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THE INTERSECTION OF THE COMPATIBLE LINEAR EXTENSIONS OF A NATURAL PARTIAL ORDER

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Abstract. For an acyclic function $f : A \longrightarrow A$ we define a natural f-compatible partial order $\hat{f}$ on A and determine the intersection of the f-compatible linear extensions of $\hat{f}$.

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1. INTRODUCTION

One of the central problems in the theory of ordered algebras is to find necessary and sufficient conditions for the existence of a compatible linear extension of r in a partially ordered algebraic structure $(A, F, \leq_r)$. If $F = \varnothing$, then Szpilrajn proved that any partial order r ($\leq_r$) on a set A can always be extended to a linear order R and any partial order is the intersection of its linear extensions (see [5]). In the present paper we consider a partial order r on the set A and an order endomorphism $f : A \longrightarrow A$ with the natural compatibility property: $x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$. Clearly, the pair (A, f) is a unary algebra and the above f-compatibility condition allows us to view the triple $(A, f, \leq_r)$ as a partially ordered unary algebra. For $F = \{f\}$ ($f : A \to A$ is a unary operation) the above mentioned extension problem has been thoroughly investigated. Szigeti and Nagy proved that the partial order r of a unary algebra $(A, f, \leq_r)$ has an f-compatible linear extension if and only if the function $f : A \longrightarrow A$ is acyclic (see [3]). For an acyclic $(A, f, \leq_r)$ the intersection of the f-compatible linear extensions of r is determined in [2]. For an arbitrary $(A, f, \leq_r)$ the maximal f-compatible partial order extensions of r and the intersection are investigated in [1] and [4]. The aim of the present paper is to determine the intersection of the f-compatible linear extensions of $\hat{f}$, where the f-compatible partial order $\hat{f}$ on A can be defined in a natural way starting from an acyclic function $f : A \longrightarrow A$.

2. PRELIMINARIES

Let (A, f) be a (mono-)unary algebra. A partial order r on A is called *f-compatible* if the function (unary operation) $f : A \longrightarrow A$ is an order endomorphism with respect to r. We shall make use of the following notation

$$\mathcal{L}(A, f, \leq_r) = \{R \mid r \subseteq R \subseteq A \times A \text{ is an } f\text{-compatible linear order on } A\}.$$

The intersection

$$r_f = \mathbf{cl}(A, f, \leq_r) = \bigcap_{R \in \mathcal{L}(A, f, \leq_r)} R$$

is called the *closure* of r with respect to f. Indeed, the above definition gives a closure operator (with the monotone, idempotent and extensive properties) on the set of the f-compatible partial orders of A.

Definition 1 (see [3])**.** Let $A \neq \varnothing$ be a set and $N \geq 0$ be an integer. A function $f : A \to A$ takes N steps on the element $x \in A$, if

$$x, f(x), f^2(x), ..., f^N(x)$$

are different elements in A, and

$$f^{N+1}(x) = f^N(x)$$

(by convention $f^0(x) = x$). Take $N = \infty$ if

$$f^m(x) \neq f^n(x)$$

for all integers $0 \leq m < n$.

Definition 2 (see [3])**.** The function $f : A \to A$ is called acyclic, if for each element $x \in A$ there is an integer $0 \leq N = N(x) \leq \infty$ such that f takes $N = N(x)$ steps on x.

A partially ordered unary algebra $(A, f, \leq_r)$ is called acyclic, if $f : A \to A$ is acyclic. It is easy to see that any linearly ordered unary algebra is acyclic. If $(A, f, \leq_r)$ is acyclic, then r_f is an f-compatible partial order on A.

Lemma 1 (see [2])**.** *Let $(A, f, \leq_r)$ be an acyclic partially ordered unary algebra. If $a, b \in A$ and $a \leq_r f(a)$ or $f(a) \leq_r a$, then $(a, b) \in r_f$ implies that $a = b$ or $f^m(a) \leq_r f^m(b)$ and $f^m(a) \neq f^m(b)$ for some integer $m \geq 0$.*

3. THE ACYCLIC PARTIALLY ORDERED UNARY ALGEBRA $(A, f, \hat{f})$

Let

$$\langle x \rangle_f = \{x, f(x), f^2(x), ...\}$$

denote the f-orbit of x and define the following reflexive and transitive relation $\hat{f} \subseteq A \times A$ as follows:

$$x \hat{f} y \quad \Leftrightarrow \quad \langle x \rangle_f \subseteq \langle y \rangle_f .$$

Clearly, $\langle x\rangle_f \subseteq \langle y\rangle_f$ if and only if $x \in \langle y\rangle_f$, i.e. we can find an integer $k \geq 0$ such that $x = f^k(y)$.

