. The state of the union of the two systems can be described by a vector in the tensor product $\mathbb{C}^d \otimes \mathbb{C}^e$. If the two systems in states $\mathbf{x}$ and $\mathbf{y}$ are "independent" (unentangled), their joint state is $\mathbf{x} \circ \mathbf{y}$, which is then called a *product state*.

Entanglement leads to rather paradoxical behavior of particles; this was pointed out by Einstein, Podolsky and Rosen in 1935. Consider a pair particles with a 2-dimensional state space (say two electrons, whose spin can be either "up" or "down") in the maximally entangled state $\frac{1}{\sqrt{2}}\mathbf{e}_1 \circ \mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2 \circ \mathbf{e}_2$. Such a pair is often called an *EPR pair*.

Suppose that Alice and Bob split an EPR pair between themselves (while it remains in the same entangled state), and they travel to different faraway places. If Alice measures the state of her particle, she will find it in one of the states $\mathbf{e}_1$ and $\mathbf{e}_2$ with the same probability. Say it is in state $\mathbf{e}_1$, then the entangled state collapses to $\mathbf{e}_1 \circ \mathbf{e}_1$ immediately. This seems to mean a long-range action faster than light, contradicting special relativity.

Hidden variables. One way out of this paradox is the theory of *hidden variables.* This interpretation, which arose from the objection to the nondeterministic–random interpretation of quantum events, suggests that if we knew the exact state of each particle (its "hidden parameters"), then we could predict quantum events with certainty.

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Based on the work in [4], we describe a connection between orthogonal representations and the theory of hidden variables. Consider a quantum system, and let $e_1, \dots, e_n$ be observable events. Construct a graph G on $V = [n]$ in which $ij \in E$ if and only if e_i and e_j are exclusive (cannot occur simultaneously). We call G the *exclusivity graph* of the events.

We start the system in a state $\mathbf{u}$, and observe an event e_i. As we know, this observation changes the state, so we cannot observe all of the other events. But if we choose the event i uniformly at random from the set $\{e_1, \dots, e_n\}$, and repeat the experiment many times, then we can find the probability p that event e_i occurs, experimentally, with arbitrary good precision.

In the classical setting, when $e_1, \dots, e_n$ are observable events in a probability space (no quantum effects), the number pn would be the expected number of events that occur simultaneously. Hence

$$p \le \frac{\alpha(G)}{n}. \tag{1}$$

The same inequality can be derived in quantum physics, if we assume that it makes sense to talk about the number of events e_i that actually hold in the given experiment. In the "hidden variable" interpretation of quantum physics this is the case. From basic quantum physical principles (not using hidden variables) one can only derive the weaker inequality

$$p \le \frac{\vartheta(G)}{n}, \tag{2}$$

using the theta function from graph theory.

Bell was the first to suggest that inequalities related to (1) could be experimentally verified (or rather falsified). In the special case called the Clauser–Horne–Shimony–Holt experiment, two observers do measurements on an EPR pair of particles, observing events whose exclusivity graph is the Wagner graph W_8. Inequalities (1) and (2) give the bounds $p \le 0.375$ and $p \le 0.427\dots$, respectively.

After a long line of increasingly sophisticated experiments, recent reports claim to have eliminated all the implicit assumptions ("loopholes"), and show that the bound (1) does not hold in general. The experiment in [7] provides the value $p \approx 0.401$. This value is about half way between the bounds above, and it can be considered as a disproof of the "hidden variable" interpretation of quantum physics (at least in its basic form).

Capacity of quantum channels. The most successful area of applying quantum physics in computer science has been quantum information theory. While splitting an EPR pair cannot be used to transmit information faster than light, such a strange behavior can be utilized to create communication channels more efficient than classical communication channels [5].

One can generalize the Shannon capacity to quantum information theory. It turns out [2, 6] that *the theta function provides an upper bound on the quantum physical version Shannon capacity just like it does for classical channels.*

Unextendible product systems. This application of orthogonal representations leads to the construction of highly entangled states. Consider a system of mutually orthogonal product states in $\mathbb{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$. Such a system is *unextendible*, if there is no product state orthogonal to all of them. Unextendible product systems were introduced in [3] in order to construct highly entangled states.

The standard basis in $\mathbb{H}$ is a trivial example. Our goal is to construct an unextendible product system as small as possible. It is not hard to see that the cardinality n of such a system must satisfy $n \geq 1 + \sum_{i=1}^{m}(d_i - 1)$. It was shown in [1] that in the case of equality, the existence (and construction) of an unextendible product system can be translated to a pure graph-theoretic condition:

There exists an unextendible product system with n elements if and only if there exists a decomposition $K_n = G_1 \cup \cdots \cup G_m$ into edge-disjoint graphs such that G_i does not contain a complete bipartite subgraph with $d_i + 1$ nodes ($i = 1, \ldots, m$).

The proof is based on orthogonal representations of graphs. Condition $n = 1 + \sum_{i=1}^{m}(d_i - 1)$ is not always sufficient for the existence of such a decomposition of K_n: if n is odd and any d_i is even, or $m = 2$ and $d_1 = 2 \leq d_2$, then there is no decomposition as above. It was shown in [1] that *these are the only exceptional cases.*

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How small orbits of periodic homeomorphisms of spheres can be?

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Dedicated to Imre Bárány.

1. Introduction

A well known theorem of Newman [1] states that periodic homeomorphisms of manifolds cannot have all orbits small. The purpose of this talk is to make this result precise for the case of spheres by exploring how small orbits of periodic homeomorphisms of the sphere can be. Our methods uses techniques form topology and discrete geometry.

We will denote by $\mathbb{S}^n$ the unit sphere of euclidean space $\mathbb{R}^{n+1}$. Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a homeomorphism, we will denote by $h^i : \mathbb{S}^n \to \mathbb{S}^n$ the i-th iteration of h and we will suppose that h^0 is the identity. A homeomorphism $h : \mathbb{S}^n \to \mathbb{S}^n$ is *periodic* if h^n is the identity for some integer $n > 1$. The minimal integer $n > 1$ for which h^n is the identity is called the *period* of h. For every $x \in \mathbb{S}^n$ the set

$$\{h^i(x)\}_{i>0}$$

is called the *orbit* of x and it is denoted by $h^*(x)$.

The norm $\|h\|$, of a homeomorphism $h : \mathbb{S}^n \to \mathbb{S}^n$ is defined as its distance from the origin, that is, $\|h\| = \mathrm{Sup}\{\|h(x) - x\| \mid x \in \mathbb{S}^n\}$, and the *orbital diameter* $\Theta(h)$, of h is defined as the maximal diameter of its orbits. We will be mainly interested in lower estimations of these two metric characteristics of periodic homeomorphism.

Let us denote by ρ_n the length of the side of a planar regular n-gon inscribed in the unit circle $\mathbb{S}^1$ and by d_n its diameter. Finally, let us denote by τ_n the length of the edge of a regular $(n+1)$-simplex inscribed in $\mathbb{S}^n$.

2. The topological result

Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a homeomorphism of period p. An orbit $h^*(x)$ is *balanced* if its barycentre coincide with the centre of the sphere, that is, if

$$\sum_1^p h^i(x) = 0.$$

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For every periodic homeomorphism h without balanced orbits we define its *barycentric mapping* $\beta : \mathbb{S}^n \to \mathbb{S}^n$ as:

$$\beta(x) = \frac{\sum_1^p h^i(x)}{\| \sum_1^p h^i(x) \|},$$

for every $x \in \mathbb{S}^n$. Note that β is well defined and continuous.

Our main topological result is the following:

Theorem 1. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism of prime period p, without balanced orbits and let $\beta : \mathbb{S}^n \to \mathbb{S}^n$ be its barycentric mapping. Then, the degree of β is divisible by p.*

Proof. Let us denote by $\mathbb{S}^n/h$ the orbit space of h and by $\Pi : \mathbb{S}^n \to \mathbb{S}^n/h$ the natural projection. Because p is prime, all non fixed points of h have orbits consisting of p elements. Hence the complement of the set of fixed points, in the orbit space is a manifold.

Next, it is possible to approximate $\beta\Pi^{-1} : \mathbb{S}^n/h \to \mathbb{S}^n$ arbitrarily close by mappings $g : \mathbb{S}^n/h \to \mathbb{S}^n$ with the property that the restriction $g \mid: g^{-1}(V) \to V$ is a finite covering map, where V is an open subset of $\mathbb{S}^n$ in the complement of the set of fixed points. If g is sufficiently close to $\beta\Pi^{-1}$, then β is homotopic to $g\Pi$ and hence we can use $g\Pi$ and the fact that the restriction $g\Pi \mid: \Pi^{-1}(g^{-1}(V) \to V$ is a finite covering map to calculate the degree of β. Note that the degree of $g\Pi$ is the sum of the signs of the preimages of a point in V, where the sign of a preimage is $+1$ if the orientation is locally preserved and -1 if the orientation is locally reversed. If h is an orientation preserving homeomorphism, the corresponding sum for the finite covering map $\Pi \mid: \Pi^{-1}(g^{-1}(V) \to V$ is p, because all elements of the same orbit have the same sign, therefore the whole sum for $g\Pi$ is a multiple of p. This implies that in this case the degree of h is divisible by p. If $p > 2$, then h is an orientation preserving homeomorphism and hence the degree of β is divisible by p. If $p = 2$ and h is a orientation reversing homeomorphism, then h has a balanced orbit, otherwise h is homotopic to the identity. □

The first consequence of our main theorem is the following well known fact [3].

Corollary 2. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism of period 2. Then there exists an $x \in \mathbb{S}^n$ such that $h(x) = -x$.*

For homeomorphism of prime period greater than 2, we have the analogous result

Corollary 3. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism of prime period p. Then, there is a point $x \in \mathbb{S}^n$ and $\lambda \geq 1$ such that*

$$\lambda x + \sum_1^{p-1} h^i(x) = 0.$$

In particular, there exists an $x \in \mathbb{S}^n$ such that the origin lies in the convex hull of $h^*(x)$.

Proof. Suppose not. Hence $\sum_1^p h^i(x) \neq 0$, for every $x \in \mathbb{S}^n$ and therefore, by Theorem 2.1, the barycentric mapping β of h is well defined and hence has degree divisible by p. In particular, β is not homotopic to identify. Consequently there is $x \in \mathbb{S}^n$ with the property that $\beta(x) = -x$, that is $\sum_1^p h^i(x) = -\delta x$, for $\delta > 0$, but hence $(1+\delta)x + \sum_1^{p-1} h^i(x) = 0$. □

3. The discrete geometric results

Theorem 4. *Let $\{x_1 \dots x_p\} \subset \mathbb{S}^n$ be such that $\lambda x_1 + \sum_2^p x_i = 0$, for some $\lambda \geq 1$, and for $1 \leq i \leq p$ suppose that $\|x_{i+1} - x_i\|$ is smaller or equal that ρ_p, the length of the side of a planar regular p-gon inscribed in the unit circle $\mathbb{S}^1$, where $x_{p+1} = x_p$. Then, $\{x_1 \dots x_p\}$ are the vertices of a planar regular p-gon inscribed in a maximal circle of $\mathbb{S}^n$.*

Proof. Intuitively the proof of the theorem consists of hanging the set $\{x_1 \dots x_p\}$ from the north pole. Let $\{q_1 \dots q_p\} \subset \mathbb{S}^1 \subset \mathbb{S}^n$ be the ordered vertices of a regular convex p-gon. Suppose first that $p = 2k$. Assume without loss of generality that $x_1 = p_1 = (1, 0, \dots, 0)$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^1$ be the orthogonal projection. Since $\|x_{i+1} - x_i\| \leq \rho_p$, for every $i = 1, \dots, p$ and $x_1 = p_1 = (1, 0, \dots, 0)$, we have that $\pi(q_{p-1}) \leq \pi(x_{p-1})$ and $\pi(q_2) \leq \pi(x_2)$. By the same reason, $\pi(q_i) \leq \pi(x_i)$, for $i = 1, \dots, p$. Since by hypothesis $\sum_1^p \pi(x_i) \leq 0 = \sum_1^p \pi(q_i)$, then $\pi(q_i) = \pi(x_i)$, $i = 1, \dots, p$. Consequently $\|x_{i+1} - x_i\| = \rho_p$, for $i = 1, \dots, p$. Furthermore, we have that $\{x_1 \dots x_{k+1}\}$ lies in a plane and similarly $\{x_{k+1} \dots x_{2k}\}$ lies in a plane. Finally in order to show that the set $\{x_1 \dots x_{2k}\}$ is planar note that its barycentre lies in $\mathbb{R}^1$. The case $p = 2k + 1$ is similar. □

Next, we have the following Jung's Theorem for spheres.

Theorem 5. *Let $F \subset \mathbb{S}^n$ be a set with diameter smaller that τ_n, the length of the edge of a regular $(n+1)$-simplex inscribed in $\mathbb{S}^n$. Then F is contained in a spherical n-disk cap of radius δ_n, $\delta_n^2 + \tau_n^2 = 4$. In particular, the convex hull of F does not contain the origin.*

4. Our results on periodic homeomorphism of sphere

Theorem 6. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism of prime period p. Then*

$$\|h\| \geq \rho_p.$$

Furthermore, if $\|h\| = \rho_p$, then there is a point $x \in \mathbb{S}^n$ such that its orbit $h^*(x)$ consists of a planar regular p-gon inscribed in a maximal circle of $\mathbb{S}^n$.

The proof of Theorem 4.1 follows immediately from Corollary 2.2 and Theorem 3.1. The next theorem gives a bound for the orbital diameter in terms of the dimension n.

Theorem 7. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism and let τ_n be the length of the edge of a regular $(n+1)$-simplex inscribed in $\mathbb{S}^n$. Then*

$$\Theta(h) \geq \tau_n$$

Furthermore, for $n \neq 1, 3, 7$,

$$\Theta(h) \geq \tau_{n-1}.$$

The proof of the first part of Theorem 4.2 follows from Corollary 2.2 and Theorem 3.2. The second part is more delicate. Assuming the oposite, Carathéodory's Theorem is used to give a trivialization of the tangent space of $\mathbb{S}^n$, which implies that $\mathbb{S}^n$ is parallelizable. This is a contradiction because $n \neq 1, 3, 7$.

In the case in which the period is prime we conjecture that the orbital diameter of a homeomorphism $h : \mathbb{S}^n \to \mathbb{S}^n$ is at least d_p, the diameter of a planar regular p-gon inscribed in the unit circle $\mathbb{S}^1$. In this direction we have:

Theorem 8. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a periodic homeomorphism of prime period p. Then*

$$\Theta(h) \geq d_p.$$

Theorem 9. *Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be a periodic homeomorphism of prime period* 3. *Then*

$$\Theta(h) \geq d_3 = \sqrt{3}.$$

Theorem 10. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a periodic isometry of prime period p, different from the identity. Then*

$$\Theta(h) = d_p.$$

For example, the isometries of of $\mathbb{S}^3$ of period 5 are basically the rotations around a plane, the product with the quaternionics and the homeomorphism produced by a cyclic permutation of a 4-simplex

Question. Does there exist a periodic homeomorphism of $\mathbb{S}^3$ of period 5 such that the convex hull of any orbit is a non degenerate 4-simplex containing the origin on its interior.

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Random normal vectors are normal

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To Imre Bárány for his 70th birthday.

Abstract

Let $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ be $n-1$ iid vectors in $\mathbb{R}^n$ (or $\mathbf{C}^n$), chosen from a general distribution. We study $\mathbf{x}$, the unit normal vector of the hyperplane spanned by the $\mathbf{v}_i$. Our main result confirms the natural conjecture that this normal vector looks like a random vector chosen uniformly from the unit sphere. In other words, it looks like a random vector with iid normal coordinates.

Our result has applications in random matrix theory. Consider an $n \times n$ random matrix with iid entries. We first prove an exponential bound on the upper tail for the least singular value, improving the earlier linear bound by Rudelson and Vershynin. Next, we derive optimal delocalization for the eigenvectors corresponding to eigenvalues of small modulus.

1. Introduction

Let $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ be $n-1$ iid vectors in $\mathbb{R}^n$ (or $\mathbf{C}^n$), chosen from a general distribution. Our object of study is the (random) hyperplane H spanned by these vectors.

As all information about a hyperplane is contained in its normal vector, the problem reduces to understanding $\mathbf{x}$, the unit normal vector of H. In this paper, we are going to assume that the coordinates of the $\mathbf{v}_i$ are iid copies of a random variable ξ with mean zero and variance 1. In matrix term, we let $A = (a_{ij})_{1\le i\le n-1, 1\le j\le n}$ be a random matrix of size $n-1$ by n where the entries a_{ij} are iid copies of ξ; the $\mathbf{v}_i$ are the row vectors of A. Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{F}^n$ be a unit vector that is orthogonal to the $\mathbf{v}_i$ (Here and later $\mathbf{F}$ is either $\mathbb{R}$ or $\mathbf{C}$, depending on the support of ξ.) Recent studies in the singularity probability of random non-Hermitian matrices (see for instance [6, 25]) show that under very general conditions on ξ, with extremely high probability A has rank $n-1$. In this case $\mathbf{x}$ is uniquely determined up to the sign ± 1 when $\mathbf{F} = \mathbb{R}$ or by a uniformly chosen rotation $\exp(\mathbf{i}\theta)$ when $\mathbf{F} = \mathbf{C}$. Throughout the

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paper, we use asymptotic notation under the assumption that n tends to infinity. In particular, $X = O(Y)$, $X \ll Y$, or $Y \gg X$ means that $|X| \leq CY$ for some fixed C.

When the entries of A are iid standard Gaussian $\mathbf{g_F}$, it is not hard to see that $\mathbf{x}$ is distributed as a random unit vector sampled according to the Haar measure in S^{n-1} of $\mathbf{F}^n$. One then deduces the following properties (see for instance [24][Section 2])

Theorem 1 (Random gaussian vector)**.** *Let* $\mathbf{x}$ *be a random vector uniformly distributed on the unit sphere* S^{n-1}*. Then,*

- *(joint distribution of the coordinates)* $\mathbf{x}$ *can be represented as*

$$\mathbf{x} := \left(\frac{\xi_1}{S}, \dots, \frac{\xi_n}{S}\right) \tag{1}$$

where ξ_i *are iid standard Gaussian* $\mathbf{g_F}$*, and* $S = \sqrt{\sum_{i=1}^n |\xi_i|^2}$*;*

- *(inner product with a fixed vector) for any fixed vector* $\mathbf{u}$ *on the unit sphere,*

$$\sqrt{n}\mathbf{x}^* \mathbf{u} \overset{d}{\to} \mathbf{g_F}; \tag{2}$$

- *(the largest coordinate) for any* $C > 0$*, with probability at least* $1 - n^{-C}$

$$\|\mathbf{x}\|_\infty \leq \sqrt{\frac{8(C+1)^3 \log n}{n}}; \tag{3}$$

- *(the smallest coordinate) for* $n \geq 2$*, any* $0 \leq c < 1$*, and any* $a > 1$*,*

$$\|\mathbf{x}\|_{\min} = \min\{|x_1|, \dots, |x_n|\} \geq \frac{c}{a} \frac{1}{n^{3/2}} \tag{4}$$

with probability at least $\exp(-2c) - \exp\left(-\frac{a^2 - \sqrt{2a^2 - 1}}{2} n\right)$.

It is natural to expect that even when ξ is not Gaussian, $\mathbf{x}$ still looks like a random vector from the unit sphere. The goal of this notes is to quantitatively confirm this belief. We say that ξ is sub-gaussian if there exists a parameter $K_0 > 1$ such that for all t

$$\mathcal{P}(|\xi| \geq t) = O(\exp(-\frac{t^2}{K_0})). \tag{5}$$

Definition 1 (Frequent events)**.** *Let* $\mathcal{E}$ *be an event depending on* n *(which is assumed to be sufficiently large).*

- *$\mathcal{E}$ holds asymptotically almost surely if $\mathcal{P}(\mathcal{E}) = 1 - o(1)$.*
- *$\mathcal{E}$ holds with high probability if there exists a positive constant δ such that $\mathcal{P}(\mathcal{E}) \geq 1 - n^{-\delta}$.*
- *$\mathcal{E}$ holds with overwhelming probability, and write $\mathcal{P}(\mathcal{E}) = 1 - n^{-\omega(1)}$, if for any $K > 0$, with sufficiently large n, $\mathcal{P}(\mathcal{E}) \geq 1 - n^{-K}$.*

We are now ready to state the main theorem.

