Miskolc Mathematical Notes
Vol. 15 (2014), No 2, pp. 327-332

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2014.1059

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A NEW RESULT ON THE QUASI POWER INCREASING SEQUENCES

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Received 14 November, 2013

Abstract. In this paper, we prove a general theorem dealing with generalized absolute convolution Cesàro mean summability factors under weaker conditions by using a general class of increasing sequences instead of an almost increasing sequence. Some new results have also been obtained.

2010 *Mathematics Subject Classification:* 26D15; 40D15; 40F05; 40G99

Keywords: convolution Cesàro mean, power increasing sequences, summability factors, Hölder inequality, Minkowski inequality, infinite series

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$(see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$ and $0 < \sigma < 1$. Every almost increasing sequence is quasi-σ-power increasing sequence for any nonnegative σ, but the converse is not true for $\sigma > 0$ (see[9]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha*\beta}$ the nth convolution Cesàro mean of order $(\alpha * \beta)$, with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [5])

$$t_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta v a_v, \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{1.2}$$

The second author is supported by NSF of China (10901044), and Program for Excellent Young Teachers in HZNU.

The third author is supported by NSERC of Canada.

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Let $(\theta_n^{\alpha*\beta})$ be a sequence defined by

$$\theta_n^{\alpha*\beta} = \begin{cases} \left|t_n^{\alpha*\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\leq v\leq n}\left|t_v^{\alpha*\beta}\right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha*\beta;\delta\mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha*\beta} \mid^k < \infty. \tag{1.4}$$

If we take $\delta = 0$, then $\mid C,\alpha*\beta;\delta\mid_k$ summability reduces to $\mid C,\alpha*\beta\mid_k$ summability (see [6]). Also, if we take $\beta = 0$ and $\delta = 0$, then $\mid C,\alpha*\beta;\delta\mid_k$ summability reduces to $\mid C,\alpha\mid_k$ summability (see [7]). If we set $\beta = 0$, then we get $\mid C,\alpha;\delta\mid_k$ summability (see [8]).

2. The known result

Theorem 1 ([4])**.** *Let (X_n) be an almost increasing sequence and let there be sequences (η_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \eta_n, \tag{2.1}$$

$$\eta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\eta_n \mid X_n < \infty, \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2.4}$$

If the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{2.5}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha*\beta;\delta\mid_k$, $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, $\delta \geq 0$ and $(\alpha+\beta-\delta) > 0$.

It should be noted that if we take $\beta = 0$, then we get the result of Bor (see [2]).

3. The main result

The aim of this paper is to generalize Theorem 1 under weaker conditions to the $\mid C,\alpha*\beta;\delta\mid_k$ summability by using a quasi-σ-power increasing sequence, which is a wider class of sequences, instead of an almost increasing sequence.
We shall prove the following main theorem.

Theorem 2. *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$ and let there be sequences (η_n) and (λ_n) such that conditions (2.1)-(2.4) of Theorem A are satisfied. If the condition*

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{3.1}$$

is satisfies, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha*\beta; \delta \mid_k$ for $0 < \alpha \leq 1$, $\delta \geq 0$, $\beta > -1$, $k \geq 1$ and $(\alpha+\beta-\delta-1) > 0$.

Remark 1. It should also be noted that condition (3.1) is the same as condition (2.5) when k=1. When $k > 1$, condition (3.1) is weaker than condition (2.5) but the converse is not true. As in [10] we can show that if (2.5) is satisfied, then we get that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = O(X_m).$$

Also if (3.1) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = \sum_{n=1}^{m} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} X_n^{k-1} = O(X_m^{k-1}) \sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}}$$
$$= O(X_m^k) \neq O(X_m).$$

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). *If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{3.2}$$

Lemma 2 ([9])). *Under the conditions on (X_n), (η_n) and (λ_n) as expressed in the statement of the theorem, we have the following ;*

$$n X_n \eta_n = O(1), \tag{3.3}$$

$$\sum_{n=1}^{\infty} \eta_n X_n < \infty. \tag{3.4}$$

4. Proof of the theorem

Let $(T_n^{\alpha*\beta})$ be the nth $(C, \alpha*\beta)$ mean of the sequence $(n a_n \lambda_n)$. Then, by (1.1), we have