Proposition 1. *If (A, f) is a unary algebra, then the following are equivalent.*

(1) *f is an acyclic function.*
(2) *$\hat{f}$ is antisymmetric.*

Proof. *(1)⇒(2)*: Suppose that $x \hat{f} y$ and $y \hat{f} x$ for $x, y \in A$. Then $\langle x\rangle_f = \langle y\rangle_f$ imply that $x = f^m(y)$ and $y = f^n(x)$ for some $m \geq 0$ and $n \geq 0$. Thus

$$x = f^m(f^n(x)) = f^{m+n}(x),$$

whence

$$x = f(x) = ... = f^n(x) = ... = f^{n+m}(x)$$

follows from the acyclic property of f. Obviously,

$$x = f^n(x) = y.$$

(2)⇒(1): Suppose that $f^m(x) = f^n(x)$ for some integers $0 \leq m < n$. Clearly, $\langle f^{m+1}(x)\rangle_f \subseteq \langle f^m(x)\rangle_f$ and $\langle f^m(x)\rangle_f \subseteq \langle f^{m+1}(x)\rangle_f$ is a consequence of

$$f^m(x) = f^{n-m-1}(f^{m+1}(x)).$$

Thus we have

$$f^m(x) \hat{f} f^{m+1}(x) \quad \text{and} \quad f^{m+1}(x) \hat{f} f^m(x).$$

Since $\hat{f}$ is antisymmetric, we get $f^m(x) = f^{m+1}(x)$ and

$$f^m(x) = f^{m+1}(x) = ... = f^n(x).$$

□

Proposition 2. *Let (A, f) be an acyclic unary algebra. Then the partial order $\hat{f}$ is f-compatible.*

Proof. If the function f is acyclic, then $\hat{f}$ is a partial order by Proposition 1. If $x \hat{f} y$ for $x, y \in A$, then we can find an integer $m \geq 0$ such that

$$x = f^m(y),$$

whence $f(x) = f^m(f(y))$ and $f(x) \hat{f} f(y)$ follows. □

4. The intersection

Proposition 3. *If (A, f) is an acyclic unary algebra and $\langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing$ for $x, y \in A$, then there exists a unique $z \in A$ such that*

$$\langle x\rangle_f \cap \langle y\rangle_f = \langle z\rangle_f.$$

This element $z = x \triangle y$ is called the f-intersection of x and y.

Proof. Since $\langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing$, we can define an integer n as follows:

$$n = \min\{k \geq 0 \mid f^k(x) \in \langle y\rangle_f\}.$$

We claim that $\langle x\rangle_f \cap \langle y\rangle_f = \langle z\rangle_f$ for $z = f^n(x)$. Obviously,

$$\langle z\rangle_f \subseteq \langle x\rangle_f \cap \langle y\rangle_f.$$

On the other hand if $u \in \langle x\rangle_f \cap \langle y\rangle_f$ then $u = f^k(x) \in \langle y\rangle_f$ for some $k \geq 0$. Thus $k \geq n$ and $u = f^{k-n}(f^n(x)) = f^{k-n}(z) \in \langle z\rangle_f$. The fact that there is only one $z \in A$ with

$$\langle x\rangle_f \cap \langle y\rangle_f = \langle z\rangle_f$$

is a consequence of Proposition 1. □

Definition 3. Let (A, f) be an acyclic unary algebra. If $\langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing$ for $x, y \in A$, then define the distance of x and y as follows:

$$\delta(x,y) = |(\langle x\rangle_f \setminus \langle y\rangle_f) \cup (\langle y\rangle_f \setminus \langle x\rangle_f)| = |\langle x\rangle_f \setminus \langle y\rangle_f| + |\langle y\rangle_f \setminus \langle x\rangle_f|.$$

We note that

$$\delta(x,y) = |\langle x\rangle_f \setminus \langle x \triangle y\rangle_f| + |\langle y\rangle_f \setminus \langle x \triangle y\rangle_f|.$$

immediately follows from

$$\langle x\rangle_f \setminus \langle y\rangle_f = \langle x\rangle_f \setminus (\langle x\rangle_f \cap \langle y\rangle_f) = \langle x\rangle_f \setminus \langle x \triangle y\rangle_f$$

and

$$\langle y\rangle_f \setminus \langle x\rangle_f = \langle y\rangle_f \setminus \langle x \triangle y\rangle_f.$$

Proposition 4. *If (A, f) is an acyclic unary algebra and $\langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing$ for $x, y \in A$, then*

$$|\langle x\rangle_f \setminus \langle y\rangle_f| = |\langle x\rangle_f \setminus \langle x \triangle y\rangle_f| = \min\{k \geq 0 \mid f^k(x) \in \langle y\rangle_f\} \leq N(x),$$

where the integer $N(x)$ is the number of f-steps on x.