Theorem 2 (Main result)**.** *Suppose that $A = (a_{ij})_{1\leq i\leq n-1, 1\leq j\leq n}$, where a_{ij} are iid copies of a normalized sub-gaussian random variable ξ. Let $\mathbf{x}$ be the normal vector of the rows of A, then the followings hold.*

- *(the largest coordinate) There are constants $C, C_1 > 0$ such that for any $m \geq C_1 \log n$*
$$\mathcal{P}(\|\mathbf{x}\|_\infty \geq \sqrt{m/n}) \leq Cn^2 \exp(-m/C). \tag{6}$$
In particularly, for any $\alpha > 0$ there exists a constant C_α such that
$$\mathcal{P}(\|\mathbf{x}\|_\infty \geq C_\alpha \sqrt{\frac{\log n}{n}}) \leq n^{-\alpha}.$$
- *(the smallest coordinate) with high probability*
$$\|\mathbf{x}\|_{\min} \geq \frac{1}{n^{3/2} \log^{O(1)} n}. \tag{7}$$
- *(joint distribution of the coordinates) There exists a positive constant c such that the following holds: for any d-tuple $(i_1, \ldots, i_m)$, with $d = n^c$, the joint law of the tuple $(\sqrt{n}x_{i_1}, \ldots, \sqrt{n}x_{i_d})$ is asymptotically independent standard normal. More precisely, there exists a positive constant c' such that for any measurable set $\Omega \in \mathbf{F}^d$,*
$$|\mathcal{P}((\sqrt{n}x_{i_1}, \ldots, \sqrt{n}x_{i_d}) \in \Omega) - \mathcal{P}((\mathbf{g}_{\mathbf{F},1}, \ldots, \mathbf{g}_{\mathbf{F},d}) \in \Omega)| \leq d^{-c'}, \tag{8}$$
where $\mathbf{g}_{\mathbf{F},1}, \ldots, \mathbf{g}_{\mathbf{F},d}$ are iid standard Gaussian.
- *(inner product with a fixed vector) Assume furthermore that ξ is symmetric, then for any fixed vector $\mathbf{u}$ on the unit sphere,*
$$\sqrt{n}\mathbf{x}^* \mathbf{u} \xrightarrow{d} \mathbf{g}_{\mathbf{F}}. \tag{9}$$

It also follows easily from (6) and (8) that with high probability $\|\mathbf{x}\|_\infty = \Theta(\sqrt{\frac{\log n}{n}})$. Indeed, it is clear that with high probability, with $m = n^c$ for some sufficiently small c, $\max\{|\mathbf{g}_{\mathbf{F},1}|, \dots, |\mathbf{g}_{\mathbf{F},m}|\} \gg \sqrt{\log m} = \sqrt{c\log n}$. Thus by (8), with high probability $\max\{|x_1|, \dots, |x_m|\} \gg \sqrt{\frac{\log n}{n}}$.

Our approach can be extended to unit vectors orthogonal to the rows of an iid matrix A of size $(n-k)\times n$, for any fixed k or even k grows slowly with n; the details will appear in a later paper.

As random hyperplanes appear frequently in various areas, including random matrix theory, high dimensional geometry, statistics, and theoretical computer science, we expect that Theorem 2 will be useful.

For the rest of this section, we discuss two direct applications in random matrix theory; we also refer the reader to the works of Garnaev-Gluskin [18] and of Kashin [20] for a different version of delocalization, which has found fundamental applications in compressive sensing.

2. Tail bound for the least singular value of a random iid matrix

Given an $n\times n$ random matrix $M_n(\xi)$ with entries being iid copies of a normalized variable ξ. Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ be its singular values. The two extremal σ_1 and σ_n are of special interest, and was studied by Goldstein and von Neumann, as they tried to analyze the running time of solving a system of random equations $M_n x = b$.

In [17], Goldstein and von Neumann speculated that σ_n is of order $n^{-1/2}$, which turned out to be correct. In particular, $\sqrt{n}\sigma_n$ tends to a limiting distribution, which was computed explicitly by Edelman in [8] in the gaussian case.

Theorem 3. *For any $t\geq 0$ we have*

$$\mathcal{P}(\sigma_n(M_{\mathbf{g}_\mathbb{R}}) \leq tn^{-1/2}) = \int_0^t \frac{1+\sqrt{x}}{2\sqrt{x}} e^{-x/2+\sqrt{x}} dx + o(1)$$

as well as

$$\mathcal{P}(\sigma_n(M_{\mathbf{g}_\mathbf{C}}) \leq tn^{-1/2}) = \int_0^t e^{-x} dx.$$

In other words, $\mathcal{P}(\sigma_n(M_{\mathbf{g}_\mathbb{R}}) \leq tn^{-1/2}) = 1 - e^{-t/2+\sqrt{t}} + o(1)$ and $\mathcal{P}(\sigma_n(M_{\mathbf{g}_\mathbf{C}}) \leq tn^{-1/2}) = 1 - e^{-t}$. These distributions have been confirmed to be universal (in the asymptotic sense) by Tao and the second author [34].

In applications, one usually needs large deviation results, which show that the probability that σ_n is far from its mean is very small. For the lower bound, Rudelson and Vershyin [25] proved that for any $t>0$

$$\mathcal{P}(\sigma_n \leq tn^{-1/2}) \leq Ct + .999^n, \tag{10}$$

which is sharp up to the constant C. For the upper bound, in a different paper [27], the same authors showed

$$\mathcal{P}(\sigma_n \geq tn^{-1/2}) \leq C\frac{\log t}{t}. \tag{11}$$

Using Theorem 2, we improve this result significantly by proving an exponential tail bound.

Theorem 4 (Exponential upper tail for the least singular values)**.** *Assume that the entries of $M_n = (m_{ij})_{1\leq i,j\leq n}$ are iid copies of a normalized sub-gaussian random variable ξ in either $\mathbb{R}$ or $\mathbf{C}$. Then there exist absolute constants C_1, C_2 depending on K_0 such that*

$$\mathcal{P}(\sigma_n \geq tn^{-1/2}) \leq C_1 \exp(-C_2 t).$$

Our proof of Theorem 4 is totally different from that of [27]. As showed in the gaussian case, the exponential bound is sharp, up to the value of C_2.

3. Eigenvectors of random iid matrices

Our theorem is closely related to (and in fact was motivated by) recent results concerning delocalization and normality of eigenvectors of random matrices. For random Hermitian matrices, there have been many results achieving almost optimal delocalization of eigenvectors, starting with the work [16] by Erdős et al. and and continued by Tao et al. and by many others in [35, 39, 9, 10, 11, 12, 13, 38, 2, 3, 4]. Thanks to new universality techniques, one also proved normality of the eigenvectors; see for instance the work [21] by Knowles and Yin, [36] by Tao and Vu, and [5] by Bourgade and Yau.

For non-Hermitian random matrix $M_n(\xi) = (m_{ij})_{1\leq i,j\leq n}$, much less is known. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues with $|\lambda_1| \geq \dots \geq |\lambda_n|$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the corresponding unit eigenvectors (where $\mathbf{v}_i$ are chosen according to the Haar measure from the eigensphere if the corresponding roots are multiple). Recently, Rudelson and Vershynin [29] proved that with overwhelming probability all of the eigenvectors satisfy

$$\|\mathbf{v}_i\|_\infty = O(\frac{\log^{9/2} n}{\sqrt{n}}). \tag{12}$$

By modifying the proof of Theorem 2, we are able sharpen this bound for eigenvectors of eigenvalues with small modulus.

Theorem 5 (Optimal delocalization for small eigenvectors)**.** *Assume that the entries of $M_n = (m_{ij})_{1\leq i,j\leq n}$ are iid copies of a normalized sub-gaussian random variable ξ in either $\mathbb{R}$ or $\mathbf{C}$. Then for any fixed $\varepsilon > 0$, with overwhelming probability the following holds for any unit eigenvector $\mathbf{x}$ corresponding to an eigenvalue λ of A with $|\lambda| = O(1)$*

$$\|\mathbf{x}\|_\infty = O(\sqrt{\frac{\log n}{n}}).$$

We believe that the individual eigenvector in Theorem 5 satisfies the normality property (8), which would imply that the bound $O(\sqrt{\frac{\log n}{n}})$ is optimal up to a multiplicative constant. Figure 1 below shows that the first coordinate of the eigenvector corresponding to the smallest eigenvalue behaves like a gaussian random variable.

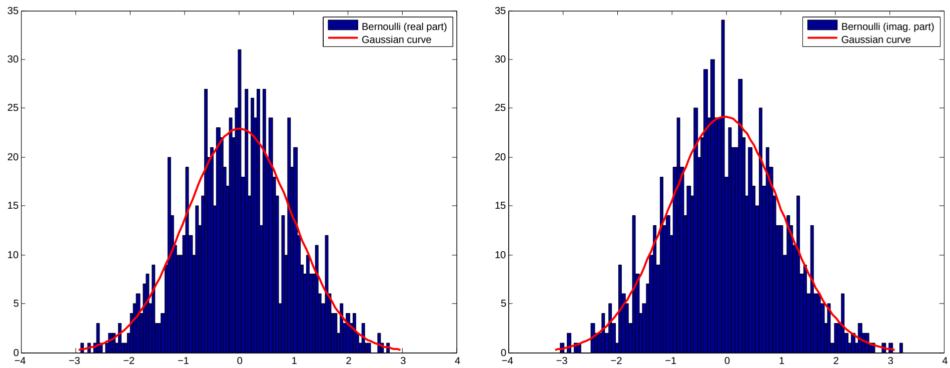

Figure 1: We sampled 1000 random complex iid Bernoulli matrices of size $n = 500$. The histograms represent the normalized real and imaginary parts $\sqrt{2n}\Re(\mathbf{v}(1))$ and $\sqrt{2n}\Im(\mathbf{v}(1))$ of the first coordinate of the unit eigenvector $\mathbf{v}$ associated with the eigenvalue of smallest modulus.

Finally, let us mention that all of our results hold (with logarithmic correction) under a weaker assumption that the variable ξ is sub-exponential, namely there are positive constants C, C' and α such that for all t, we have $\mathcal{P}(|\xi| \geq t) \leq C \exp(-C' t^\alpha)$. The detailed proofs and more discussion can be found in [23].

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Families of curves with many touchings

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Abstract

Consider a family of n continuous open curves in the plane, no three of which pass through the same point and no two of which intersect in infinitely many points. We say that a pair of curves is *touching* if they have only one interior point in common and at this point the first curve does not get from one side of the second curve to its other side. Otherwise, if the two curves intersect, they are said to form a *crossing* pair. Let t and c denote the number of touching pairs and crossing pairs, respectively. We prove that $c \geq \frac{1}{10^5}\frac{t^2}{n^2}$, provided that $t \geq 10n$.

Dedicated to Imre Bárány on his 70th birthday.

1. Introduction

In the context of the theory of topological graphs and graph drawing, many interesting questions have been raised concerning the adjacency structure of a family of curves in the plane or in another surface [5]. In particular, during the past four decades, various important properties of string graphs (i.e., intersection graphs of curves in the plane) have been discovered, and the study of different crossing numbers of graphs and their relations to one another has become a vast area of research. A useful tool in these investigations is the so-called crossing lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton [1, 6]. It states the following: Given a graph of n vertices and $e > 4n$ edges, no matter how we draw it in the plane by not necessarily straight-line edges, there are at least constant times e^3/n^2 crossing pairs of edges.

This lemma has inspired a number of results establishing the existence of many crossing subconfigurations of a given type in sufficiently rich geometric or topological structures. Another statement in similar spirit is the following extension of the

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Boros–Füredi–Bárány theorem: For every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that given a family of εn^3 triangles spanned by n points in the plane, one can always find $\varepsilon' n^3$ of them that have a point in common; see [3, 2].

In this note, we will be concerned with families of curves in the plane. By a *curve*, we mean a non-selfintersecting continuous arc in the plane, that is, a homeomorphic image of the open interval $(0, 1)$. Two curves are said to *touch* each other if they have precisely one point in common and at this point the first curve does not pass from one side of the second curve to the other. Any other pair of curves with nonempty intersection is called *crossing*. A family of curves is in *general position* if any two of them intersect in a finite number of points and no three pass through the same point.

The aim of this note to prove the following theorem.

Theorem. *Consider a family of n curves in general position in the plane which determines t touching pairs and c crossing pairs.*

If $t \geq 10n$, then we have $c \geq \frac{1}{10^5} \frac{t^2}{n^2}$. This bound is best possible up to a constant factor.

We make no attempt to optimize the constants in the Theorem.

Pach, Rubin, and Tardos [7] established a similar relationship between t, the number of touching pairs, and C, the number of crossing *points* between the curves. They proved that $C \geq t(\log \log(t/n))^{\delta}$, for an absolute constant $\delta > 0$. Obviously, we have $C \geq c$. There is an arrangement of n red curves and n blue curves in the plane such that every red curve touches every blue curve, and the total number of crossing points is $C = \Theta(n^2 \log n)$; cf. [4]. Of course, the number of crossing pairs, c, can never exceed $\binom{n}{2}$.

2. Proof of the Theorem

We start with an easy observation.

Lemma. *Given a family of $n \geq 3$ curves in general position in the plane, no two of which cross, the number of touchings, t, cannot exceed $3n - 6$.*

Proof. Pick a different point on each curve. Whenever two curves touch each other at a point p, connect them by an edge (arc) passing through p. In the resulting drawing, any two edges that do not share an endpoint are represented by disjoint arcs. According to the Hanani-Tutte theorem [9], this means that the underlying graph is planar, so that its number of edges, t, satisfies $t \leq 3n - 6$. □

Proof of the Theorem. We proceed by induction on n. For $n \leq 20$, the statement is void. Suppose that $n > 20$ and that the statement has already been proved for all values smaller than n.

We distinguish two cases.

CASE A: $t \le 10n^{3/2}$.

In this case, we want to establish the stronger statement

$$c \ge \frac{1}{10^4}\frac{t^2}{n^2}.$$

By the assumption, we have

$$\frac{1}{10^4}\frac{t^2}{n^2} \le \frac{n}{100}. \tag{1}$$

Let G_t (resp., G_c) denote the *touching graph* (resp., *crossing graph*) associated with the curves. That is, the vertices of both graphs correspond to the curves, and two vertices are connected by an edge if and only if the corresponding curves are touching (resp., crossing).

Let T be a minimal vertex cover in G_c, that is, a smallest set of vertices of G_c such that every edge of G_c has at least one endpoint in T. Let $\tau = |T|$. Let U denote the complement of T. Obviously, U is an *independent set* in G_c. According to the Lemma, the number of edges in $G_t[U]$, the touching graph induced by U, satisfies

$$|\mathbb{E}(G_t[U])| < 3|U| \le 3n. \tag{2}$$

By the minimality of T, G_c has at least $|T| = \tau$ edges. That is, we have $c \ge \tau$, so we are done if $\tau \ge \frac{1}{10^4}\frac{t^2}{n^2}$.

From now on, we can and shall assume that $\tau < \frac{1}{10^4}\frac{t^2}{n^2}$. By (1), we have $\frac{1}{10^4}\frac{t^2}{n^2} \le \frac{n}{100}$. Hence, $|T| \le \frac{n}{100}$ and

$$|U| = n - |T| \ge \frac{99n}{100}. \tag{3}$$

Let $U' \subseteq U$ denote the set of all vertices in U that are not isolated in the graph G_c. By the definition of T, all neighbors of a vertex $v \in U$ in G_c belong to T. If $|U'| \ge \frac{1}{10^4}\frac{t^2}{n^2}$, then we are done, because $c \ge |U'|$.

Therefore, we can assume that

$$|U'| < \frac{1}{10^4}\frac{t^2}{n^2} \le \frac{n}{100}, \tag{4}$$

where the second inequality follows again by (1).

Letting $U_0 = U \setminus U'$, by (3) and (4) we obtain $|U_0| = |U| - |U'| \ge \frac{98n}{100}$. Clearly, all vertices in U_0 are isolated in G_c.

Suppose that $G_t[T \cup U']$ has at least $\frac{t}{10}$ edges. Consider the set of curves $T \cup U'$. We have $n_0 = |T \cup U'| \le \frac{2n}{100}$ and, the number of touchings, $t_0 = |\mathbb{E}(G_t[T \cup U'])| \ge \frac{t}{10}$. Therefore, by the induction hypothesis, for the number of crossings we have $c_0 = |\mathbb{E}(G_c[T \cup U'])| \ge \frac{1}{10^5}\frac{t_0^2}{n_0^2} \ge \frac{1}{10^4}\frac{t^2}{n^2}$ and we are done. Hence, we assume in the sequel that $G_t[T \cup U']$ has fewer than $\frac{t}{10}$ edges.

Consequently, for the number of edges in G_t running between T and U_0, we have

$$|\mathbb{E}(G_t[T,U_0])| \geq t - |\mathbb{E}(G_t[T \cup U'])| - |\mathbb{E}(G_t[U_0 \cup U'])| \geq t - \frac{t}{10} - 3n > \frac{t}{2}. \tag{5}$$

Here we used the assumption that $t \geq 10n$.

Let $\chi = \chi(G_c[T])$ denote the chromatic number of $G_c[T]$. In any coloring of a graph with the smallest possible number of colors, there is at least one edge between any two color classes. Hence, $G_c[T]$ has at least $\binom{\chi}{2} \geq \frac{1}{10^4}\frac{t^2}{n^2}$ edges, and we are done, provided that $\chi > \frac{1}{70} \cdot \frac{t}{n}$.

Thus, we can suppose that

$$\chi = \chi(G_c[T]) \leq \frac{1}{70} \cdot \frac{t}{n}. \tag{6}$$

Consider a coloring of $G_c[T]$ with χ colors, and denote the color classes by $I_1, I_2, \ldots, I_\chi$. Obviously, for every j, $I_j \cup U_0$ is an independent set in G_c. Therefore, by the Lemma, $G_t[I_j \cup U_0]$ has at most $3n$ edges. Summing up for all j and taking (6) into account, we obtain

$$|\mathbb{E}(G_t[T,U_0])| \leq \sum_{j=1}^{\chi} |\mathbb{E}(G_t[I_j \cup U_0])| \leq \frac{1}{70} \cdot \frac{t}{n} 3n \leq \frac{t}{20},$$

contradicting (5). This completes the proof in CASE A.

CASE B: $t \geq 10n^{3/2}$.

Set $p = \frac{10n^3}{t^2} \leq \frac{1}{10}$. Select each curve independently with probability p. Let $\mathbf{n}'$, $\mathbf{t}'$, and $\mathbf{c}'$ denote the number of selected curves, the number of touching pairs, and the number of crossing pairs between them, respectively. Clearly,

$$\mathbb{E}[\mathbf{n}'] = pn, \;\; \mathbb{E}[\mathbf{t}'] = p^2 t, \;\; \mathbb{E}[\mathbf{c}'] = p^2 c. \tag{7}$$

The number of selected curves, $\mathbf{n}'$, has binomial distribution, therefore,

$$\mathrm{Prob}[|\mathbf{n}' - pn| > \frac{1}{4}pn] < \frac{1}{3}. \tag{8}$$

By Markov's inequality,

$$\mathrm{Prob}[\mathbf{c}' > 3p^2 c] < \frac{1}{3}. \tag{9}$$

Consider the touching graph G_t. Let $d_1, \ldots, d_n$ denote the degrees of the vertices of G_t, and let $e_1, \ldots, e_t$ denote its edges, listed in any order. We say that an edge e_i is *selected* (or belongs to the random sample) if both of its endpoints were selected. Let X_i be the *indicator variable* for e_i, that is,

$$X_i = \begin{cases} 1 & \text{if } e_i \text{ was selected,} \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathbb{E}[X_i] = p^2$. Let $\mathbf{t}' = \sum_{i=1}^{t} X_i$. It follows by straightforward computation that for every i,

$$\mathrm{var}[X_i] = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2] = p^2 - p^4,$$

If e_i and e_j have a common endpoint for some $i \neq j$, then

$$\mathrm{cov}[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = p^3 - p^4.$$

If e_i and e_j do not have a common vertex, then X_i and X_j are independent random variables and $\mathrm{cov}[X_i, X_j] = 0$. Therefore, we obtain

$$\sigma^2 = \mathrm{var}[\mathbf{t}'] = \sum_{i=1}^{t} \mathrm{var}[X_i] + \sum_{1 \le i \neq j \le t} \mathrm{cov}[X_i, X_j]$$

$$= \sum_{i=1}^{t} \mathrm{var}[X_i] + \sum_{1 \le i \neq j \le t} \mathrm{cov}[X_i, X_j]$$

$$= (p^2 - p^4)t + (p^3 - p^4)\sum_{i=1}^{n} d_i(d_i - 1) < p^2 t + 2p^3 nt.$$

From here, we get $\sigma < \sqrt{p^2 t} + \sqrt{2p^3 nt} < p^2 t = \mathbb{E}[\mathbf{t}']$. By Chebyshev's inequality,

$$\mathrm{Prob}[|\mathbf{t}' - p^2 t| \ge \lambda\sigma] \le \frac{1}{\lambda^2}.$$

Setting $\lambda = \frac{1}{4}$,

$$\mathrm{Prob}[|\mathbf{t}' - p^2 t| \ge \frac{p^2 t}{4}] \le \frac{1}{4^2} < \frac{1}{3}. \tag{10}$$

It follows from (8), (9), and (10) that, with positive probability, we have

$$|\mathbf{n}' - pn| \le \frac{1}{4}pn, \quad \mathbf{c}' \le 3p^2 c, \quad |\mathbf{t}' - p^2 t| \le \frac{1}{4}p^2 t. \tag{11}$$

Consider a fixed selection of n' curves with t' touching pairs and c' crossing pairs for which the above three inequalities are satisfied. Then we have

$$t' \ge \frac{3}{4}p^2 t = \frac{300}{4} \cdot \frac{n^6}{t^3},$$

$$n' \le \frac{5}{4}pn = \frac{50}{4} \cdot \frac{n^4}{t^2},$$

and, hence,

$$t' \ge 10n'. \tag{12}$$

On the other hand,

$$t' \le \frac{5}{4}p^2 t = \frac{500}{4} \cdot \frac{n^6}{t^3},$$

$$n' \geq \frac{3}{4}pn = \frac{30}{4} \cdot \frac{n^4}{t^2},$$

so that

$$10(n')^{3/2} \geq 10 \cdot \frac{30^{3/2}}{4^{3/2}} \cdot \frac{n^6}{t^3} > t'. \tag{13}$$

According to (12) and (13), the selected family meets the requirements of the Theorem in CASE A. Thus, we can apply the Theorem in this case to obtain that $c' \geq \frac{1}{10^4}\frac{t'^2}{n'^2}$. In view of (11), we have

$$3p^2c \geq c', \quad t' \geq \frac{3}{4}p^2t, \quad n' \leq \frac{5}{4}pn.$$

Thus,

$$3p^2c \geq c' \geq \frac{1}{10^4}\frac{t'^2}{n'^2} \geq \frac{1}{10^4}\frac{(3p^2t/4)^2}{(5pn/4)^2} = \frac{1}{10^4}\left(\frac{3}{5}\right)^2\frac{p^2t^2}{n^2}.$$

Comparing the left-hand side and the right-hand side, we conclude that

$$c \geq \frac{1}{10^5}\frac{t^2}{n^2},$$

as required. This completes the proof of the Theorem. □

The following construction shows that the order of magnitude of the bound in the Theorem is best possible. Suppose that $t < \frac{n^2}{4}$. Take a collection A of $n - \frac{2t}{n} > \frac{n}{2}$ pairwise disjoint curves, and another collection B of $\frac{2t}{n}$ curves such that every element of B touches precisely $\frac{n}{2}$ elements of A, but does not cross any of them. The family $A \cup B$ consists of n curves, and the number of touching pairs is t. The only pairs of curves that may cross each other belong to B. Thus, the number of crossing pairs is at most $\binom{2t/n}{2} \leq \frac{2t^2}{n^2}$.