$$T_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 1, we have that

$$T_n^{\alpha*\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

thus,

$$\begin{aligned}
\mid T_n^{\alpha*\beta}\mid &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\mid\\
&\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\theta_v^{\alpha*\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid\theta_n^{\alpha*\beta}\\
&=T_{n,1}^{\alpha*\beta}+T_{n,2}^{\alpha*\beta}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}n^{\delta k-1}\mid T_{n,r}^{\alpha*\beta}\mid^k<\infty,\quad for\quad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
&\sum_{n=2}^{m+1}n^{\delta k-1}\mid T_{n,1}^{\alpha*\beta}\mid^k\\
&\leq\sum_{n=2}^{m+1}n^{\delta k-1}(A_n^{\alpha+\beta})^{-k}\{\sum_{v=1}^{n-1}(A_v^{\alpha+\beta})^k(\theta_v^{\alpha*\beta})^k\mid\Delta\lambda_v\mid^k\}\times\{\sum_{v=1}^{n-1}1\}^{k-1}\\
&=O(1)\sum_{n=2}^{m+1}n^{\delta k-2+k-(\alpha+\beta)k}\{\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(\theta_v^{\alpha*\beta})^k\eta_v^k\}\\
&=O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\theta_v^{\alpha*\beta})^k\eta_v^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha+\beta-\delta-1)k}}\\
&=O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\theta_v^{\alpha*\beta})^k\eta_v^k\int_v^{\infty}\frac{dx}{x^{2+(\alpha+\beta-\delta-1)k}}\\
&=O(1)\sum_{v=1}^{m}(\theta_v^{\alpha*\beta})^k\eta_v\eta_v^{k-1}v^{\delta k+k-1}\\
&=O(1)\sum_{v=1}^{m}(\theta_v^{\alpha*\beta})^k\eta_v\left(\frac{1}{vX_v}\right)^{k-1}v^{\delta k+k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v\eta_v)\sum_{r=1}^{v} r^{\delta k}\frac{(\theta_r^{\alpha*\beta})^k}{rX_r^{k-1}} + O(1)m\eta_m\sum_{v=1}^{m} v^{\delta k}\frac{(\theta_v^{\alpha*\beta})^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\eta_v)\mid X_v + O(1)m\eta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\eta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\eta_v X_v + O(1)m\eta_m X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1}\mid T_{n,2}^{\alpha*\beta}\mid^k &= \sum_{n=1}^{m}\mid\lambda_n\mid^{k-1}\mid\lambda_n\mid n^{\delta k}\frac{(\theta_n^{\alpha*\beta})^k}{n}\\
&= O(1)\sum_{n=1}^{m-1}\Delta\mid\lambda_n\mid\sum_{v=1}^{n} v^{\delta k}\frac{(\theta_v^{\alpha*\beta})^k}{vX_v^{k-1}}\\
&+ O(1)\mid\lambda_m\mid\sum_{n=1}^{m} n^{\delta k}\frac{(\theta_n^{\alpha*\beta})^k}{nX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\eta_n X_n + O(1)\mid\lambda_m\mid X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

Remark 2. If we take (X_n) as an almost increasing sequence, $\beta = 0$ and $\delta = 0$, then we obtain a theorem dealing with the $\mid C,\alpha\mid_k$ summability factors. Also, if we take $\delta = 0$, then we get a new result concerning the $\mid C,\alpha*\beta\mid_k$ summability factors of infinite series.

ACKNOWLEDGEMENT

The authors express their thanks to the referee for his/her valuable comments and suggestions.

REFERENCES

[1] N. K. Bari and S. B. Stečkin, "Best approximation and differential properties of two conjugate functions," *Trudy. Moskov. Mat. Obšč.*, vol. 5, no. 1, pp. 483–522, 1956.

[2] H. Bor, "An application of almost increasing sequences," *Math. Inequal. Appl.*, vol. 5, no. 1, pp. 79–83, 2002.

[3] H. Bor, "On a new application of quasi power increasing sequences," *Proc. Est. Acad. Sci.*, vol. 57, no. 4, pp. 205–209, 2008.

[4] H. Bor, "An application of almost increasing sequences," *Appl. Math. Lett.*, vol. 24, no. 3, pp. 298–301, 2011.
[5] D. Borwein, "Theorems on some methods of summability," *Quart. J. Math. Oxford Ser.(2)*, vol. 9, no. 1, pp. 310–316, 1958.
[6] G. Das, "A tauberian theorem for absolute summability," *Proc. Camb. Phil. Soc.*, vol. 67, no. 2, pp. 321–326, 1970.
[7] T. M. Flett, "On an extension of absolute summability and some theorems of littlewood and paley," *Proc. London Math. Soc.*, vol. 7, no. 1, pp. 113–141, 1957.
[8] T. M. Flett, "Some more theorems concerning the absolute summability of fourier series," *Proc. London Math. Soc.*, vol. 8, no. 3, pp. 357–387, 1958.
[9] L. Leindler, "A new application of quasi power increasing sequences," *Publ. Math. Debrecen*, vol. 58, no. 4, pp. 791–796, 2001.
[10] W. T. Sulaiman, "A note on $|A|_k$ summability factors of infinite series," *Appl. Math. Comput.*, vol. 216, no. 9, pp. 2645–2648, 2010.

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