Proof. If $n = \min\{k \geq 0 \mid f^k(x) \in \langle y\rangle_f\}$, then we have

$$x \triangle y = f^n(x)$$

(see the proof of Proposition 3). We distinguish two cases.
Case 1 If $N(x) = \infty$, then $n < N(x)$ and

$$\langle x\rangle_f \setminus \langle f^n(x)\rangle_f = \{f^k(x) | k \geq 0\} \setminus \{f^k(x) | k \geq n\}.$$

Thus

$$|\langle x\rangle_f \setminus \langle f^n(x)\rangle_f| = |\{x, f(x), ..., f^{n-1}(x)\}| = n.$$

Case 2 If $N = N(x) < \infty$ and $N < n$, then

$$\langle x\rangle_f \cap \langle y\rangle_f = \{x, f(x), ..., f^N(x)\} \cap \langle y\rangle_f = \varnothing,$$

a contradiction. Thus $n \leq N$ and

$$\langle x\rangle_f \setminus \langle f^n(x)\rangle_f = \{x, f(x), ..., f^N(x)\} \setminus \{f^n(x), ..., f^N(x)\},$$

i.e.

$$|\langle x\rangle_f \setminus \langle f^n(x)\rangle_f| = |\{x, f(x), ..., f^{n-1}(x)\}| = n.$$

□

The next theorem gives a complete description of the intersection of the f-compatible linear extensions of the partial order $\hat{f}$.

Theorem 1. *If (A, f) is an acyclic unary algebra and $x, y \in A$, then:*

$$(x, y) \in (\hat{f})_f \quad \Leftrightarrow \quad x = y, \text{ or } \langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing \text{ and } \delta(x, x\triangle y) < \delta(y, x\triangle y).$$

Proof. Since $\langle f(x)\rangle_f \subseteq \langle x\rangle_f$ (i.e. $f(x) \leq_{\hat{f}} x$) for all $x \in A$, the application of Lemma 1 gives that

$$(\hat{f})_f = \{(x, y) \in A\times A | (\exists m) 0 \leq m, f^m(x) \hat{f} f^m(y), f^m(x) \neq f^m(y)\} \cup \{(x, x) | x \in A\}.$$

If $(x, y) \in (\hat{f})_f$ and $x \neq y$, then $\langle f^m(x)\rangle_f \subseteq \langle f^m(y)\rangle_f$ and $f^m(x) \neq f^m(y)$ for some integer $m \geq 0$. Suppose that

$$k = \delta(x, x\triangle y) \geq \delta(y, x\triangle y) = l.$$

Since $f^m(x) \in \langle y\rangle_f$, we have $m \geq k$ (see Proposition 4). Now

$$f^m(x) = f^{m-k}(f^k(x)) = f^{m-k}(x\triangle y)$$

and

$$f^m(y) = f^{m-l}(f^l(y)) = f^{m-l}(x\triangle y)$$

imply that

$$f^m(y) = f^{k-l}(f^m(x)).$$

It follows that $\langle f^m(y)\rangle_f \subseteq \langle f^m(x)\rangle_f$. The antisymmetric property of $\hat{f}$ gives that $f^m(x) = f^m(y)$, a contradiction.

If $\langle x\rangle_f \cap \langle y\rangle_f \neq \varnothing$ and

$$m = \delta(x, x\triangle y) < \delta(y, x\triangle y) = l,$$

then

$$f^m(x) = x\triangle y = f^l(y) = f^{l-m}(f^m(y))$$

ensures that $\langle f^m(x)\rangle_f \subseteq \langle f^m(y)\rangle_f$. On the other hand, using Proposition 4,

$$m < l \leq N(y)$$

implies that

$$f^m(y) \neq f^l(y) = f^m(x).$$

Thus we have $(x, y) \in (\hat{f})_f$. □

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