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k-monotone interpolation

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Abstract

A real function f is called *k-monotone* if its $(k-2)$nd derivative is convex. A set of points P in the plane is *k-monotone interpolable* if it lies on the graph of a k-monotone function. These notions have been studied in analysis and approximation theory since the 1940s.

We show a Ramsey-type result: for every n, k there exists N such that every set P of N points in the plane with distinct x coordinates has a subset Q of size n such that Q or its vertical mirror reflection is k-monotone interpolable. The cases $k = 1$ and $k = 2$ are classical results of Erdős and Szekeres, while the case $k = 3$ was show by Cibulka, Matousek and Patak in [1] (2014). They claim this to be an interesting result because of the very non-local property of k-monoton interpolability. In the case of $k = 1$ and $k = 2$ a set P is k-monotone interpolable if and only if every $(k+1)$ element subset of P is k-monotone interpolable. In fact it is enough to consider (k+1)-element sets with consecutive x coordinates. But for $k = 3$ they show that for every n there exists a set P of size n that is not 3-monotone interpolable, but every proper subset of P is 3-monotone interpolable. They claim that this example works for every odd k as well.

Joint work with Martin Balko, Géza Tóth and Pavel Valtr.

We describe the characterization of k-monotone interpolability. We use most of the definitions and notations from [1] and [2].

Divided differences. The kth divided difference of a real function f at points $x_0, \ldots, x_k$ is denoted by $[x_0, \ldots, x_k]f$ and is defined recursively by

$$[x_0]f = f(x_0), \;\; [x_0, \ldots, x_k]f = \frac{[x_1, \ldots, x_k]f - [x_0, \ldots, x_{k-1}]f}{x_k - x_0},$$

It is known that a function $f : I \to \mathbb{R}$ is k-monotone on the open interval I if and only if for every $x_0 < \cdots < x_k \in I$ the divided difference is non-negative, $[x_0, \ldots, x_k]f \geq 0$.

B-splines. Let $X = \{x_1, \ldots, x_{n+k}\} \subset \mathbb{R}$, $x_1 < \cdots < x_{n+k}$, be a set of $n+k$ real numbers. The *B-Splines* of degree $k-1$ corresponding to X are the functions $M_1(t), \ldots, M_n(t)$ defined as follows:

$$M_i(t) = k[x_i, \ldots, x_{i+k}] \max(0, x - t)^{k-1} \;\; .$$

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The $M_i(t)$ are functions which are zero outside the interval $[x_i, x_{i+k}]$ and are strictly positive in the interior of the interval.

Characterization of k-monotone interpolability. The following lemma is Corollary 6.5 in [2] and gives a characterization of k-monotone interpolability.

Lemma 1. *Let $X = (x_1, \dots, x_{n+k}) \subset \mathbb{R}$, $x_1 < \dots, x_{n+k}$, be a node sequence, let $f : X \to \mathbb{R}$ be a function, and let the vector $v = (v_1, \dots, v_n)$ be given by $v_i = [x_i, \dots, x_{i+k}]f$. Then $P = (X, f) = \{(x, f(x)) \mid x \in X\}$ is k-monotone interpolable if and only if the following holds for every $a = (a_1, \dots, a_n) \in \mathbb{R}^n$: If $\sum_{i=1}^n a_i M_i(t) \geq 0$ for every $t \in [x_1, x_{n+k}]$, then $\sum_{i=1}^n a_i v_i \geq 0$.*

The following alternative characterization of k-monotone interpolability, derived from the previous lemma, was described in [1]:

Lemma 2. *Let $X = (x_1, \dots, x_{n+k}) \subset \mathbb{R}$, $x_1 < \dots < x_{n+k}$, be a node sequence, let $f : X \to \mathbb{R}$ be a function, and let the vector $v = (v_1, \dots, v_n)$ be given by $v_i = [x_i, \dots, x_{i+k}]f$. Then (X, f) is k-monotone interpolable if and only if there exist $c_1, \dots, c_n \geq 0 \in \mathbb{R}$ and $t_1, \dots, t_n \in [x_1, x_{k+n}]$ satisfying $v_i = \sum_{j=1}^n c_j M_i(t_j)$ for all $i = 1, \dots, n$.*

In [1] Ramsey-type result for three-interpolability is proven by only considering B-splines of the form $t \in X$. If $t_j = x_j$ for $j \in [1, n]$ then $M_j = (M_1(t_j), \dots, M_n(t_j)) \in \mathbb{R}^n$ is a vector with all zero components except for at most two $(k-1)$ in general: $M_{j-2}(t_j)$ and $M_{j-1}(t_j)$. If there exists a positive solution of $Mc = v$, where $M = (M_1, \dots, M_n)$ and $c = (c_1, \dots, c_n)$ then (X, f) is k-monotone interpolable. For the classical cases $k = 1$ and $k = 2$ it is enough that all the v_i are positive, but for $k \geq 3$ this not sufficient. The authors realize that this linear system does have a positive solution if after it is transformed such that all the $v_i = 1$ either always the first non zero element in each column of M is larger then the second non zero element or the other way around. For $k \geq 4$ simply comparing these elements is not enough.

We show that

Theorem 3. *For every $k, n \geq 4$ there exists $N = N_k(n)$ such that for every N-element set X and function f there exists an n-element set $X' \subset X$ such that either (X', f) or $(X', -f)$ is k-monotone interpolable.*

The proof is based on the result that there exists a positive solution of $Mc = v$ for the *middle* (by removing the first and last k rows) if for any two columns the ratios of the $k-1$ non zero elements are *almost* the same.

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Polytopes and cones in random hyperplane tessellations

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Under suitable assumptions, a system of countably many hyperplanes in Euclidean space $\mathbb{R}^d$ divides the space into countably many convex polytopes. Similarly, finitely many linear hyperplanes (i.e., hyperplanes through the origin) divide the space into finitely many polyhedral cones (or, equivalently, the unit sphere into finitely many spherically convex polytopes). When the hyperplanes are random, this leads to various types of random polytopes or random cones. We present some recent results on the shapes of such random polyhedra and on some geometric functions related to them. Before being more concrete, it is necessary to specify the stochastic models that we shall consider.

Let $\mathcal{H}^d$ be the space of hyperplanes in $\mathbb{R}^d$, with its usual topology. We write hyperplanes in the form $u^\perp + tu$, with $u \in \mathbb{S}^{d-1}$ (the unit sphere). A hyperplane process X is a measurable mapping from some probability space into the measurable space of locally finite subsets of $\mathcal{H}^d$ (with a suitable σ-algebra). Its intensity measure is defined by $\Theta(A) = \mathbb{E}\,\mathrm{card}(X \cap A)$ for $A \in \mathcal{B}(\mathcal{H}^d)$ ($\mathcal{B}$ = Borel sets). We assume in the following that X is a *stationary Poisson hyperplane process*, that is, the intensity measure $\Theta \not\equiv 0$ is translation invariant, locally finite, satisfies

$$\mathbb{P}\{\mathrm{card}(X \cap A) = n\} = e^{-\Theta(A)} \frac{\Theta(A)^n}{n!} \quad \text{for } n \in \mathbb{N}_0 \text{ and } A \in \mathcal{B}(\mathcal{H}^d),$$

and the restrictions of X to pairwise disjoint sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H}^d)$ are stochastically independent.

The properties of the process X and the polytopes it defines will essentially depend on the directions of the hyperplanes appearing in the process. Therefore, we have to take the *directional distribution* of X into account. This is the even probability measure φ on $\mathbb{S}^{d-1}$ that is defined by the decomposition

$$\Theta(\cdot) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_{(\cdot)}\{u^\perp + tu\}\,\mathrm{d}t\,\varphi(\mathrm{d}u),$$

where the constant $\gamma > 0$ is the intensity of X. The intuitive meaning of the directional distribution is revealed by

$$\varphi(A) = \frac{\mathbb{E}\,\mathrm{card}\{u^\perp + tu \in X : u \in A,\, t \in [0,1]\}}{\mathbb{E}\,\mathrm{card}\{u^\perp + tu \in X : t \in [0,1]\}}$$

for $A \in \mathcal{B}(\mathbb{S}^{d-1})$. It is assumed that φ is not concentrated on a great subsphere. Under this assumption, a realization of X induces, with probability one, a tessellation

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of $\mathbb{R}^d$ into compact convex polytopes. The tessellation is called the *mosaic* induced by X and denoted by M_X. Its d-dimensional polytopes are called the *cells* of the mosaic. We shall be interested in their shape and in some geometric functionals associated with them.

The second model we study is much simpler. We consider n independent, identically distributed random linear hyperplanes. They divide $\mathbb{R}^d$ into finitely many polyhedral cones. Picking one of these at random, we define a random polyhedral cone; this will be studied in Section 3.

1. The cells in a stationary hyperplane mosaic

The possible shapes of the cells in a stationary Poisson hyperplane mosaic M_X depend, of course, on the directional distribution of the underlying hyperplane process X. For example, if X is a *parallel process* (its directional distribution is concentrated in n antipodal pairs of points), then all cells are parallelepipeds. On the other hand, if the process is *isotropic* (its directional distribution is rotation invariant), then in a realization one will see many different shapes (and sizes) of cells. In recent joint work [5] with Matthias Reitzner, this impression was confirmed in a strong sense. The directional distribution φ need only satisfy the following assumption (which is satisfied in the isotropic case).

Assumption (*). The support of the directional distribution φ is the whole unit sphere $\mathbb{S}^{d-1}$, and φ assigns measure zero to each great subsphere of $\mathbb{S}^{d-1}$.

With probability one, the hyperplanes of X are in general position. This implies that all cells of the mosaic M_X are simple polytopes. In a precise sense, this is the only restriction, as manifested in the subsequent two theorems. To express that the cells can approximate all possible shapes, we use the space $\mathcal{K}^d$ of convex bodies in $\mathbb{R}^d$ with the Hausdorff metric.

Theorem. *If assumption $(*)$ is satisfied, then with probability one the set of all translates of the cells of M_X is dense in $\mathcal{K}^d$.*

Theorem. *If assumption $(*)$ is satisfied, then with probability one, for every simple d-polytope P there are infinitely many cells of M_X that are combinatorially isomorphic to P.*

Crucial for the proof are the independence properties of the Poisson process. Besides geometric constructions, the proof uses a generalization of the Borel–Cantelli lemma (to not necessarily independent events), due to Erdös and Rényi (1959).

2. Typical faces, especially their vertex number

In contrast to the previous section, we now consider typical cells of the mosaic M_X or, more generally, typical k-faces. Typical cells of stationary random hyperplane mosaics belong to the most studied models of random polytopes. By k-faces we mean all k-dimensional faces of all cells of the mosaic M_X, $k = 0, \dots, d$. Let $\mathcal{F}_k(X)$ be the set of these k-faces. To every translation-invariant positive measurable function w on k-polytopes, one can define a random polytope $Z_w^{(k)}$, the *w-weighted typical k-face* of M_X. The heuristic idea is to pick out, within a 'large' bounded region of space, at random one of the k-faces, with chances proportional to the weight function w. A correct definition can be given by using either grain distributions of stationary particle processes, or Palm distributions. We mention here only a consequence, which mirrors the intuitive idea of selecting, with weights, from large bounded regions. If s denotes the Steiner point (say) and $B(o,r)$ is the ball with center o and radius r, then the distribution of $Z_w^{(k)}$ is given by

$$\mathbb{P}\{Z_w^{(k)} \in A\} = \lim_{r\to\infty} \frac{\mathbb{E}\sum_{F\in\mathcal{F}_k(X),\, F\subset B(o,r)} \mathbf{1}_A\{F - s(F)\} w(F)}{\mathbb{E}\sum_{F\in\mathcal{F}_k(X),\, F\subset B(o,r)} w(F)}$$

for Borel sets A in the space of polytopes.

For the weight $w \equiv 1$, the w-weighted typical k-face is just called the *typical k-face* and denote by $Z^{(k)}$. Besides this, we consider the volume-weighted typical k-face, $Z_{\mathrm{vol}}^{(k)}$, where the weight is the k-dimensional volume.

Expectations of several geometric functionals of the typical cell $Z^{(d)}$ were already determined by Miles [4]. Perhaps the simplest of these functionals is the vertex number, commonly denoted by f_0. It was noted by Joseph Mecke (1981) that the expectation

$$\mathbb{E}\, f_0(Z^{(k)}) = 2^k$$

does not depend on the directional distribution of X. This changes drastically if the typical k-face is replaced by the volume-weighted typical k-face. It was proved in [6] that

$$2^k \le \mathbb{E}\, f_0(Z_{\mathrm{vol}}^{(k)}) \le 2^{-k} k! \kappa_k^2 \tag{1}$$

for $k \in \{2, \dots, d\}$, where κ_k is the volume of the k-dimensional unit ball. Equality on the left side of (1) holds if and only if X is a parallel process. On the right side, equality holds if X is isotropic. In [7], these inequalities were extended to the weights L_j defined below.

Here we want to point out the recent result that a similar phenomenon occurs for the typical k-face if not the expectation, but the variance is considered. It is proved in [8] that

$$0 \le \operatorname{Var} f_0(Z^{(k)}) \le 2^k k! \left(\sum_{j=0}^{k} \frac{\kappa_j^2}{4^j (k-j)!} \right) - 2^{2k} \tag{2}$$

for $k \in \{2, \dots, d\}$. Equality on the left side of (2) holds if and only if X is a parallel process. Equality on the right holds if X is isotropic with respect to a suitable scalar product on $\mathbb{R}^d$, and for $k = d$ it holds only in this case.

More generally than $\mathbb{E}\, f_0(Z^{(k)})^2$, the following second moments were determined in [8]. For a convex polytope P in $\mathbb{R}^d$ and for $r \in \{0, \dots, d\}$, let $L_r(P)$ be the sum of the r-dimensional volumes of the r faces of P. Thus, L_0 is the vertex number f_0, L_1 is the total edge length, L_{d-1} is the surface area, and L_d is the volume. In [8], explicit formulas are given for the mixed second moments $\mathbb{E}\,(L_r L_s)(Z^{(k)})$. They involve the associated zonoid of the hyperplane process X. In the isotropic case for $k = d$, they reduce to formulas of Miles [4]. Together with the known expectations, our formulas explicitly provide the covariance matrix of the random vector $(L_0(Z^{(k)}), \dots, L_k(Z^{(k)}))$.

3. Random cones

For considering the second model, n independent, identically distributed random linear hyperplanes and the cones of the induced tessellation, there are (at least) two motivations. First, it yields the natural analogue, in spherical space, of random hyperplane tessellations, and, second, questions on random cones have recently found particular interest in certain applications of convex optimization. For example, the question “When does a randomly oriented cone strike a fixed cone?”, verbally quoted from [3], means the following. Let $C, D \subset \mathbb{R}^d$ be polyhedral convex cones, not both subspaces. Let $\boldsymbol{\theta} \in \mathrm{SO}_d$ be a random rotation, with distribution given by the normalized Haar measure. The question asks for the probability $\mathbb{P}\{C \cap \boldsymbol{\theta} D \neq \{o\}\}$. The answer involves the conic intrinsic volumes. For a polyhedral cone C, a simple approach to these is as follows. Let Π_C denote the nearest-point map (metric projection) of C, and let $\mathbf{g}$ be a standard Gaussian random vector in $\mathbb{R}^d$. Then

$$V_k(C) := \mathbb{P}\{\Pi_C(\mathbf{g}) \in F \text{ for some } k\text{-face } F \text{ of } C\}$$

defines the kth conic intrinsic volume V_k, $k \in \{0, \dots, d\}$. Now the spherical kinematic formula and the spherical Gauss–Bonnet formula can be combined to yield the following (well-known) answer to the question above:

$$\mathbb{P}\{C \cap \boldsymbol{\theta} D \neq \{o\}\} = 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{i=2k+1}^{d} V_i(C) V_{d+2k+1-i}(D).$$

This formula gives the probability of non-trivial intersection of a random cone with a fixed cone. The random cone is of a special type: the randomness comes only from the random rotation, which is applied to a fixed cone. It is certainly of interest to have similar results for more flexible types of random cones, where also the shape can be random and not only the position. On the other hand, it is to be expected that only very special models can lead to explicit results. We provide here one such class of random cones which are suitable for this purpose.

Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be stochastically independent random linear hyperplanes, each with rotationally invariant distribution. They induce a tessellation of $\mathbb{R}^d$ into d-dimensional polyhedral cones. We pick one of these at random (with equal chances) and call it the (isotropic) *random Schläfli cone* S_n, with parameter n. The name has been chosen since Schläfli has shown that n linear hyperplanes in general position in $\mathbb{R}^d$ divide the space into

$$C(n,d) := 2\sum_{r=0}^{d-1}\binom{n-1}{r}$$

d-dimensional cones. Since the Schläfli cone S_n has an isotropic distribution and since the expectations of its conic intrinsic volumes are explicitly known (see below), one can prove the following formula of 'kinematic type':

$$\mathbb{P}\{C \cap S_n \neq \{o\}\} = \frac{2}{C(n,d)}\sum_{j=1}^{n}\sum_{k=0}^{\lfloor\frac{j-1}{2}\rfloor}\binom{n}{j-2k-1}V_j(C). \tag{3}$$

One can also compute the probability that two stochastically independent random Schläfli cones, with parameters n and m, have a non-trivial intersection.

To obtain (3), one needs to know the expectations $\mathbb{E}\, V_k(S_n)$. They were obtained as special cases in joint work [2] with Daniel Hug, investigating random central hyperplane arrangements and their induced cones. Let ϕ^* be a probability measure on the Grassmannian $G(d,d-1)$ that assigns measure zero to the set of hyperplanes in $G(d,d-1)$ containing some given line through the origin. Let $\mathcal{H}_1,\dots,\mathcal{H}_n$ be independent random hyperplanes in $G(d,d-1)$ with distribution ϕ^*. Let S_n denote the induced random Schläfli cone. In [2], the expectations are determined for a series of geometric functionals of S_n. These functionals comprise the numbers f_k of k-faces (for which the expectations were already found by Cover and Efron [1]), the conic intrinsic volumes V_k, and the total face contents Λ_k, defined as follows. For a polyhedral cone C, $\Lambda_k(C)$ is the sum of the internal angles of the k-faces of C ($k = 1,\dots,d$). Equivalently, $\Lambda_k(C)$ is the total $(k-1)$-dimensional spherical volume of the $(k-1)$-faces of the spherical polytope $C \cap \mathbb{S}^{d-1}$. Thus, these are the spherical analogues of the functionals $L_0,\dots,L_d$ considered for Euclidean space in Section 2. The expectation

$$\mathbb{E}\,\Lambda_k(S_n) = \frac{2^{d-k}\binom{n}{d-k}}{C(n,d)}$$

is, as the more general ones determined in [2], independent of the distribution. This reveals the combinatorial core of these results.

In contrast, if second moments are considered, we have to assume that the distribution ϕ^* is rotation invariant. Under this assumption, the main result of [2] is the determination of the mixed second moments

$$\begin{aligned}\mathbb{E}\,(\Lambda_r\Lambda_s)(S_n) \quad &= \quad \frac{1}{C(n,d)}\sum_{p\in\mathbb{N}}2^{d-p}\binom{n}{d-p}\binom{n-d+p}{p-r,p-s,n-d-p+r+s}\\ &\qquad \times\theta(n-d-p+r+s,p),\end{aligned}$$

where

$$\theta(n,d) := \frac{(d-1)\kappa_{d-1}}{d\kappa_d} \int_0^\pi \left(1 - \frac{x}{\pi}\right)^n \sin^{d-2} x \,\mathrm{d}x.$$

This can be considered as the spherical counterpart to the results in Section 2 for $\mathbb{E}\,(L_r L_s)(Z^{(d)})$, which in the isotropic case go back to Roger Miles.

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Tensors, colors, and convex hulls

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Abstract

We describe some recent affine variations of Tverberg's theorem, the colorful Carathéodory theorem, and the link between them via Sarkaria's tensor product technique.

To Imre Bárány, who was a splendid advisor, in honor of this 70th birthday.

1. Introduction

An important part of discrete geometry deals with the intersection of the convex hulls of finite sets of points. This gave birth to the classic theorems of Helly, Tverberg, and Carathéodory. As our understanding of these results improved, it became clear that stronger combinatorial conditions could be imposed on these theorems, in the form of colors for the sets and conditions regarding the colors (consult [12] for a general introduction).

Moreover, a clear connection between colorful Carathéodory results and (monochromatic) Tverberg results was established by Sarkaria's seminal proof of Tverberg's theorem [13], giving rise to a strong linear-algebraic way to approach Tverberg-type results, surveyed in [4]. The aim of this note is to present some of the recent advances surrounding this technique and the related theorems.

2. The colorful Carathéodory theorem

Given a finite set $X \subset \mathbb{R}^d$, and a point $c \in \mathbb{R}^d$, Carathéodory's theorem bounds the complexity of checking whether $c \in \operatorname{conv} X$. We say that X *captures* c if $c \in \operatorname{conv}(X)$. The colorful Carathéodory theorem is a very neat extension of this result.

Theorem 1 (Colorful Carathéodory, Bárány 1982 [3])**.** *Given $c \in \mathbb{R}^n$ and $n+1$ sets $F_1, F_2, \dots, F_{n+1}$ in $\mathbb{R}^n$ such that each F_i captures c, we can find $n+1$ points $x_1 \in F_1, x_2 \in F_2, \dots, x_{n+1} \in F_{n+1}$ such that the set $\{x_1, \dots, x_{n+1}\}$ captures c.*

The classic Carathédory theorem is the case $F_1 = \dots = F_{n+1}$. The reason it is called a colorful version is because we can consider each set F_i to be a color class, we are simply finding a colorful set that captures the origin. This result has far-reaching generalizations [11]. Recently, it was shown that if each F_i satisfies

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$|F_i| = n+1$, we can always find at least n^2+1 different sets satisfying the theorem's conclusion [14].

The case when $|F_i| = 2$ for all i and c is the origin is particularly interesting. Let us show a short proof. Without loss of generality, we assume that for every i there is a vector v_i such that $F_i = \{v_i, -v_i\}$. Then, there is a nontrivial linear dependence of the set $v_1, v_2, \ldots, v_{n+1}$, $\sum_{i=1}^{n+1} \alpha_i v_i = 0$. If any $\alpha_i < 0$, we can swap v_i and $-v_i$ and change the sign of α_i. Once every $\alpha_i \geq 0$, as they are not all zero, we may assume that their sum of 1.

Thus, the cases when there is an $r > 2$ such that $|F_i| = r$ for every i can be interpreted as an extension of the statement that *every $n+1$ vectors in $\mathbb{R}^n$ have a non-trivial linear dependence.* Indeed, that is how we are going to use it.

3. Tverberg's theorem

Tverberg's theorem is a compelling result in discrete geometry.

Theorem 2 (Tverberg 1966 [17]). *Let r, d be positive integers. For any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r sets whose convex hulls intersect.*

The number of points in the result above is optimal. Before jumping into the linear-algebraic proof methods for this theorem, let us show how we can use Carathéodory's theorem to get a similar result.

Claim 3 (weak Tverberg). *Given $(r-1)d(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r sets whose convex hulls intersect.*

Proof. Let S be a set of $(r-1)d(d+1)+1$ points and let p be a centerpoint of S. In other words, every closed half-space that contains p has at least $\lceil \frac{|S|}{d+1} \rceil = (r-1)d+1$ points of S. By Carathéodory's theorem, we can find a set of at most $d+1$ points whose convex hull contains p. Let A_1 be an inclusion-minimal set satisfying this property. Then, if we remove A_1, every closed half-space that contains p in its boundary must contain at least $(r-2)d+1$ of the remaining points. We can construct A_2 in a similar manner and continue until we get the r desired sets. □

The technique above shows a link between the existence of centerpoints (which follows from Helly's theorem) and Tverberg-type problems. Moreover, since the dependence in r is linear, this version is strong enough to prove classic consequences of Tverberg's theorem, such as the existence of weak epsilon-nets for convex sets [2]. The method also works for other convexity spaces, such as the integer lattice, where other proof methods for Tverberg's theorem fail [10].

Among the many proofs from Tverberg's theorem, a linear-algebraic technique by Sarkaria stands out by its simplicity. The version we present is a sketch of a further simplification by Bárány and Onn [6].

Proof of Tverberg's theorem. Let $n = (r-1)(d+1)$ and $u_1, u_2, \dots, u_r$ be the vertices of a regular simplex in $\mathbb{R}^{r-1}$ centered at the origin. For any $a \in \mathbb{R}^d$, consider the points of the form $(a,1) \otimes u_i \in \mathbb{R}^{(r-1)(d+1)} = \mathbb{R}^n$ for $i = 1, 2, \dots, r$, where $\otimes$ is the tensor product.

Then, a set S of $n+1$ points in $\mathbb{R}^d$ yields $n+1$ sets in $\mathbb{R}^n$, each capturing the origin. The set found by applying the colorful Carathéodory theorem in this family induces a partition of S into r sets, which turns out to be a Tverberg partition. $\square$

This is a very malleable proof. Indeed, many variations of the colorful Carathéodory theorem translate to a variation of Tverberg's theorem. For example, if we are given a set S of N points in $\mathbb{R}^d$, then for each $a \in S$, and any closed half-space $H \subset \mathbb{R}^{(r-1)(d+1)} = \mathbb{R}^n$ containing the origin, we can pick randomly one of the points $(a,1) \otimes u_i \in \mathbb{R}^n$ for $i = 1, 2, \dots, r$. By doing this we have, in expectation, at least $\frac{N}{r}$ of those points in H. A more careful analysis shows that there should be a choice of this type that *every hyperplane H that contains the origin has at least $\frac{N}{r} - \sqrt{\frac{nN\ln(Nr)}{2}}$ points of the selection.* If we apply Sarkaria's technique with this claim instead of colorful Carathéodory, we obtain the following.

Theorem 4 (Tverberg with tolerance [16])**.** *Let r, t, d be positive integers. Then, there is an $N = rt + \tilde{O}(r^2\sqrt{td} + r^3 d)$ such that for any N points in $\mathbb{R}^d$, there is a partition of them into r parts such that even if we remove any t points, the convex hulls of what's left in each set still intersect. Here the $\tilde{O}$ notation only hides polylogarithmic factors in terms of t, d, r.*

4. The colorful Tverberg theorem

Just as it is the case with the theorems by Carathéodory and Helly, there are "colorful" versions of Tverberg's theorem. In each case, there are color classes introduced; and when the color classes are equal, they yield the original theorem. For Tverberg's theorem, such a version is given in [1] (where the proof, again, uses Sarkaria's transformation). However, the most well-known colorful version for Tverberg's theorem does not follow this recipe, and remains open.

Conjecture 5 (Colorful Tverberg theorem, Bárány, Larman 1992 [5])**.** *Let r, d be positive integers. Given $d+1$ sets $F_1, \dots, F_{d+1}$, each of r points of $\mathbb{R}^d$, there is a partition of them into r sets, each with exactly one point of each F_i, whose convex hulls intersect.*

This conjecture has been proven for $d = 2$ and any r [5], and when $r+1$ is a prime number for any d [8, 9]. Recently, a new approach to variations of Tverberg's theorem was devised in [7]. The idea is to apply Tverberg's theorem or its topological version in a higher dimension. The excess dimensions can be used to impose properties in the partition in the original space. This implies the colorful Tverberg theorem if each F_i has $2r-1$ elements instead of r. The case $r = 2$ of the colorful Tverberg theorem was originally proven by Lovász using the Borsuk-Ulam theorem, with the proof appearing in [5]. However, we can also prove this result with linear-algebraic arguments as in previous sections.

Proof of colored Tverberg for $r = 2$. Denote by x_i and y_i the elements of F_i. Then, consider the $d+1$ vectors $x_i - y_i$. There must be a non-trivial linear dependence $\sum_{i=1}^{d+1} \alpha_i(x_i - y_i) = 0$. If any $\alpha_i < 0$, we can swap the names of x_i and y_i and the sign of α_i. Once all $\alpha_i \geq 0$, as they are not all are zero, we assume without loss of generality that their sum is 1. Thus, we have $\sum_{i=1}^{d+1} \alpha_i x_i = \sum_{i=1}^{d+1} \alpha_i y_i$, which gives the desired partition and the intersection of the convex hulls. □

This argument can be extended using Sarkaria's technique, but now we modify the tensoring trick instead of the colorful Carathéodory theorem. Consider $[r] = \{1, 2, \ldots, r\}$, and $u_1, \ldots, u_r$ as before. Given a set $F = \{z_1, \ldots, z_r\}$, and a permutation $\sigma : [r] \to [r]$, we construct $F(\sigma) = \sum_{i=1}^{r} z_i \otimes u_{\sigma(i)} \in \mathbb{R}^{d(r-1)}$. This new transformation implies a version of Conjecture 5 where we are given $(r-1)d+1$ colored sets of r points each, instead of just $d+1$, but with the added benefit that the convex hulls of the colorful sets intersect using the same coefficients [15].

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On the number of non-intersecting hexagons in 3-space

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Abstract

Two hexagons in the space are said to intersect *badly* if the intersection of their convex hulls consists of at least one common vertex as well as an interior point. We are going to show that the number of hexagons on n points in 3-space without bad intersections is $o(n^2)$, under the assumption that the hexagons are "fat".

Dedicated to Imre Bárány on the occasion of his 70th birthday

1. Introduction

The general problem of finding the maximum number of hyperedges in a geometric hypergraph in d-dimensional space with certain forbidden configurations (intersections) was considered by Dey and Pach in [1]. In this note we are interested in finding the maximum number of (convex planar) polygons on some vertex set of n points in 3-space, where no two of them are allowed to intersect in certain ways.

It was asked by Gil Kalai and independently by Günter Ziegler what is the maximum number of triangles spanned by n points such that any two are almost disjoint:

Definition 1 (Almost disjoint polygons)**.** *Two planar polygons in 3-space are said to be* almost disjoint *if they are either disjoint or their intersection consists of one common vertex.*

The maximum number of pairwise almost disjoint triangles on n points is bounded above by $O(n^2)$. Indeed, in a set of such triangles, any given point can be a vertex of at most $(n-1)/2$ triangles. It is not known whether the maximum number of such triangles is $o(n^2)$ or not. Károlyi and Solymosi constructed configurations with $\Omega(n^{3/2})$ almost disjoint triangles on n points [2]. Finding sharper bounds seems like a very hard problem. In fact, it is not even known if the genus of a polytope on n-vertices can have order n^2. If so, there would be a configuration of order n^2 almost disjoint triangles on n points. The best lower bound of the largest genus is $n \log n$, due to a construction of McMullen, Schulz and Wills [3]. For more details, we refer the interested readers to [5] where Ziegler gives a simplified construction providing the same bound. In this note, we study the maximum number of polygons without bad intersections, defined below.

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Definition 2 (Badly intersecting polygons)**.** *Two planar polygons in* 3*-space are said to* intersect badly *if the intersection of their convex hulls consists of at least one common vertex as well as an interior point.*

In what follows by polygons in space we mean planar polygons, i.e. the vertices are co-planar. We say that a collection of polygons has no bad intersections if no two of these polygons intersect badly. In such arrangements if two hexagons share a vertex then this is the only common point they have, but hexagons not sharing a vertex might intersect. In particular, they cannot share a diagonal and so the maximum number of k-gons ($k \geq 4$) without bad intersection is, again, $O(n^2)$. This trivial upper bound is actually sharp for quadrilaterals ($k = 4$). One can give a construction of $\Omega(n^2)$ quadrilaterals without a bad intersection as follows: Suppose we are given $n/2$ points $P_1, \ldots, P_{n/2}$ in general position (no three points collinear) on a plane π. Fix any vector v not parallel to π. Then the n points $P_1, \ldots, P_{n/2}, P_1 + v, \ldots, P_{n/2} + v$ are incident to $\Theta(n^2)$ desired quadrilaterals with vertices $P_i, P_j, P_j + v, P_i + v$, where $1 \leq i < j \leq n$.

When $k = 6$, we can show that the number of hexagons without bad intersections in 3-space is $o(n^2)$, under an extra assumption on the 'fatness' of the hexagons defined below. We conjecture that Theorem 1 holds for any set of hexagons or even for pentagons.

Definition 3 (Fat hexagons)**.** *Let $c \geq 1$ and $0 < \alpha < \pi/2$. A hexagon is* (c, α)-fat *if*

1. *it is convex;*
2. *the ratio of any two sides is bounded between $1/c$ and c; and*
3. *it has three non-neighbour vertices having interior angles between α and $\pi - \alpha$.*

Our main tool is the Triangle Removal Lemma of Ruzsa and Szemerédi, which states that for any $\varepsilon > 0$, there exists $\delta > 0$ such that any graph on n vertices with at least εn^2 pairwise edge-disjoint triangles has at least δn^3 triangles in total. See [4] for the original formulation of this result. The precise statement of our theorem is as follows.

Theorem 1. *For any $c \geq 1$ and $0 < \alpha < \pi/2$ numbers there is a function $F_{(c,\alpha)}(n)$,*

$$\frac{F_{(c,\alpha)}(n)}{n^2} \to 0 \text{ as } n \to \infty,$$

such that any family of (c, α)-fat hexagons in 3*-space on n points without bad intersections has size at most $F_{(c,\alpha)}(n)$.*

Note that, since almost disjoint hexagons don't intersect badly, the same upper bound, $o(n^2)$, holds for pairwise almost disjoint hexagons.

2. Proof of Theorem 1

Suppose there are εn^2 (c, α)-fat hexagons on n vertices in 3-space. We will show that two of these hexagons intersect badly.

To reduce the dimension of the ambient space, we project these hexagons onto a random plane such that a positive fraction is (c', α')-fat. Indeed, if we project a (c, α)-fat hexagon H to a plane making an angle at most $\theta < \pi/2$ with the plane containing H, some simple calculations show that the projected hexagon is (c', α')-fat, where

$$c' = \frac{c}{\cos\theta} \qquad \text{and} \qquad \alpha' = \cos^{-1}\left(\frac{\cos\alpha + \sin^2\theta}{\cos^2\theta}\right).$$

The existence of badly intersecting hexagons relies on a similar-slope property. This can be described quantitatively by the difference of two angles of inclination. To this end, let $\phi > 0$ be the smallness of such difference which is to be determined later.

We choose from the (c', α')-fat projected hexagons the most popular family consisting of $\varepsilon' n^2$ hexagons, which have inscribed triangles of similar shapes and orientations.

More precisely, let us enumerate by any order the projected hexagons as $\{H_i\}$ and label their vertices as $A_i, B_i, C_i, D_i, E_i, F_i$, oriented counter-clockwise, where B_i, D_i, F_i are the three non-neighbour vertices having angles between α' and $\pi - \alpha'$.

There exists a positive fraction of these hexagons so that for any i, j, the inclined angles of the diagonals A_iC_i and A_jC_j differ by at most ϕ. Similarly the same property holds true for the diagonals C_iE_i and E_iA_i in yet a sub-collection of $\varepsilon' n^2$ hexagons.

We define G to be the graph whose vertices are the n projected points and whose edges are from the triangles formed by the vertices A_i, C_i, E_i chosen above. Then, G contains $\varepsilon' n^2$ edge-disjoint triangles. An application of the Triangle Removal Lemma yields a triangle T whose edges come from three different hexagons, say H_1, H_2 and H_3. For each $i = 1, 2, 3$, let T_i be the triangle $A_iC_iE_i$.

We are ready to study the intersection properties of these three hexagons in the 3-space. In other words, we now 'unproject' the n points.

Two of the triangles, say T_1 and T_2, lie on the same side of T and let T_1 be the triangle making a larger angle with T. Then, as shown in Figure 1, the hexagon H_2 intersects badly with the triangle T_1, and hence with the hexagon H_1, as long as the three non-neighbour vertices B_1, D_1, F_1 lie outside of the triangle T on the plane of projection, which is guaranteed if we choose

$$\phi < \tan^{-1}\left(\frac{\sin\alpha'}{c' + \cos\alpha'}\right),$$

the right hand side being a lower bound of the six angles $B_1A_1C_1$ etc. under the (c', α')-fatness assumption. This completes the proof.

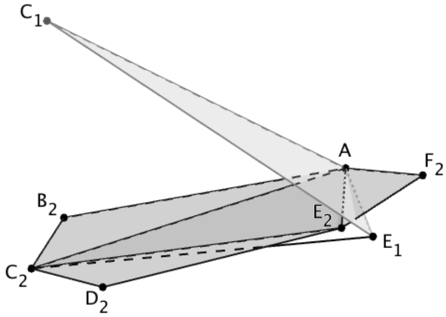

Figure 1: The triangle $T_1 = AC_1E_1$ and the hexagon $H_2 = AB_2C_2D_2E_2F_2$ intersect badly. Here the triangle T is AC_2E_1.

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A vector-sum theorem and the Fermat–Torricelli problem in normed planes

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Abstract

We show the following result on vector sums in strictly convex normed planes. Let $x_1, \ldots, x_n$ be $n \geq 3$ vectors in a strictly convex normed plane such that $\|x_i\| \geq 1$ and $\|\sum_{j \neq i} x_j\| \leq 1$ for all $i = 1, \ldots, n$. Then necessarily $\sum_{i=1}^{n} x_i = o$. As a consequence, we deduce a result on Fermat–Torricelli points in smooth two-dimensional normed spaces.

Dedicated to Imre Bárány on the occasion of his 70th birthday

1. Introduction

Imre Bárány has published many interesting and beautiful papers on subset sums, signed sums and rearrangements of vector sequences in normed spaces [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this note we prove a simple vector-sum result for strictly convex normed planes (Theorem 2) that has an application to Fermat–Torricelli points in smooth normed planes (Corollary 7). We obtain two further corollaries on the norms of sums of unit vectors in any normed plane (Corollaries 4 and 5).

For another result on two-dimensional vector sums, with a closely related result on Fermat–Torricelli points, see [16].

2. Vector sums

The following observation is a simple exercise in expanding inner products (see [12, Lemma 5] for a generalization).

Proposition 1. *If $x_1, \ldots, x_n$ $(n \geq 3)$ are vectors in Euclidean space such that $\|x_i\|_2 \geq 1$ and $\|\sum_{j \neq i} x_j\|_2 \leq 1$ for all $i = 1, \ldots, n$, then $\sum_{i=1}^{n} x_i = o$.*

We show that this statement also holds for strictly convex normed planes.

Theorem 2. *Let $x_1, \ldots, x_n$ $(n \geq 3)$ be vectors in a strictly convex normed plane such that $\|x_i\| \geq 1$ and $\|\sum_{j \neq i} x_j\| \leq 1$ for all $i = 1, \ldots, n$. Then $\sum_{i=1}^{n} x_i = o$.*

The proof needs the following technical result.

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Lemma 3. *Let x_1 and x_2 be linearly independent vectors in a strictly convex normed plane with $\|x_1\|, \|x_2\| \geq 1$. Let v be given such that $\|x_1 - v\|, \|x_2 - v\| \leq 1$. Then v is in the closed strip bounded by the line through $2x_1$ and $2x_2$ and its parallel through o.*

Proof. Let ℓ_i be a line through the origin parallel to a supporting line of the unit ball B at $\frac{1}{\|x_i\|}x_i$, $i = 1, 2$. Since x_1 and x_2 are linearly independent and the norm is strictly convex, ℓ_1 and ℓ_2 are not parallel. Then the unit ball $B + x_i$ is contained in the half plane H_i bounded by ℓ_i that contains x_i, $i = 1, 2$. It follows that v is in the cone $H_1 \cap H_2$, which in turn is contained in the half plane H bounded by the line through o parallel to x_1x_2 and containing x_1 and x_2.

Since $v' := x_1 + x_2 - v$ also satisfies $\|v' - x_i\| \leq 1$ $(i = 1, 2)$, by what was proved above, v' also lies in H. It follows that $v \in x_1 + x_2 - H$, which is the half plane bounded by the line through x_1 and x_2 that contains o. □

Proof of Theorem 2. Let $v := \sum_{i=1}^{n} x_i$, and suppose that $v \neq o$. Let ℓ_1 and $-\ell_1$ be the two lines parallel to ov that support the unit ball B. By strict convexity, ℓ_1 touches B in a unique point p. Since $\|x_i - v\| \leq 1$ for all $i = 1, \dots, n$, the x_i are all in the open half-plane bounded by op containing v. It follows that $o \notin \operatorname{conv}\{x_i\}$. Thus, without loss of generality, $x_1, \dots, x_n$ are in the cone generated by x_1 and x_2 and by Lemma 3, v is in the closed strip bounded by the line ℓ_2 through $2x_1$ and $2x_2$ and its parallel ℓ_3 through o. However, since $x_3, \dots, x_n$ all lie on the same side of ℓ_3 as x_1 and x_2, $\sum_{i=1}^{n} x_i = v$ is in the open half-plane bounded by ℓ_2 opposite o. This is a contradiction, and we conclude that $v = o$. □

We obtain the following corollaries that are valid for all normed planes.

Corollary 4. *Let $x_1, \dots, x_n$ $(n \geq 3)$ be vectors in a normed plane such that $\|x_i\| \geq 1$ for all $i = 1, \dots, n$. Then there exists an i such that $\|\sum_{j \neq i} x_j\| \geq 1$.*

Proof. Suppose that $\|\sum_{j \neq i} x_j\| < 1$ for all i. We may perturb the x_i's and the norm so that it becomes strictly convex, $\|x_i\|$ remains ≥ 1 and $\|\sum_{j \neq i} x_j\|$ remains < 1 for all i, and $\sum_i x_i \neq o$. This contradicts Theorem 2. □

The case $n = 3$ of Corollary 4 was shown by Katona, Mayer and Woyczynski [13].

Corollary 5. *Let $x_1, \dots, x_n$ $(n \geq 3)$ be vectors in a normed plane such that $\|x_i\| \geq 1$ for all $i = 1, \dots, n$. Then there exists a permutation π of $\{1, \dots, n\}$ such that $\|\sum_{i=1}^{k} x_{\pi(i)}\| \geq 1$ for all $k = 2, \dots, n - 1$.*

3. Fermat–Torricelli points

Let points $p_1, \dots, p_n$ be given in a normed space X. A point $p \in X$ is called a *Fermat–Torricelli point* of $\{p_i\}$ if p minimizes the function $x \mapsto \sum_{i=1}^{n} \|x - p_i\|$ on X. Answering a question of Fermat, Torricelli showed that if p minimizes the sum of the distances to the vertices of a triangle $\triangle p_1p_2p_3$ in the Euclidean plane, then the rays pp_i form three $120°$ angles if all angles of the triangle are $< 120°$. On the other hand, if $\triangle p_1p_2p_3$ has an angle that is at least $120°$, say at p_1, then p_1 minimizes the sum of the distances to p_1, p_2 and p_3. See [14] for a discussion of the history of this problem.

The following characterization of Fermat–Torricelli points in finite-dimensional normed spaces is well known [14, 11, 15]. We denote the dual of a normed space X by X^*. A *norming functional* of $x \in X$, $x \neq o$, is an element $\varphi \in X^*$ such that $\|\varphi\|^* = 1$ and $\varphi(x) = \|x\|$. By the separation theorem, any non-zero x has a norming functional, which is unique if X is smooth. Note that a finite-dimensional X is smooth iff its dual X^* is strictly convex.

Lemma 6. *Let $p, p_1, \dots, p_n$ be distinct points in a finite-dimensional normed space X.*

1. *The point p is a Fermat–Torricelli point of $p_1, \dots, p_n$ iff $p_i - p$ has a norming functional $\varphi_i \in X^*$ $(1 \leq i \leq n)$ such that $\sum_{i=1}^n \varphi_i = o$.*

2. *The point p is a Fermat–Torricelli point of $p, p_1, \dots, p_n$ iff $p_i - p$ has a norming functional $\varphi_i \in X^*$ $(1 \leq i \leq n)$ such that $\|\sum_{i=1}^n \varphi_i\| \leq 1$.*

Note the trivial fact that if p is a Fermat–Torricelli point of $\{p_1, \dots, p_n\}$, then p is also a Fermat–Torricelli point of $\{p, p_1, \dots, p_n\}$ (and is unique). It follows from Lemma 6 that if p is a Fermat–Torricelli point of $\{p_1, \dots, p_n\}$ with $p \neq p_i$ for all i, then p is a Fermat–Torricelli point of $\{p, p_1, \dots, p_n\} \setminus \{p_i\}$ for any i. The following corollary of Theorem 2 gives a converse, but only for smooth normed planes.

Corollary 7. *Let $p, p_1, \dots, p_n$ $(n \geq 3)$ be distinct points in a smooth normed plane. If p is a Fermat–Torricelli point of $\{p\} \cup \{p_j : j \neq i\}$ for each $i = 1, \dots, n$, then p is a Fermat–Torricelli point of $\{p_1, \dots, p_n\}$.*

Proof. This follows from Theorem 2 and Lemma 6. □

We end this section by observing that the above corollary holds in Euclidean space of any dimension. This is a consequence of Proposition 1 and Lemma 6.

Corollary 8. *Let $p, p_1, \dots, p_n$ $(n \geq 3)$ be distinct points in a finite-dimensional Euclidean space. If p is a Fermat–Torricelli point of $\{p\} \cup \{p_j : j \neq i\}$ for each $i = 1, \dots, n$, then p is a Fermat–Torricelli point of $\{p_1, \dots, p_n\}$.*

4. The hypotheses of two-dimensionality, strict convexity and smoothness

Theorem 2 becomes false for any norm that is not strictly convex and any even $n \geq 4$, and also for any odd $n \geq 3$ if there is a segment on the unit circle of length $> 2/(n-1)$.

Indeed, let $\operatorname{conv}\{a,b\}$ be a segment on the unit circle of length $\|a-b\| =: \lambda$. Let $e = \lambda^{-1}(b-a)$ and $m = \frac{1}{2}(a+b)$.

If $n \geq 4$ is even, let $k = n/2$ and $\varepsilon = \frac{\lambda}{2(n-1)}$. For the $n = 2k$ vectors, take $x_1 = \dots = x_k = m + \varepsilon e$ and $x_{k+1} = \dots = x_n = -m + \varepsilon e$. Then $\sum_{i=1}^n = n\varepsilon e \neq o$. For each $i = 1,\dots,k$, $\sum_{j\neq i} x_j = -m + (n-1)\varepsilon e$, which is on the segment $\operatorname{conv}\{-a,-b\}$, since $(n-1)\varepsilon = \lambda/2$. Similarly, for each $i = k+1,\dots,2k$, $\sum_{j\neq i} x_j = m+(n-1)\varepsilon e \in \operatorname{conv}\{a,b\}$, and therefore, $\|\sum_{j\neq i} x_j\| = 1$.

If $n \geq 3$ is odd and $\lambda > \frac{2}{n-1}$, let $k = (n-1)/2$ and $\varepsilon \in (\frac{1-\lambda/2}{2k-1}, \frac{1}{2k})$, and let the $n = 2k+1$ vectors be $x_1 = \dots = x_k = m+\varepsilon e$, $x_{k+1} = \dots = x_{2k} = -m+\varepsilon e$, $x_n = -e$. It can again be checked that for all $i = 1,\dots,n$, $\|\sum_{j\neq i} x_j\| = 1$, but $\sum_{i=1}^n x_i = (2k\varepsilon - 1)e \neq o$.

For each dimension $d \geq 3$, there exists a strictly convex norm for which Theorem 2 and Corollary 4 become false. In fact, it is easy to construct $m \geq 5$ unit vectors $x_1,\dots,x_m$ in ℓ_∞^d where $d \geq 3$, such that $\|\sum_{j\neq i} x_i\|_\infty < 1$ for all $i = 1,\dots,m$. (This was shown by Katona, Mayer and Woyczynski [13] for the case $m = d+1$.) We indicate the construction in dimension 3. Consider a $3\times m$ matrix. In each row we set $k := \lfloor (m-1)/2 \rfloor$ entries equal to 1 and $m-k$ entries equal to $-1+\varepsilon$, where $\varepsilon := 4/(m+1)$ if m is even, and $\varepsilon := 2/m$ if m is odd. We place these entries in an arbitrary order in each of the 3 rows, with the only requirement that there is at least one 1 in each column (which is satisfiable since $m \geq 5$). Then the m columns are unit vectors $x_1,\dots,x_m \in \ell_\infty^3$ such that $\|\sum_{j\neq i} x_i\|_\infty < 1$ for each $i = 1,\dots,m$. By perturbing ℓ_∞^3, we obtain a strictly convex norm with the same property.

The smoothness assumption cannot be dropped in Corollary 7. For instance, consider the normed plane with a regular hexagon as unit ball. Let p be the origin o, and let p_1, p_2, p_3 be three consecutive vertices of the unit ball. Then it follows easily from Lemma 6 that p is a Fermat–Torricelli point of $\{p, p_i, p_j\}$ for all distinct i, j. Nevertheless, p is clearly not a Fermat–Torricelli point of $\{p_1, p_2, p_3\}$.

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Holes in planar point sets

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Let X be a finite set of points in the plane. We say that X is *in general position* if no three points of X lie on a line. We say that X is *in convex position*, if each point of X is a vertex of the convex hull of X.

A classical result both in discrete geometry and in Ramsey theory is the Erdős–Szekeres (Happy Ending) theorem.

Theorem 1 (Erdős–Szekeres Theorem [4]). *For every $k \geq 3$ there is a (smallest) integer* $\mathrm{ES}(k)$ *such that any set of at least* $\mathrm{ES}(k)$ *points in general position in the plane contains k points in convex position.*

Let P be a finite set of points in general position in the plane. A convex k-gon G is called a *k-hole (*or *empty convex k-gon) of P*, if all vertices of G lie in P and no point of P lies inside G.

Erdős [3] asked if, for a fixed k, any sufficiently large point set in general position contains a k-hole. Harborth [6] proved that any set of 10 points in general position in the plane contains a 5-hole and gave a construction of 9 points with no 5-hole.

Horton constructed, for any n, a set of n points in general position in the plane with no 7-hole. The remaining case $k = 6$ became a well-known open problem. Gerken [5] and Nicolás[8] independently solved this problem by showing that any sufficiently large planar point set in general position in the plane contains a 6-hole.

Theorem 2 (The Empty Hexagon Theorem [5, 8]). *There is an integer n such that any set of at least n points in general position in the plane has a 6-hole.*

Here we give a sample of open problems on holes in planar point sets. Throughout this extended abstract, all point sets are assumed to be finite and in general position.

Empty pentagons

Although Harborth's paper has only three pages, the proof that any set of 10 points in general position in the plane contains a 5-hole is based on a relatively complicated case analysis. It would be very interesting to find a simple proof of this fact. We remark that it is very easy to prove that a sufficiently large point set contains a 5-hole. Here is a hint. The Erdős–Szekeres Theorem says that a sufficiently large point set P contains 6 points in general position. Let X be a set of 6 points of P in

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convex position such that the convex hull of X has minimal area, say. Then it is a little exercise to show that for some $l \in \{3,4,5\}$, there are l points of X consecutive in the clockwise order of X which forms a 5-hole together with some $5-l$ points of P inside the convex hull of X.

Problem 1. *Give a simple proof that any set of* 10 *points in general position in the plane contains a* 5*-hole.*

Minimum number of empty polygons

Let $h_k(n)$ be the minimum number k-holes in a set of n points in general position in the plane. Horton [7] proved that $h_k(n) = 0$ for any $k \geq 7$ and any positive integer n.

It is known that $h_3(n)$ and $h_4(n)$ are of order $\Theta(n^2)$. For $h_5(n)$, a first superlinear lower bound was proved very recently in a paper of Aichholzer et al. [1] by a complicated proof which relies on several lemmas with computer assisted proofs. The obtained bound is $h_5(n) = \Omega(n \log^{4/5}(n))$ but it seems very likely that $h_5(n)$ grows much faster. The best known upper bound on $h_5(n)$ is quadratic in n.

Problem 2. *Show that* $h_5(n) = \Omega(n^{1+\varepsilon})$, *for some constant* $\varepsilon > 0$.

For the minimum number of empty hexagons, no superlinear lower bound is known.

Problem 3. *Show that* $\liminf_{n\to\infty} h_6(n)/n = \infty$.

For the minimum number of empty triangles, $\liminf_{n\to\infty} h_3(n)/n \geq 1$ is known. It is a challenging open problem to show a strict inequality.

Problem 4. *Show that* $\liminf_{n\to\infty} h_3(n)/n > 1$.

A closely related open problem of Imre Bárány asks to prove the existence of a pair of points appearing at the same time in many empty triangles.

Problem 5 (Imre Bárány)**.** *Show that for every k there is an n such that any set P of n points in general position in the plane contains a pair of points a, b such that there are (at least) k distinct* 3*-holes abx with $x \in P \setminus \{a,b\}$.*

Minimum number of empty polygons: Upper bounds

The following upper bounds on $h_k(n), k = 3,4,5,6$, are proved in a paper of Imre Bárány and myself [2].

$$\begin{aligned} h_3(n) &\leq 1.6195...n^2 + o(n^2), \\ h_4(n) &\leq 1.9396...n^2 + o(n^2), \\ h_5(n) &\leq 1.0206...n^2 + o(n^2), \\ h_6(n) &\leq 0.2005...n^2 + o(n^2), \end{aligned}$$

All the four upper bounds are obtained using the so-called squared Horton set. Although these bounds are far from the best-known lower bounds (especially for $k = 5, 6$), I conjecture that they are very close to the actual values of $h_k(n), k = 3, 4, 5, 6$, as their structure seems to be suitable for the purpose of minimizing the number of holes.

Three dimensions

Holes can be defined also for point sets in higher dimensions. In R^3, a set is in general position if it contains no three collinear points and no four coplanar points. A subset H of a finite point set P in general position is a *hole*, if it is a subset in convex position such that the interior of its convex hull contains no point of P. It is not very difficult to show that a sufficiently large point set contains a hole of size 7; see the paper [9]. I strongly believe that 7 can be significantly improved but even the possibility to improve it to 8 is an open problem.

Problem 6. *Show that there is an integer n such that every set of at least n points in general position in R^3 contains an 8-hole.*

An upper bound of 22 was given in the paper [9]. It is obtained by a 3-dimensional analogue of the so-called Horton sets in the plane.

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On a Helly-type question for central symmetry

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1. Introduction

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The second author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

If $\mathcal{F}$ contains the family of all line-segments, then the respective $\mathcal{F}$-convexity is a generalization of the usual convexity. This is so for the mentioned arc-wise or polygonal connectedness.

If every member of $\mathcal{F}$ is convex, then the respective $\mathcal{F}$-convexity is a particularization of the usual convexity. As examples, we mention the rectangular convexity studied by Blind, Valette and the second author [1], and by Böröczky Jr [2], and also the right convexity investigated by the second author [8].

For other families $\mathcal{F}$, the $\mathcal{F}$-convexity is neither implied, nor implies the usual convexity. This is the case, for example, if $\mathcal{F}$ consists of special types of finite sets. We survey here some recent results obtained in this direction.

As usual, for $M \subset \mathbb{R}^d$, $\mathrm{bd}M$ denotes its boundary, $\mathrm{int}M$ its interior, $\mathrm{diam}M = \sup_{x,y\in M} \|x-y\|$ its diameter, and $\mathrm{conv}\, M$ its convex hull.

For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x and y, xy the line-segment from x to y, H_{xy} the hyperplane through x orthogonal to $\overline{xy}$, and C_{xy} the hypersphere of diameter xy.

For $S_1,\ S_2 \subset \mathbb{R}^d$, let $d(S_1, S_2) = \inf\{d(x,y) \mid x \in S_1,\ y \in S_2\}$ denote the *distance* between S_1 and S_2.

The d-dimensional unit ball (centred at $\mathbf{0}$) is denoted by B_d $(d \geq 2)$.

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2. Isosceles Triple Convexity

Three points $x,\ y,\ z \in \mathbb{R}^d$ (always $d \geq 2$) form an *isosceles triple* $\{x,\ y,\ z\}$ if one of them is equidistant from the others.

Let $S \subset \mathbb{R}^d$. A pair of points $x,\ y \in S$ is said to enjoy the *it-property in* S if there exists a third point $z \in S$, such that $\{x,\ y,\ z\}$ is an isosceles triple. The set S is called *isosceles triple convex*, for short *it-convex*, if every pair of its points enjoys the *it*-property in S.

We repost in this section results from [7].

2.1. Nondiscrete *it*-convexity

Theorem 1. *All sets in* $\mathbb{R}^d$ *which cannot be strictly separated by any hyperplane are it-convex.*

Corollary 2. *All connected sets in* $\mathbb{R}^d$ *are it-convex.*

Theorem 3. *Let* S *be a set with at least* 2 *components. If the union of any two components is it-convex, then* S *is it-convex.*

Let h denote the Pompeiu-Hausdorff distance.

Theorem 4. *If the compact set* K *has two connected components* A, B, *and* $h(A,B) < r(A)/2$, *then* K *is it-convex.*

The continua A, B are called *unseparable* if, for any hyperplane H disjoint from $A \cup B$, one of the two open half-spaces determined by H includes $A \cup B$.

Theorem 5. *Let* $A_1, A_2, ..., A_n$ *be continua, and* G *be a tree with vertex set* $V(G) = \{v_1, ..., v_n\}$. *Suppose for every edge* $(v_i, v_j) \in E(G)$, *the sets* A_i, A_j *are unseparable. Then* $\bigcup_{i=1}^n A_i$ *is rq-convex.*

2.2. *it*-convexity of Archimedean tilings

A *plane tiling* $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \cdots\}$ which cover the plane without gaps or overlaps. And every closed set $T_i \in \mathcal{T}$ is called a *tile of* $\mathcal{T}$. We consider a special case of tilings in which each tile is a polygon. If the corners and sides of a polygon coincide with the vertices and edges of the tiling, we call the tiling *edge-to-edge*. A so-called *type of vertex* describes its neighbourhood. If, for example, in some cyclic order around a vertex there are a triangle, then another triangle, then a square, next a third triangle, and last another square, then its type is $(3^2.4.3.4)$. We consider plane edge-to-edge tilings in which all tiles are regular polygons, and all vertices are of the same type. Thus, the vertex type will be defining our tiling.

There exist precisely eleven such tilings [3]. These are (3^6), $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), (4^4), $(4.6.12)$, (4.8^2), and (6^3). They are called *Archimedean tilings.*

We shall say that a tiling is *rq-convex* if its vertex set is *rq*-convex.

Theorem 6. *The Archimedean tilings* (3^6), (4^4), (6^3), $(3.6.3.6)$ $(3^2.4.3.4)$ *and* $(3^4.6)$ *are it-convex.*

Despite the encouraging Theorem 6, not all Archimedian tilings are *rq*-convex.

Theorem 7. *The Archimedian tilings* (4.8^2), $(3.4.6.4)$, $(3^3.4^2)$, $(4.6.12)$, (3.12^2) *are not it-convex.*

2.3. *it*-convexity of finite subsets of the square lattice

Let x, y be points of the square lattice, and P, Q be shortest paths from x to y in the graph defined by the lattice. These paths, considered as arcs in $\mathbb{R}^2$, form the boundary of an open set U, the unique unbounded component of $\mathbb{R}^2 \setminus (P \cup Q)$.

All lattice points in $\mathbb{R}^2 \setminus U$ form a set that we call *monotone.* It is clear from the definition that both P, Q and the whole monotone set determined by them lie in the rectangle (with horizontal and vertical sides) with diagonal xy, and contain x, y, which are called *endpoints* of the set.

Let $\mathcal{T}(m,n)$ be the family of all monotone sets with endpoints $(0,0)$ and(m,n).

Theorem 8. *There are precisely* 8 *pairwise non-congruent it-convex monotone sets in* $\bigcup_{m,n=0}^{\infty} \mathcal{T}(m,n)$*, namely one in* $\mathcal{T}(2,0)$*, two in* $\mathcal{T}(1,1)$*, three in* $\mathcal{T}(2,2)$*, one in* $\mathcal{T}(4,4)$*, and one in* $\mathcal{T}(5,5)$ (*see Figure* 1).

2.4. *it*-convexity of other finite sets

Are the vertex sets of all regular polygons *rq*-convex?

Theorem 9. *The vertex set of a regular n-polygon is it-convex if and only if* $n \not\equiv 2 \pmod 4$.

Also the vertex sets of the five Platonic polyhedra behave differently. While the vertex sets of the regular tetrahedron and regular octahedron are *rq*-convex, those of the cube, regular dodecahedron and regular icosahedron are not.

Let $S = \{x_1, x_2, \cdots, x_n\} \subset \mathbb{R}^d$. A matrix $A(S) = [a_{ijk}]_{n\times n\times n}$ is called the *it-trimatrix of* S in case $a_{ijk} = 1$ if and only if $\{x_i, x_j, x_k\}$ form an isosceles triple, otherwise $a_{ijk} = 0$. Particularly, $a_{ijk} = 0$, when at least two of i, j, k are equal.

Thus, $S = \{x_1, x_2, \cdots, x_n\}$ is *it*-convex if and only if for any distinct i, $j \in \{1, 2, \cdots, n\}$, there is a k such that, in $A(S)$, $a_{ijk} \neq 0$.

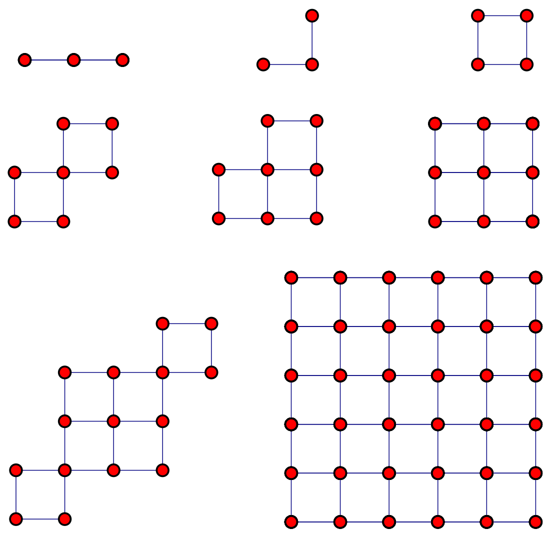

Figure 1: All the non-congruent *it*-convex monotone sets.

Let $A(S) = [a_{ijk}]_{n\times n\times n}$ be the *it*-trimatrix of $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$. The matrix $B(S) = [b_{ij}]_{n\times n}$ is called the *it-matrix of* S, if $b_{ij} = \sum_{k=1}^{n} a_{ijk}$.

Obviously, $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$ is *it*-convex if and only if its *it*-matrix is non-zero outside the main diagonal.

An n-point *it*-convex set is called *poor* if the number of isosceles triples in it is minimal among all n-point *it*-convex sets. The number of isosceles triples in a poor n-point *it*-convex set is denoted by $N(n)$.

Theorem 10. *$N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil$, when n is odd; $N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil + 1$, when n is even.*

Theorem 11. *$N(3) = 1$, $N(4) = 3$, $N(5) = 4$, $N(6) = 6$.*

3. Right quadruple convexity

In this section we present a discretization of rectangular convexity, the right quadruple convexity, which constitutes a generalization of rectangular convexity, see [4].

A set of four points $w, x, y, z \in \mathbb{R}^d$ forms a *rectangular quadruple* if $\operatorname{conv}\{w, x, y, z\}$ is a non-degenerate rectangle. Let $\mathcal{R}$ be the family of all rectangular quadruples. Here, we shall choose $\mathcal{F}$ to be this family $\mathcal{R}$.

Let $M \subset \mathbb{R}^d$. A pair of points $x, y \in M$ is said to enjoy the *rq-property in M* if there exists another pair of points $z,\ w \in M$, such that $\{w, x, y, z\}$ is a rectangular quadruple. The set M is called *rq-convex*, if every pair of its points enjoys the *rq*-property in M. This property is the *right quadruple convexity.*

Let $A \subset \mathbb{R}^d$. We call A^* an *rq-convex completion* of A, if A^* is *rq*-convex, $A^* \supset A$ and $\operatorname{card}(A^* \backslash A)$ is minimal (but possibly infinite). Let $\gamma(A) = \operatorname{card}(A^* \backslash A)$, which is called the *rq-convex completion number* of A, in case A is finite. For finite n, let $\gamma(n) = \sup\{\gamma(A) : \operatorname{card} A = n\}$.

3.1. Not simply connected *rq*-convex sets

Theorem 12. *If* $\operatorname{conv} M$ *is a disc and* $(\operatorname{conv} M) \smallsetminus M$ *lies in a circular disc of radius* r *at distance at least* $(\sqrt{3}-1)r$ *from* $\operatorname{bd} \operatorname{conv} M$, *then* M *is rq-convex.*

This theorem gives a useful sufficient condition for the *rq*-convexity of a set M which is not simply connected, regardless the shape of $(\operatorname{conv} M) \smallsetminus M$. Notice that it allows both M and its complement to have arbitrarily many components.

3.2. Unbounded *rq*-convex sets

An infinite family $\mathcal{K}$ of closed convex sets is said to be *uniformly bounded below* if, for some $\lambda > 0$, each of the sets contains a translate of the disc λB_2.

Theorem 13. *Let* $\mathcal{K}$ *be a family of pairwise disjoint closed convex sets in* $\mathbb{R}^d$. *If* $\mathcal{K}$ *is finite or uniformly bounded below, then the closure of the complement of* $\bigcup \mathcal{K}$ *is rq-convex.*

We can drop the convexity condition if the considered sets are bounded.

Theorem 14. *The complement of any bounded set in* $\mathbb{R}^d$ *is rq-convex.*

Theorem 15. *The Archimedean tilings* (4^4), (3^6), (6^3), $(3.6.3.6)$, $(3^4.6)$, $(3.3.4.3.4)$, $(4.8.8)$ *have rq-convex vertex sets.*

Theorem 16. *The vertex sets of the Archimedean tilings* $(3.3.3.4.4)$, $(3.4.6.4)$, $(4.6.12)$, $(3.12.12)$ *are not rq-convex.*

3.3. *rq*-convex skeleta of parallelotopes

As already remarked in [1], for $d \geq 3$, there is not even any conjectured characterization of rectangularly convex sets in $\mathbb{R}^d$. Among the sets mentioned in [1] as rectangularly convex we find the cylinder $K \times [0,1]$ with a $(d-1)$-dimensional compact convex set K as basis. In particular, any *right parallelotope*, i.e. the cartesian product of d pairwise orthogonal line-segments, is rectangularly convex and, a fortiori, rq-convex.

Theorem 17. *The* 1*-skeleton of any right parallelotope is rq-convex.*

Contrary to the case of an arbitrary cylinder, the following is true.

Theorem 18. *The boundary of any right parallelotope is rq-convex.*

Theorem 19. *Not every convex cylinder has an rq-convex boundary.*

3.4. rq-convexity of finite sets

Let $\mathcal{A}$ be the family of all finite point sets in $\mathbb{R}^2$.

Theorem 20. *For any set $A \in \mathcal{A}$ with* $\mathrm{card}A = n \geq 3$, *we have* $\gamma(A) \leq n^2 - 2n$.

Theorem 21. *There are precisely two kinds of* 6*-point rq-convex sets in* $\mathcal{A}$*, shown in Figure* 2.

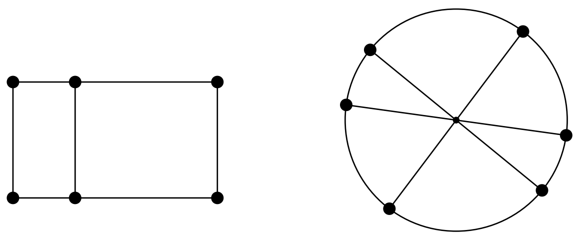

Figure 2: 6-point rq-convex sets.

Theorem 22. *There are precisely three kinds of* 8*-point rq-convex sets, shown in Figure* 3.

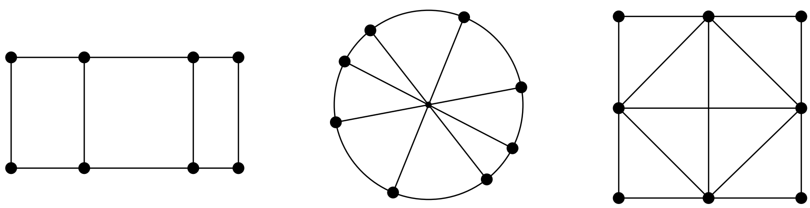

Figure 3: 8-point rq-convex sets.

Theorem 23. *The smallest odd cardinality of an rq-convex set in* $\mathbb{R}^2$ *is* 9.

3.5. rq-convexity of the vertex sets of Platonic solids

Due to their symmetry, the vertex sets of the cube, regular octahedron, regular dodecahedron, and regular icosahedron are all rq-convex. Among the Platonic solids, only the regular tetrahedron lacks this property. But what is the rq-convex completion number of the vertex set of the regular tetrahedron?

Theorem 24. *The rq-convex completion number of the vertex set of the regular tetrahedron is* 3.

Theorem 24 reveals the existence of 7-point *rq*-convex sets in $\mathbb{R}^3$, in contrast with the inexistence of such sets in $\mathbb{R}^2$. What happens in higher dimensions?

For space restrictions, we have to abandon an initially planned fourth section about right triple convexity, introduced and studied in [5], [6].

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Contributed articles

Algebraic vertices of non-convex polyhedra

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Based on joint work with Imre Bárány and Sinai Robins.

We study the vertices of **non-convex polyhedra**, which we also call **generalized polyhedra**, and which we define as the finite union of convex polyhedra in $\mathbb{R}^d$. There are many different ways to define a vertex of a generalized polyhedron P, most of them based on properties of the tangent cone to P at a point $\mathbf{v} \in P$. The tangent cone at $\mathbf{v}$, which we write as $\operatorname{tcone}(P, \mathbf{v})$, is intuitively the collection of all directions that we can 'see' if we stand at $\mathbf{v}$ and look into P. We furthermore define a **line-cone** to be a cone that is the union of parallel lines. Throughout, we denote the **indicator function** of any set $S \subset \mathbb{R}^d$ by $[S]$. In other words $[S](x) = 1$ if $x \in S$, and $[S](x) = 0$ if $x \notin S$.

Definition. *For a generalized polyhedron P, a point $\mathbf{v} \in P$ is called an* **algebraic vertex** *of P if the indicator function of its tangent cone* $\operatorname{tcone}(P, \mathbf{v})$ *cannot be represented (up to a set of measure zero) as a linear combination of indicator functions of line-cones.*

The theorem of D. Frettlöh and A. Glazyrin [3] states that the indicator function of a convex cone which is not a line-cone cannot be represented as a sum of indicator functions of line-cones, implying that the vertices of an ordinary convex polytope are indeed algebraic vertices.

Our main result is the following description of algebraic vertices, showing that in some sense these generalized vertices form a minimal set of points needed to describe a **generalized polytope**, which is by definition a bounded generalized polyhedron.

Theorem 1. *Let $\mathcal{V}_P$ be the set of algebraic vertices of a generalized polytope $P \subset \mathbb{R}^d$, and let $\mathcal{T}_P$ be the set of simplices whose vertices lie in $\mathcal{V}_P$. Then*

$$[P] = \sum_{T_i \in \mathcal{T}_P} \alpha_i [T_i], \tag{1}$$

where the α_i are integers and the equality holds throughout $\mathbb{R}^d$, except perhaps for a set of measure zero.

Moreover, if $[P]$ is represented (up to measure zero) as a linear combination of indicator functions of some finite number of simplices, then the set of vertices of these simplices must contain $\mathcal{V}_P$.

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Generalized (non-bounded) polyhedra also can be described through their algebraic vertices with the following theorem.

Theorem 2. *Let $\mathcal{V}_P$ be the set of algebraic vertices of a generalized polyhedron $P \subset \mathbb{R}^d$. Then*

$$[P] = \sum_{i=1}^{k} \alpha_i [D_i] + \sum_{\mathbf{v} \in \mathcal{V}_P} [\operatorname{tcone}(P, \mathbf{v})],$$

for some integers α_i and line-cones D_i, $i = 1, \dots, k$. The equality holds almost everywhere, except perhaps on a set of measure zero.

Moreover, if $[P]$ is represented (up to a set of measure zero) as a linear combination of indicator functions of line-cones and simplicial cones, then the set of apices of these simplicial cones should contain $\mathcal{V}_P$.

It is well known that the indicator function of a line-cone has vanishing Fourier–Laplace transform (see [1] or [2]). Therefore if $\mathbf{v}$ is not an algebraic vertex, then the Fourier–Laplace transform of the indicator function of its tangent cone also vanishes, because it is a finite linear combination of indicator functions of line-cones. We show that the opposite also holds. We formulate this fact in a more general form.

Theorem 3. *If P is a generalized polyhedron with zero Fourier–Laplace transform, then it does not have algebraic vertices.*

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Longest convex chains and subadditive ergodicity

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Let $T \subset \mathbb{R}^2$ be a triangle with vertices p_0, p_1, p_2 and let $X \subset T$ be a finite point set. A subset $Y \subset X$ is a *convex chain* in T (from p_0 to p_2) if the convex hull of $Y \cup \{p_0, p_2\}$ is a convex polygon with exactly $|Y| + 2$ vertices. The *length* of the convex chain Y is just $|Y|$. We are interested in the situation when $X = X_n$ is a random sample of n random, uniform, independent points from T. Let L_n be the length of a longest convex chain in X_n. The random variable L_n is a distant relative of the "longest increasing subsequence" problem, cf. [3].

In our paper with I. Bárány [1] we showed that the order of magnitude of L_n is $n^{1/3}$, moreover, its expectation does not fluctuate too much:

Theorem 1. *There exists a positive constant α for which*

$$\lim_{n\to\infty} \frac{\mathbb{E}L_n}{\sqrt[3]{n}} = \alpha\,.$$

We also proved the estimates $1.57 < \alpha < 3.43$, and (based on strong experimental evidence) we conjectured that $\alpha = 3$. This would match nicely the result concerning longest increasing chains [3], where the expectation converges to $2\sqrt{n}$.

We show that Theorem 1 may be proved effortlessly using subadditive ergodic theory. We will apply the celebrated result of Kingman [2]:

Theorem 2 (Subadditive ergodic theorem, Kingman)**.** *Assume $X_{n,m}$, $n, m \in \mathbb{N}$, is a family of random variables satisfying the following conditions:*

S1) $X_{l,n} \le X_{l,m} + X_{m,n}$ whenever $0 \le l < m < n$;

S2) The joint distributions of the process $\{X_{m+1,n+1}\}$ are the same as those of $\{X_{m,n}\}$;

S3) For each n, $\mathbb{E}|X_{0,n}| < \infty$ and $\mathbb{E}X_{0,n} > -cn$ for some constant c.

Then

$$\gamma = \lim_{n\to\infty} \frac{\mathbb{E}X_{0,n}}{n}$$

exists,

$$X = \lim_{n\to\infty} \frac{X_{0,n}}{n}$$

exists almost surely, and $\mathbb{E}X = \gamma$.

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In order to make use of this result, first we change our probabilistic model. Let $\mathcal{P}$ be a Poisson process on $\mathbb{R}^2$ of intensity 4. Denote by Γ the parabola arc $\{y = x^2, x \geq 0\}$ in $\mathbb{R}^2$. For $a \geq 0$, let $p_a = (a, a^2)$, and let ℓ_a be the tangent line to Γ at p_a, i. e., the equation of ℓ_a is $y = 2ax - a^2$. Furthermore, let $T_{a,b}$ denote the triangle determined by the lines ℓ_a, ℓ_b and $p_a p_b$, see Figure 1. The third vertex of $T_{a,b}$ is $((a+b/2), ab)$, and the area of $T_{a,b}$ is $(a-b)^3/4$. Hence, the expected number of points of $\mathcal{P}$ falling in $T_{a,b}$ is $(a-b)^3$. Let now denote by $X_{m,n}$ the longest convex chain from p_m to p_n in $T_{m,n}$ containing points of $\mathcal{P}$. Since a concatenation of two convex chains in $T_{l,m}$ and $T_{m,n}$ is a convex chain in $T_{l,n}$, condition S1) is satisfied. Translation invariance of $\mathcal{P}$ shows the validity of S2), and S3) holds trivially. Thus, we obtain that $\lim EX_{0,n}/n$ exists. An easy argument shows that $\mathbb{E}X_{0,n} = \mathbb{E}L_{n^3}$, thus Theorem 1 follows with $\alpha = \gamma$. $\square$

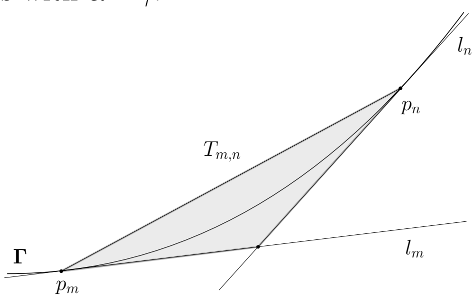

Figure 1: Supporting triangle of the parabola.

Note that Theorem 2 implies a stronger result than that of Theorem 1. However, it does not provide an estimate on α.

Standard concentration inequalities of Talagrand[4] show that the random variable $X_{0,n}$ is exponentially concentrated about its expectation. This probabilistic statement further implies a geometric concentration: the longest convex chains from p_0 to p_n are very close to Γ. Quantitative estimates are to be found in [1].

Instead of convex chains, one may consider higher order convexity, e.g. k-monotone interpolability. The same technique can be applied to determine the order of magnitude of the expectation of longest k-monotone chains within a triangle, which proves to be $n^{1/(k+1)}$. The determination of the exact constants in the asymptotics remain an open question.

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Random polytopes and the affine surface area

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1. Affine surface area

By a convex body K in $\mathbb{R}^d$, we mean a compact convex set with non-empty interior. The affine surface area of K has various definitions. One is based on the classical theorem of Aleksandrov stating that for almost every point $x \in \partial K$ with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$, there exists a paraboloid osculating ∂K. Then the (generalized) Gaussian curvature $\kappa(x)$ of ∂K at such an x is the Gaussian curvature of this paraboloid at x, and the function $\kappa(x)$ is known to be integrable with respect to $\mathcal{H}^{d-1}$. Hence going back to Blaschke [10] and [11], affine surface area can be defined as (see Schütt, Werner [23])

$$\Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mathcal{H}^{d-1}(x). \tag{1}$$

Another approach considers the floating body K_t assigned to K for $t > 0$. More precisely, K_t is the intersection of all half-spaces whose complements intersect K in a set of volume at most t, which is non-empty for $t \le \frac{1}{e} V(K)$. According to Schütt, Werner [23], we have

$$\Omega(K) = c_d \lim_{t\to 0^+} \frac{V(K) - V(K_t)}{t^{\frac{2}{d+1}}} \tag{2}$$

where $c_d = 2\left(\frac{\omega_{d-1}}{d+1}\right)^{\frac{2}{d+1}}$ and ω_m is the m-volume of the m-dimensional unit ball. The affine surface area satisfies the following properties where the first property (i) explains the name.

(i) $\Omega(K)$ is invariant under volume preserving affine transformations (see (2)); namely, $\Omega(\Phi(K)) = \Omega(K)$ if $\Phi x = Ax + b$ where $A \in \mathrm{GL}(n,\mathbb{R})$ with $\det A = \pm 1$, and $b \in \mathbb{R}^d$.

(ii) $\Omega(K)$ is a valuation; namely,

$$\Omega(K) + \Omega(C) = \Omega(K \cup C) + \Omega(K \cap C)$$

if $K \cup C$ is convex for convex bodies K, C (see (1)),

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(iii) $\Omega(K)$ is upper semicontinuous; namely, if the convex bodies K_n tend to K, then

$$\Omega(K) \geq \limsup_{n\to\infty} \Omega(K_n)$$

according to Dolzmann, Hug [12], Leichtweiß [15] and Lutwak [17],

(iv) $\Omega(K)$ is positive for convex bodies with C^2 boundary, and zero for polytopes (see (1)),

(v) $\Omega(\lambda K) = \lambda^{\frac{d(d-1)}{d+1}} \Omega(K)$ for $\lambda > 0$ (see (1)).

For lower dimensional compact convex sets, the affine surface area is defined to be zero. Ludwig, Reitzner [16] proved that the properties (i)-(iv) characterize affine surface area.

Theorem 1 (Ludwig, Reitzner). *If Z is an upper semicontinuous* $\mathrm{SL}(n,\mathbb{R})$ *and translation invariant real valued valuation on convex compact sets in* $\mathbb{R}^d$*, then there exist* $c_1, c_2 \in \mathbb{R}$ *and* $c_0 \geq 0$ *such that*

$$Z(K) = c_1 + c_2 V(K) + c_0 \Omega(K)$$

for any convex body K where $V(K)$ is the volume.

According to the Affine Isoperimetric Inequality due to Blaschke [10, 11], whose proof for three dimensional convex bodies with C^2 boundaries readily extends to general dimension and to general convex bodies, we have

Theorem 2 (Affine Isoperimetric Inequality (Blaschke)). *If K is a convex body in* $\mathbb{R}^d$*, then*

$$\Omega(K)^{d+1} \leq d^{d+1} \omega_d^2 V(K)^{d-1}$$

with equality if and only if K is an ellipsoid.

Here the equality case among general convex bodies was actually characterized by Petty [18].

2. Random polytopes in a convex body

Let K be a convex body in $\mathbb{R}^d$ with $V(K) = 1$. First, we consider the convex hull $P_n = P_{n,K}$ of n points chosen randomly from K according to the uniform distribution, and we are interested in the expected volume of $K \backslash P_n$ and the expectation of the number $f_i(P_n)$ of i-faces of P_n for large n (see Bárány [5] for an extensive survey). The problem was initiated by Rényi and Sulanke [19, ?], who described the asymptotics of $V(K \backslash P_n)$ and $f_0(P_n)$ as n tends to infinity if $d = 2$ and ∂K is C^3_+. For higher dimensions, the breakthrough came by Bárány [1] for convex bodies with C^3_+ boundary, and his result was extended to any convex body by C. Schütt [22].

Theorem 3 (Bárány, Schütt). *If K is a convex body in $\mathbb{R}^d$ with $V(K)=1$, then*

(i) $\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}\,V(K\backslash P_n)=c(d)\Omega(K)$

(ii) $\lim_{n\to\infty} n^{\frac{-(d-1)}{d+1}}\,\mathbb{E}f_0(P_n)=c(d)\Omega(K)$

where $c(d)$ depends only on d.

Let us fix a ball B of volume one, and a simplex T of volume one in $\mathbb{R}^d$. As the Affine Isoperimetric Inequality suggest, ellipsoids are worst approximable in the sense of volume; namely, for any n, $V(K\backslash P_{n,K})$ is maximized by ellipsoids according to Groemer [13]. On the other hand, Bárány, Buchta [6] proved that asymptotically, simplices are best approximable in a very strong sense.

Theorem 4 (Bárány, Buchta). *If K is a convex body and not a simplex in $\mathbb{R}^d$ with $V(K)=1$, then*

$$\liminf_{n\to\infty}\frac{\mathbb{E}\,V(K\backslash P_{n,K})}{\mathbb{E}\,V(T\backslash P_{n,T})}\geq 1+\frac{1}{d+1}$$

Concerning i-faces for $i\geq 1$, Bárány [2] determined the order of approximation.

Theorem 5 (Bárány). *There exist constants $c_1,c_2>0$ depending on d such that if K is a convex body in $\mathbb{R}^d$ with $V(K)=1$ and $n\geq n_K$, then*

$$c_1V(K\backslash K_{1/n})\leq \mathbb{E}f_i(P_n)\leq c_2V(K\backslash K_{1/n})\quad \textit{for } i=0,\dots,d-1.$$

In other words, $\mathbb{E}f_i(P_n)$ is approximately $\Omega(K)n^{\frac{(d-1)}{d+1}}$ for $i=0,\dots,d-1$. The role of the floating body is even more apparent in a result by Bárány and Vitale [9]. For a convex body K in $\mathbb{R}^d$ with $V(K)=1$, let $\mathbb{E}P_n$ be the expected random polytope defined in terms of the expectation of the support function.

Theorem 6 (Bárány, Vitale). *There exist constants $b>a>0$ depending only on d such that if K is convex body in $\mathbb{R}^2$ with $V(K)=1$ and $n>n_K$, then*

$$K_{b/n}\subset \mathbb{E}P_n\subset K_{a/n}.$$

In 1864, Sylvester [24] asked for the probability that 4 random points in a planar convex body is in convex position; namely, all the four are vertices of their convex hull. Here the probability measure and K was not specified. The generally accepted meaning of the question is the following. For $n\geq 4$ and a planar convex body K of area one, we search for the probability $p(n,K)$ that n random points according to the uniform probability measure on K is in convex position. If $n=4$, then the extremal values of $p(n,K)$ were determined by Blaschke [11]; namely,

$$p(4,T)\leq p(4,K)\leq p(4,B).$$

In general, $p(n,K)$ is not known, but Bárány [4] was able to describe the asymptotics of $p(n,K)$. In order to state his result, we note that given a convex body K in any $\mathbb{R}^d$, the upper semicontinuity of the affine surface area and the Blaschke Selection Theorem yield that the affine surface area of any convex bodies contained in K attains its maximum. In the planar case, Bárány [3] managed to prove that the maximum is attained at a unique convex body contained in K.

Theorem 7 (Bárány). *For any convex body K in $\mathbb{R}^2$, there exists a unique convex body $K_0 \subset K$ such that*

$$\Omega(K_0) = \max\{\Omega(C) : C \subset K \quad \text{is a convex body}\}.$$

This K_0 comes up in various affine invariant extremal problems, say in Bárány's asymptotic solution of Sylvester's problem in [4].

Theorem 8 (Bárány). *For any convex body K in $\mathbb{R}^2$, we have*

$$\lim_{n\to\infty} n^2 \sqrt[n]{p(n,K)} = \frac{e^2}{4}\Omega(K_0)^3.$$

Unfortunately, the higher dimensional analogue of Theorem 7 is still not known.

Conjecture 9 (Bárány). *For any $d \geq 3$ and convex body K in $\mathbb{R}^d$, there exists a unique convex body $K_0 \subset K$ such that*

$$\Omega(K_0) = \max\{\Omega(C) : C \subset K \quad \text{is a convex body}\}.$$

This conjecture was verified by Schneider [21] for convex bodies with elliptic type, in which case $K_0 = K$.

3. Lattice polytopes in a convex body with respect to $\varepsilon\mathbb{Z}^d$

Random polytopes in a convex body with K in $\mathbb{R}^d$ can be modelled by polytopes whose vertices are chosen from $\varepsilon\mathbb{Z}^d$ as ε tends to zero. We write $Q_\varepsilon = Q_{\varepsilon,K}$ to denote the convex hull of $K \cap \varepsilon\mathbb{Z}^d$.

In the case of the unit ball B^d, Bárány, Larman [7] proved that the convex hull Q_ε behaves as P_n for $n = |B^d \cap \varepsilon\mathbb{Z}^d|$ where $|\cdot|$ stands for cardinality.

Theorem 10 (Bárány, Larman). *There exist $c_1, c_2 > 0$ depending on d such that $Q_\varepsilon = Q_{\varepsilon,B^d}$ satisfies*

(i) $c_1\varepsilon^{\frac{2d}{d+1}} \leq V(B^d \backslash Q_\varepsilon) \leq c_2\varepsilon^{\frac{2d}{d+1}}$

(ii) $c_1\varepsilon^{\frac{-d(d-1)}{d+1}} \leq f_i(Q_\varepsilon) \leq c_2\varepsilon^{\frac{-d(d-1)}{d+1}}$ *for $i = 0, \ldots, d-1$*

For any convex body K with C^2_+ boundary, an unpublished result by Bárány, Böröczky says that $f_i(Q_\varepsilon)$ is approximately $\Omega(\frac{1}{\varepsilon}K)$ for small $\varepsilon > 0$.

Theorem 11 (Bárány, B). *There exist $c_1, c_2 > 0$ depending on d such that if K is a convex body in $\mathbb{R}^d$ with C^2_+ boundary and $i = 0, \dots, d-1$, then*

$$c_1 \Omega\left(\frac{1}{\varepsilon} K\right) \le f_i(Q_\varepsilon) \le c_2 \Omega\left(\frac{1}{\varepsilon} K\right)$$

In Theorem 11, one does not need the condition $V(K) = 1$, however, some condition on ∂K is needed.

A spectacular development about lattice polytopes is the understanding the typical lattice polygon in a planar convex body. For a convex body K with $V(K) = 1$ in $\mathbb{R}^2$ and for small $\varepsilon > 0$, let $\mathcal{P}(K, \varepsilon)$ be the family of all lattice polygons in K. Bárány [3] determined the asymptotics of at least $\log |\mathcal{P}(K, \varepsilon)|$ where ζ stands for Riemann's zeta function.

Theorem 12 (Bárány). *For a convex body K with $V(K) = 1$ in $\mathbb{R}^2$, and the K_0 in Theorem 7, we have*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-2/3} \log |\mathcal{P}(K, \varepsilon)| = 3 \sqrt[3]{\frac{\zeta(3)}{4\zeta(2)}}\, \Omega(K_0).$$

Bárány [3] determined the limit shape of lattice polygons in a planar convex body. We write $\delta_H(C, Q)$ to denote the Hausdorff distance of convex compact sets C and Q.

Theorem 13 (Bárány). *For a convex body K with $V(K) = 1$ in $\mathbb{R}^2$, and the K_0 in Theorem 7, and for any small $\eta > 0$, we have*

$$\lim_{\varepsilon \to 0^+} \frac{\{P \in \mathcal{P}(K, \varepsilon) : \delta_H(P, K_0) < \eta\}}{|\mathcal{P}(K, \varepsilon)|} = 1.$$

Finally, Bárány, Prodromou [8] considered the the maximal possible number $M(K, \varepsilon)$ of vertices of a lattice polygon in $\mathcal{P}(K, \varepsilon)$.

Theorem 14 (Bárány, Prodromou). *For a convex body K with $V(K) = 1$ in $\mathbb{R}^2$, and the K_0 in Theorem 7, we have*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-2/3} M(K, \varepsilon) = \frac{3}{(2\pi)^{2/3}}\, \Omega(K_0).$$

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Random approximations of convex bodies by ball-polytopes

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Approximation of convex bodies by random polytopes is a classical topic in stochastic geometry. One of the most frequently investigated models is when one selects n independent, identically distributed (i.i.d.) random points from a convex body K according to some prescribed probability distribution and then takes the convex hull of these random points, thus obtaining a random polytope in K. Many of the results about random polytopes are asymptotic in nature, that is, it is assumed that n tends to infinity. Such asymptotic results include estimates and also exact formulas on the expectation and variance of basic geometric quantitites (such as the number of k-faces, intrinsic volumes, etc.) associated with the random polytopes, and also laws of large numbers and central limit theorems. For more information on this extensive subject we refer to the surveys that can be found in, for example, Bárány [1], Hug [5], Reitzner [6], Schneider [7, 8, 9], Schneider and Weil [10], Weil and Wieacker [11].

In this talk we intend to prove analogues of some of the classical results about random polytopes for random sets produced by intersections of congruent closed balls. We will consider the following probability model. Let $K \subset \mathbb{R}^d$ be a convex body with the property that its boundary is C^3 smooth and that K slides freely in a ball B of radius $r > 0$, meaning that for each $x \in \partial B$, there exists $v \in \mathbb{R}^d$ such that both $x \in K + v$ and $K + v \subset B$. Let $x_1, \ldots, x_n$ be i.i.d. uniform random points in K and let $K_{(n)}$ denote the intersection of all radius r closed balls which contain $x_1, \ldots, x_n$. Then $K_{(n)}$ is a (uniform) random ball-polytope (of radius r) and it is known to be contained in K, cf. Bezdek, Lángi, Naszódi and Papez [3]. We are mainly interested in the expectation of the number of facets of $K_{(n)}$, especially in the case when K is a ball and r is equal to its radius. In this case we prove that the expected number of facets of $K_{(n)}$ tends to a constant (as $n \to \infty$) which depends only on the dimension. This phenomenon was observed earlier in the planar case ($d = 2$) by Fodor, Kevei and Vígh [4], who also established various asymptotic formulas about the expected number of sides, missed area and the perimeter of $K_{(n)}$.

One of the important tools in the proof is a recent result of Bárány, Hug, Reitzner and Schneider [2] about parallelotopes spanned by random vectors chosen from a half-sphere. Beside many other interesting results, the authors prove in [2] that if one considers a spherical random polytope generated by n i.i.d. uniform random points chosen from a half-sphere, then the expectation of the number of its

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facets tends to a constant as n goes to infinity. This phenomenon is similar to what happens in the case of random ball-polytopes in a ball.

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Globalizing groups

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Joint work with Guido Ahumada, Bernard Brighi, and Nicolas Chevallier (LMIA).

A group of bijections G acting on a set X is called a GAF if each element of G has at least one fixed point. The group G is called a GAG if there exists $x \in X$ which is fixed by all elements of G. The acronyms GAF and GAG come from the french language. Our main purpose is to explore to which extent a "gaffe" (a blunder) is a "gag" (a joke) or not. The group of all bijections of X will be denoted by $\mathrm{Bij}\, X$.

Exercise 1. An abelian group such that one of its elements has a unique fixed point is a GAG.

The group G is called *excentric* if it is a GAF but not a GAG. The group of rotations acting on the 2-dimensional sphere $\mathbb{S}_2$ is an example of excentric group. An example of excentric abelian group is the following: Let a be an irrational number and consider the tranvections $f : (x, y) \mapsto (x + y + 1, y)$ and $g : (x, y) \mapsto (x + ay, y)$. Then the group generated by f and g is abelian and excentric. One can also build a group of affine bijections on $\mathbb{R}^2$ which is not a GAG, although each of the bijections has a unique fixed point. Thse examples show that both words "abelian" and "unique" are necessary in Exercise 1.

A group G is called *globalizing* if it contains no excentric subgroup.

Exercise 2. The symmetric group $S_n = \mathrm{Bij}\,\{1, \dots, n\}$ is globalizing if only if $n \le 4$.

If X is a metric space, then $\mathrm{Isom}\, X$ denotes the subgroup of $\mathrm{Bij}\, X$ whose elements are isometries on X. If X is orientable, then the group of orientation preserving isometries on X is denoted by $\mathrm{Isom}^+ X$.

We have almost exhaustively treated the case of isometries on the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}_n$, and the elliptic spaces $\mathbb{S}_n$ and $\mathbb{R}\mathbf{P}_n$ (the space of straight lines of $\mathbb{R}^{n+1}$) for all $n \in \mathbb{N}$.

In the Euclidean case, we find that the group $\mathrm{Isom}^+ \mathbb{R}^n$ is globalizing if and only if $n \le 3$.

In the hyperbolic case, we find that $\mathrm{Isom}^+ \mathbb{H}_n$ is globalizing if $n \le 3$ and not globalizing if $n \ge 5$. We do not know whether $\mathrm{Isom}^+ \mathbb{H}_4$ is globalizing or not.

One might be led to believe that, for each family of groups $\mathrm{Isom}\, \mathbb{X}_n$ or $\mathrm{Isom}^+ \mathbb{X}_n$, with $\mathbb{X} = \mathbb{R}, \mathbb{H}, \mathbb{S}$, or $\mathbb{R}\mathbf{P}$, there is a critical value n_0 such that the group is globalizing if only if $n \le n_0$. This is true for the Euclidean and the hyperbolic spaces, but not

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for the elliptic ones! We find that $\mathrm{Isom}^+ \mathbb{S}_3$ is globalizing, whereas $\mathrm{Isom}^+ \mathbb{S}_2$ is not since it is already excentric itself.

Let $\mathbb{F}_n$ denote either $\mathbb{R}^n$ or $\mathbb{H}_n$. One main difference between $\mathbb{F}_n$ and the elliptic spaces $\mathbb{S}_n$ or $\mathbb{R}\mathbf{P}_n$ is that, if F is an affine, resp. hyperbolic, subspace of $\mathbb{F}_n$, then every point $x \in \mathbb{F}_n$ has a unique orthogonal projection on F.

Two general results concerning isometries on the Euclidean and the hyperbolic spaces are the following.

Theorem 1. *Let G be a* solvable *group of isometries of $\mathbb{F}_n$. If G is a* GAF, *then G is a* GAG.

The assumption of finite dimension is necessary:

Exercise 3. In the Hilbert space $E = \ell^2(\mathbb{N}, \mathbb{R})$ of square summable sequences of real numbers, let h_k be the symmetry of center 1 on the k-th coordinate, i.e.

$$h_k(x_0, x_1, \dots) = (x_0, \dots, x_{k-1}, 2 - x_k, x_{k+1}, \dots).$$

Let G_n be the group of isometries generated by $\{h_0, \dots, h_n\}$ and let $G = \bigcup_{n \in \mathbb{N}} G_n$. Show that G is abelian and excentric.

Theorem 2. *Let G be a group of isometries of $\mathbb{F}_n$ and let H be a subgroup of G such that G/H is cyclic. If H is globalizing, then G is globalizing.*

In particular a group of isometries of $\mathbb{F}_n$ is globalizing as soon as it contains a globalizing subgroup of index 2. Since a subgroup of a globalizing group is obviously globalizing, we then deduce that $\mathrm{Isom}\, \mathbb{R}^n$ is globalizing if only if $n \leq 3$, and that $\mathrm{Isom}\, \mathbb{H}_n$ is globalizing if $n \leq 3$ and not globalizing if $n \geq 5$.

Caution! Theorem 2 is false in the elliptic case: We find that $\mathrm{Isom}\, \mathbb{S}_n$ is globalizing if and only if $n = 1$. Therefore $\mathrm{Isom}^+ \mathbb{S}_3$ is globalizing whereas $\mathrm{Isom}\, \mathbb{S}_3$ is not!

The projective case sums up as follows:
– It is easy to prove that $\mathrm{Isom}\, \mathbb{R}\mathbf{P}_{2k}$ is excentric for all $k \geq 1$. Since $\mathbb{R}\mathbf{P}_{2k}$ is nonorientable, $\mathrm{Isom}^+ \mathbb{R}\mathbf{P}_{2k}$ does not make sense.
– For $k \geq 2$, we can construct an excentric subgroup of $\mathrm{Isom}^+ \mathbb{R}\mathbf{P}_{2k+1}$, showing that $\mathrm{Isom}^+ \mathbb{R}\mathbf{P}_{2k+1}$, hence also $\mathrm{Isom}\, \mathbb{R}\mathbf{P}_{2k+1}$, are not globalizing.
– The case $n = 3$ is more tricky: We found an excentric subgroup of $\mathrm{Isom}\, \mathbb{R}\mathbf{P}_3$, showing that $\mathrm{Isom}\, \mathbb{R}\mathbf{P}_3$ is not globalizing, but we do not know whether $\mathrm{Isom}^+ \mathbb{R}\mathbf{P}_3$ is globalizing or not.

Exercise 4. Let us call *super-globalizing* a group G such that, for every set X and every morphism $\rho : G \to \mathrm{Bij}\, X$, the pair $(X, \rho(G))$ is globalizing. Prove that a group is super-globalizing if and only if it is cyclic (finite or infinite).

Exercise 5. Prove that the set of rational numbers $\mathbb{Q}$ is *finitely super-globalizing* in the following sense: If X is a finite set and $\rho : \mathbb{Q} \to \mathrm{Bij}\, X$ a morphism, then $(X, \rho(\mathbb{Q}))$ is globalizing.

A note on a picture-hanging puzzle

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In the picture-hanging puzzle we are to hang a picture so that the string loops around n nails and the removal of any nail results in a fall of the picture. We show that the length of a sequence representing an element in the free group with n generators that corresponds to a solution of the picture-hanging puzzle must be at least $n2^{\sqrt{\log_2 n}}$. In other words, this is a lower bound on the length of a sequence representing a non-trivial element in the free group with n generators such that if we replace any of the generators by the identity the sequence becomes trivial.

For $n = 2$, the shortest solution has length four and the corresponding sequence is $s_2 = a_1 a_2 a_1^{-1} a_2^{-1}$. Already this case, suggests how to construct a sequence s_n of a solution for an arbitrary n. Namely, given s_{n_1} and s_{n_2} with disjoint symbol sets we put $s = s_{n_1} s_{n_2} s_{n_1}^{-1} s_{n_2}^{-1}$, where $s^{-1} = x_m^{-1} \dots x_1^{-1}$ for $s = x_1 \dots x_m$. The shortest sequence s_n obtained by this method is the one constructed recursively by taking $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$. This idea leads to a solution of the picture-hanging puzzle, whose corresponding sequence s has length roughly quadratic in n, and which was discovered by Chris Lusby Taylor, see [1, Section 3].

A question posed therein asks if this is the family of shortest possible solutions, possibly up to the order of magnitude. By a computer assisted proof [2], the construction is optimal up to $n = 5$. Our result can be seen as a first step towards answering this question. We establish a slightly more general result implying the claimed lower bound by elementary means. We are not aware of any better previously established lower bound than $2n$, for $n \geq 2$, which holds because every symbol must appear an even number of times in the sequence.

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Coin-weighting and different directions of lines

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Abstract

Starting with a version of coin-weighting problems by ApSimon (1984) we discuss directions of lines among lattice points. Consider a set V of k vectors on the plane with non-negative integer coordinates. Let $S(V)$ be the set of the $2^k - 1$ non-empty subset sums. We are looking for the smallest $n = n(k)$ such that V is a subset of $[0,n]^2 := \{0,1,2,3,\dots,n\} \times \{0,1,2,3,\dots,n\}$ and the slopes of the members of $S(V)$ are all distinct.

Dedicated to Imre Bárány

1. ApSimon's Mints problem

A coin weighting problem proposed by ApSimon [4] can be reformulated as follows (see Guy and Nowakowski [9]). Given k, choose non-negative integers $a_1, \dots, a_k$ and $b_1, \dots, b_k$ such a way that the fraction

$$\frac{\sum_{i\in I} a_i}{\sum_{i\in I} b_i} \tag{1}$$

takes $2^k - 1$ different values for the $2^k - 1$ possible choices of $I \subset \{1,2,3,\dots,k\}$. ApSimon showed that this is always possible, he gave the example $b_1 = .. = b_k = 1$ and $a_i = i!$ for $i = 1,2,\dots,k$. He asked what is the minimum of $A(k) := \sum_i \max\{a_i, b_i\}$.

He showed $A(1) = 1$, $A(2) = 2$, $A(3) = 4$, $A(4) = 8$, and $A(5) = 15$. Guy and Nowakowski showed (with a computer search) that $A(6) \le 38$, $A(7) \le 74$, and also $A(k) \le (4^{k+1} - 1)/3$ for $9 \le k \le 12$ by taking $a_i = i - 1$ and $b_i = 4^i$ $(i = 1,2,\dots,k)$. This example does not work for $k = 13$.

2. Vectors of distinct directions

In this note we determine the order of magnitude of ApSimon's function, we show that there are constants $1 < c_1 < c_2$ such that

$$c_1^k \le A(k) \le c_2^k \tag{2}$$

holds for all sufficiently large k.

We call a set of vectors V *multidirectional* if it satisfies (1). Let $n(k)$ be the minimum n such that one can select k multidirectional vectors from $\{0,1,\dots,n\}^2$.

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Theorem 1. *There are positive constants $c_3, \dots, c_6$ such that*

$$c_3 2^{k/2} n^{c_4} < n(k) < c_5 2^k n^{c_6}.$$

holds for every $k \geq 1$.

Note that this trivially implies the bounds on the order of magnitude of $A(k)$ in (2). The lower bound is trivial, the $2^k - 1$ sums in $S(V)$ can not fit into a square of size less than $2^{k/2}$. The upper bound can be obtained by simple random method, using the lemma below.

3. An algorithmic construction

It is an interesting question to find (small) explicit multidirectional sets. With the help of the lemma below we can show that the greedy algorithm provide a k set V with $\max\{\|v\| : v \in V\} = O((\sqrt{8})^k n^{c_7})$.

If one tries, e.g., the family $v_i := (1, 2^i)$ then very soon can be observed that it is not multidirectional,

$$\frac{1}{3}(v_0 + v_4 + v_6) = \frac{1}{4}(v_2 + v_3 + v_5 + v_6).$$

4. The main lemma on lattice lines

Any line L contains at most n points from the lattice $[n] \times [n]$, so an upper bound kn is obvious in the following Lemma.

Lemma 2. *Suppose that $L_1, L_2, \dots, L_k$ are lines on the plane, no two of them are parallel. Then they meet the $[n] \times [n]$ lattice in at most $(2 + 2\sqrt{k})n$ points,*

$$|(\cup_{1 \leq i \leq k} L_i) \cap [n] \times [n]| \leq (2 + 2\sqrt{k})n. \tag{3}$$

Let C_k be the best coefficient which can be written in the right hand side of (3) instead of $2 + 2\sqrt{k}$. The order of magnitude of C_k (i.e., $O(\sqrt{k})$) is the best possible. Actually, it is not difficult to determine C_k exactly. For $k \leq 4$ we have $C_k = k$. For $k > 4$ suppose that $t > 1$ is the smallest integer such that

$$4 < k \leq 4 + 4\left(\varphi(2) + \varphi(3) + \cdots + \varphi(t+1)\right).$$

Here φ is Euler's totient function. Let $\alpha(k,t) := k - 4 - 4\left(\varphi(2) + \varphi(3) + \cdots + \varphi(t)\right)$, (i.e., we have $\alpha \leq 4\varphi(t+1)$). Then

$$C_k = 4 + 4\left(\frac{\varphi(2)}{2} + \frac{\varphi(3)}{3} + \cdots + \frac{\varphi(t)}{t}\right) + \frac{\alpha(k,t)}{t+1}.$$

The proof of the Lemma follows from well-known elementary properties of the totient function. Also, a direct proof can be given to show that if $k \le 4t^2$ then $C_k \le 2 + 4t$. $\square$

If, instead of $[n] \times [n]$, one considers a general grid $A \times B$ where $|A| = |B| = n$ then Lemma 2 no longer holds. For example, one can take $A = B := \{q, q^2, \dots, q^n\}$ then the lines $y = q^i x$ for $i = 1, 2, \dots, k$ cover $kn - O(k^2)$ grid points of $A \times B$ (k is fixed, $n \to \infty$).

This Lemma 2, at least implicitly, has been used many other occasions by different authors. For example, Erdős and Purdy asked the following question. What is $t(n, 2)$ the minimum number t of the $[n] \times [n]$ lattice points such that the $\binom{t}{2}$ lines determined by t appropriately chosen lattice points cover all of them. Alon [2] showed that there are positive constants c and C such that

$$cn^{2/3} \le t(n, 2) \le Cn^{2/3} \log n.$$

The upper bound follows from a random choice, and the lower bound can be obtained from Lemma 2 because given any chosen point the $t - 1$ lines through on it can cover at most $4\sqrt{t}n$ lattice points, so one needs $4t^{3/2}n \ge n^2$.

5. Averaging sets and other subset sum problems

Here we mention some related questions.

Straus [13] proposed the following problem. Call a set V *non-averaging* if no member is the arithmetic mean of some others (such a set does not contain an AP_3, arithmetic progression of length 3). Let $f(x)$ denote the size of the largest non-averaging set chosen from the first x integers. There are remarkable constructions by Abbott [1] and later by Bosznay [5] ($f(x) \ge \Omega(x^{1/4})$) and upper bounds for $f(x)$ by Erdős and Straus [8]. Generalizations by Alon and Ruzsa [3] and Konyagin, Ruzsa, and Schlag [10]. Applications in coding theory by Milenkovic, Kashyap, and Leyba [11] and in extremal graph theory by Conlon, Fox, and Sudakov [6].

Straus [13] also defined $h(x)$, the largest integer such that one can select $h(x)$ numbers from $\{1, 2, \dots, x\}$ such that no two distinct subsets have the same arithmetic mean. In fact, this is the same (or almost the same) as ApSimon's question, with the additional condition that $b_1 = \dots = b_k = 1$. Straus [13] showed that

$$c_8 \frac{\log x}{\log \log x} < h(x) < \frac{\log x}{\log 2} + O(\log \log x).$$

6. Higher dimensions and other open problems

There are interesting open questions if we are looking for dense multidirectional sets in dimension d.

It is rather natural to use combinatorial geometry, incidency structures to solve problems in number theory, see, e.g., the works of Solymosi [12].

Erdős and Moser (see in [7]) showed that

$$\log_2 x \le s(x) < \log_2 x + \frac{1}{2}\log\log x + c_9,$$

where $s(x)$ is the size of the largest set $V \subset \{1, 2, \ldots, x\}$ such that all the 2^s subset sums are distinct. Erdős offered \$300 to prove or disprove that $s(x) < \log_2 x + C$ for some absolute constant C.

It would be really interesting to extend the Erdős-Moser problem to higher dimensional sums.

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Proof of László Fejes Tóth's zone conjecture

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A *plank* of width w is a part of the d-dimensional Euclidean space $\mathbb{R}^d$ that lies between two parallel hyperplanes at distance w. Given a convex body C, its *width* is the smallest w such that a plank of width w covers C. The following question solved by Bang [1] is usually attributed to Tarski.

Tarski's plank problem. If a convex body of width w is covered by a collection of planks in $\mathbb{R}^d$, then the total width of the planks is at least w.

The spherical analog of a plank is a *zone*. A *zone* of width ω on the 2-dimensional unit sphere is defined as the set of points within spherical distance $\omega/2$ of a given great circle. In 1973, Fejes Tóth [2] conjectured

Fejes Tóth's zone conjecture. The total width of any set of zones covering the sphere is at least π.

We completely resolve this conjecture and generalize it for the d-dimensional unit sphere S^d. We believe the following strengthening of our result holds.

Open problem. A *spherical segment* is the solid defined by cutting a sphere with a pair of parallel planes. Its width is the length of the shortest arc on the sphere whose endpoints touch both parallel planes. If the unit ball is covered with a collection of spherical segments, then the total width of the spherical segments is at least π.

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Dense regular horoball packings in higher dimensional hyperbolic spaces

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In this talk we consider regular ball packings of hyperbolic space for dimensions $n = 3, 4$, and 5. Classical results in this topic stem from the work of László Fejes Tóth and Károly Böröczky [1, 3]. Böröczky showed that the notion of packing density is critical in hyperbolic space, as the computed density depends on the cellular decomposition of the space, see [2] for an overview. To resolve this issue, we consider cellular decompositions of hyperbolic space into Dirichlet-Voronoi cells. In n-dimensional spaces of constant curvature, one has Böröczky-type simplicial packing density upper bounds, which state that locally the densest packing configuration by balls of a fixed radius r is attained by centering $n + 1$ balls at the vertices of a regular simplex of edge length $2r$. Fortunately in 3-dimensional hyperbolic space the regular fully asymptotic simplex gives a Coxeter tiling, and the monotonicity of the simplicial density function as $r \to \infty$ is known [4], hence this upper bound is attained. In this case the balls have ideal centers, so the packing consists of limiting objects called horoballs, instead of regular balls.

Based on this result of Böröczky, we investigate horoball packings related to the paracompact (Koszul) Coxeter simplex tilings. Such tilings have regular and ideal vertices, and exist for hyperbolic dimensions $n = 2 \dots 9$; their number is finite for $n \geq 3$. In these cases the fundamental domain of the tiling, the Coxeter simplex, is also the Dirichlet-Voronoi cell for the packing with horoballs centered at the ideal vertices of the tiling, hence density is well defined. In papers [7], [8] the notion of the horoballs of different types is introduced, and it is proved that the Böröczky type density upper bound is locally no longer valid for cases where we consider horoballs in fully asymptotic simplices for dimensions $n \geq 3$. However these ball packing configurations are only locally optimal and cannot be extended to the entirety of hyperbolic spaces $\mathbb{H}^n$.

We consider packings with symmetries given by Coxeter simplex groups, and fundamental domains by Coxeter simplices. Using the commensurability classes of the paracompact (Koszul) simplex reflection groups, studying both the arithmetic and non arithmetic cases, we classify the horoball packings and produce some of the best known packing densities in dimensions three and higher. One example of particular interest includes a class of packings where we showed that it is possible

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to exceed the conjectured 4-dimensional packing density upper bound due to L. Fejes Tóth [3] with densities of ≈ 0.71644896 [6]. This density is also realized as a horoball packing with horoballs of different types in the 4-dimensional hyperbolic 24 cell honeycomb $\{3,4,3,4\}$, see [9]. Another set of interesting examples in 3-dimensions is a class of packings that attain simplicial packing density the upper bound due to K. Böröczky, but have a distinct group of symmetries as the classical example [5]. We will also discuss new results in higher dimensions.

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Approximation of convex bodies by polytopes in the geometric distance

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Given a convex body K in $\mathbb{R}^d$ with the center of mass at the origin, a positive integer $t \geq d+1$, and $\delta, \vartheta \in (0,1)$. We discuss conditions on the parameters d, t, δ, ϑ that imply that the convex hull of some t points of K contains ϑK. One may call it approximation in the "geometric distance".

First we consider the case when the points are chosen uniformly from K [10].

Second, we turn to the case when we are free to choose a probability distribution to obtain our approximating polytope. This part is joint work with Fedor Nazarov and Dmitry Ryabogin [11].

1. Random polytopes – uniformly

One approach is to pick the t points uniformly and independently in K.

The main result of [3] concerns the case of very rough approximation, that is, where the number t of chosen points is linear in the dimension d. It states that the convex hull of $t = \alpha d$ random points chosen uniformly in a centered convex body K is a polytope P which satisfies $\frac{c_1}{d} K \subseteq P$, with probability $1 - \delta = 1 - e^{-c_2 d}$, where $c_1, c_2 > 0$ and $\alpha > 1$ are absolute constants.

We prove a slightly stronger version of this statements, where the three constants are made explicit.

Theorem 1. *Let K be a centered convex body in $\mathbb{R}^d$. Choose $t = 500d$ points $X_1, \ldots, X_t$ of K randomly, independently and uniformly. Then*

$$\frac{1}{d} K \subseteq conv\{X_1, \ldots, X_t\} \subseteq K.$$

with probability at least $1 - 1/e^d$.

Another instance of our general problem is Theorem 5.2 of [5], which concerns fine approximation, that is, where the number t of chosen points is exponential in the dimension d. It states that for any $\delta, \gamma \in (0,1)$, if we choose $t = e^{\gamma d}$ random points uniformly in any centered convex body K in $\mathbb{R}^d$, then the polytope P thus obtained satisfies $c(\delta)\gamma K \subseteq P$, with probability $1 - \delta$.

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The same argument yields Proposition 5.3 of [5], according to which for any $\delta,\vartheta \in (0,1)$, if we choose $t = c(\delta)\left(\frac{c}{1-\vartheta}\right)^d$ random points uniformly in any centered convex body K in $\mathbb{R}^d$, then the polytope P thus obtained satisfies $\vartheta K \subseteq P$, with probability $1-\delta$.

Our first main result is the following [10].

Theorem 2. *Let $\delta,\vartheta \in (0,1)$, and let K be a centered convex body in $\mathbb{R}^d$. Let*

$$t := \left\lceil C\frac{(d+1)e}{(1-\vartheta)^d}\ln\frac{e}{(1-\vartheta)^d}\right\rceil,$$

where $C \geq 2$ is such that

$$C^2\left(\frac{(1-\vartheta)^d}{e}\right)^{C-2} \leq \frac{(\delta/4)^{1/(d+1)}}{e^3}.$$

Choose t points $X_1,\ldots,X_t$ of K randomly, independently and uniformly. Then

$$\vartheta K \subseteq conv\{X_1,\ldots,X_t\} \subseteq K$$

with probability at least $1-\delta$.

By substituting $\vartheta = \frac{1}{d}, \delta = e^{-d-1}, C = 7$, we obtain Theorem 1.

By substituting $C = 3$, we obtain the two results of [5] mentioned above.

The proof is very simple: it is a combination of two results. One is a stability version of a theorem of Grünbaum [6] which is a classical fact in convexity, and which states that any hyperplane through the centroid of a convex body splits its volume into two parts none of which is less than $1/e$ times the volume of the body. Our second tool is the ε-net theorem, a result from combinatorics obtained by Haussler and Welzl [8] building on works of Vapnik and Chervonenkis [12], and then refined by Komlós, Pach and Woeginger [9].

2. Random polytopes – using a smarter measure

Next, we consider the question of existence of a polytope that approximates a given convex body K well. This time, we are free to choose a method of constructing our polytope, we do not need to select the vertices according to the uniform distribution on K. Our second main result, a joint work with Fedor Nazarov and Dmitry Ryabogin [11], reads

Theorem 3. *Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\vartheta \in \left(\frac{1}{2},1\right)$. Then there exists a convex polytope P with at most $e^{O(d)}(1-\vartheta)^{-\frac{d-1}{2}}$ vertices such that $\vartheta K \subset P \subset K$.*

Our approach uses a mixture of geometric and probabilistic tools. The main part of the proof is the construction of a probability distribution on the boundary of K, according to which caps are of large measure.

This result improves the 2012 theorem of Barvinok [2] by removing the symmetry assumption and the extraneous $(\log \frac{1}{1-\vartheta})^d$ factor.

We refer the reader to the surveys of Bárány [1], Bronshtein [4] and Gruber [7] for a detailed discussion of approximation of a convex body by polytopes.

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On the gap between translative and lattice kissing numbers of a convex body

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In a packing with convex bodies, two convex bodies are called neighbours if they touch each other (that is, if their intersection is not empty). The translative kissing number $H(K)$ of a d-dimensional convex body K is the maximum number of neighbours of a member in a packing of the d-dimensional Euclidean space with translates of K. $H(K)$ is also called the Hadwiger number of K. The lattice kisssing number $H_L(K)$ of K is the similar quantity with the further restriction to lattice packings of K.

The inequality $H_L(K) \leq H(K)$ holds trivially. It is known that $H(K) \leq 3^d - 1$ (Hadwiger [2]). For strictly convex bodies we have $H_L(K) \leq 2^{d+1} - 2$ (Minkowski [3]). The following general lower bounds are known: $H(K) \geq 2^{cd}$ (Talata [5], where c is an absolute constant, $c > 0$), and $H_L(K) \geq d^2 + d$ (Swinnerton-Dyer [4]).

It is known that $H(K) = H_L(K)$ holds in two dimensions (Grünbaum [1]). We investigate how large the gap can be between $H(K)$ and $H_L(K)$ in higher dimensions. It is known that for every $d \geq 3$, there exists a d-dimensional strictly convex body C for which $H(C) - H_L(C) \geq m\left(\sqrt{7}\right)^d$ holds, for some absolute constant $m > 0$ (Talata [6]). Now, we improve on this bound to show that for every $d \geq 3$, there exists a d-dimensional convex body D for which $H(D) - H_L(D) \geq k \cdot 3^d$ holds, for some absolute constant $k > 0$.

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A result in asymmetric Euclidean Ramsey theory

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A typical question in Euclidean Ramsey theory has the following form: is it true that for any colouring of Euclidean space E^n in two (or more) colours there exists a monochromatic copy of some fixed geometric configuration F? Research in Euclidean Ramsey theory was surveyed in [1, 2, 3] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus; for a more recent survey, see Graham [4].

I will focus on the asymmetric version of this question - is it true that for any colouring of E^n in red and blue, there exists either a red copy of F_1 or a blue copy of F_2? For $d \in \mathbb{Z}^+$, and geometric configurations F_1, F_2, let the notation $\mathbb{E}^d \to (F_1, F_2)$ mean that for any red-blue coloring of $\mathbb{E}^d$, either the red points contain a congruent copy of F_1, or the blue points contain a congruent copy of F_2. Most of the questions in this field are very easy to state, but even some simplest cases are still open. For instance, Juhász [5] proved that if ℓ_2 is a configuration of two points at unit distance, T_k is any configuration of k points, then $\mathbb{E}^2 \to (\ell_2, T_4)$. It is not known if $\mathbb{E}^2 \to (\ell_2, T_5)$.

I will give a brief overview of known results. I will also present my recent result [6], that states that $\mathbb{E}^2 \to (\ell_2, \ell_5)$, where ℓ_5 denotes 5 points on a line with distance 1 between consecutive points.

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On the Geometry of Alexandrov Surfaces

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In the following, by *surface* we always mean a compact Alexandrov surface (with curvature bounded below by κ and without boundary), as defined for example in [3]. Roughly speaking, these surfaces are 2-dimensional topological manifolds endowed with an intrinsic metric which verifies Toponogov's comparison property. Let $\mathscr{A}(\kappa)$ denote the set of all such surfaces.

For any surface A, denote by ρ its metric, and by ρ_x the *distance function* from $x \in A$, given by $\rho_x(y) = \rho(x,y)$. A *segment* between x and y in A is a path from x to y of length $\rho(x,y)$. A point $y \in A$ is called *critical* with respect to ρ_x (or to x), if for any tangent direction τ of S at y there exists a segment from y to x whose tangent direction at y makes a non-obtuse angle with τ. For any point x in A, denote by Q_x the set of all critical points with respect to x, and by Q the *critical point mapping* associating to any point x in S the set Q_x. Similarly, M_x is the set of all relative maxima of ρ_x, F_x the set of all farthest points from x (i.e., absolute maxima of ρ_x) and M, respectively F, are the corresponding set-valued mappings.

The *cut locus* $C(x)$ of a point $x \in A$ is the set of all extremities, different from x, of maximal (with respect to inclusion) segments starting at x. It is known that $C(x)$ is locally a tree and with at most countably many ramifications points.

Theorem 4. [4] *Every graph can be realized as a cut locus on a surface.*

Theorem 5. [2] *Any point on any surface is critical with respect to some point of the surface.*

Theorem 6. [2] *A smooth orientable surface A is homeomorphic to the sphere* S^2 *if and only if each point in A is critical with respect to precisely one other point of S.*

Theorem 7. [6] *For every orientable surface A and every point y in A, there exists an open and dense set $\mathscr{Q}^y$ of Riemannian metrics on A such that y is critical with respect to an odd number of points in A for every $g \in \mathscr{Q}^y$.*

Theorem 8. [6] *Let A be a smooth orientable surface of genus $g > 0$ and y a point in A. If $g = 1$ then* $\mathrm{card} Q_y^{-1} \le 5$*, and if $g \ge 2$ then* $\mathrm{card} Q_y^{-1} \le 8g - 5$*.*

Endowed with the Hausdorff-Gromov metric, $\mathscr{A}(\kappa)$ is a Baire space. Denote by $\mathscr{A}(\kappa,\chi,\omega)$ the set of all surfaces in $\mathscr{A}(\kappa)$ of Euler-Poincaré characteristic χ and orientability ω, where $\omega = 1$ if the surface is orientable and $\omega = -1$ otherwise. If non-empty, $\mathscr{A}(\kappa,\chi,\omega)$ is a connected component of $\mathscr{A}(\kappa)$, for $\kappa \in \mathbb{R}$, $\chi \le 2$ and $\omega = \pm 1$ [8]. In particular, $\mathscr{A}(0)$ has two flat components (consisting of flat surfaces).

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Theorem 9. [1] *On most Alexandrov surfaces outside flat components, most points are not interior to any segment.*

Theorem 10. [5] *Most surfaces* $A \in \mathscr{A}(\kappa)$ *have not conical points.*

Let $\underline{G}(x)$ and $\overline{G}(x)$ denote the lower and the upper curvature of the surface A at $x \in A$, as defined in [7].

Theorem 11. [5] *For most surfaces* $A \in \mathscr{A}(\kappa)$*, at most points* $x \in A$*,* $\underline{G}(x) = \kappa$*. For most surfaces* $A \in \mathscr{A}(\kappa)$ *outside flat components (if any), at most points* $x \in A$*,* $\overline{G}(x) = \infty$*.*

Theorem 12. [9] *i) We have* $\mathscr{A}(1) = \mathscr{A}(1,2,1) \cup \mathscr{A}(1,1,-1)$*.*

Most surfaces in $\mathscr{A}(1,2,1)$ *have no simple closed geodesic.*

Most surfaces in $\mathscr{A}(1,1,-1)$ *have infinitely many simple closed geodesics.*

ii) We have $\mathscr{A}(0) = \mathscr{A}(0,2,1) \cup \mathscr{A}(0,0,1) \cup \mathscr{A}(0,1,-1) \cup \mathscr{A}(0,0,-1)$*.*

All surfaces in $\mathscr{A}(0,0,1) \cup \mathscr{A}(0,1,-1)$ *are unions of pairwise disjoint simple closed geodesics.*

Most surfaces in $\mathscr{A}(0,2,1)$ *have no closed geodesics.*

Most surfaces in $\mathscr{A}(0,1,-1)$ *have infinitely many simple closed geodesics.*

iii) Most surfaces in $\mathscr{A}(-1)$ *admit infinitely many, pairwise disjoint, simple closed geodesics.*

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