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OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MT-CONVEX FUNCTIONS

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Abstract. Some inequalities of Ostrowski type for MT-convex functions via fractional integrals are obtained. These results not only generalize those of [25], but also provide new estimates on these types of Ostrowski inequalities for fractional integrals.

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1. INTRODUCTION

The following result is known in the literature as the Ostrowski inequality (see [17, page 468] or [18]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a,b]$.

Theorem 1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a,b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall\, x \in [a,b]. \tag{1.1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [1, 4, 5, 8, 10–15, 20, 24] and the references cited therein). In [23] (see also [25, 26]), Tunç and Yidirim defined the following so-called MT-convex function:

Definition 1. A function: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of MT(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.2}$$

In [25], Tunç derived some inequalities of Ostrowski type for MT-convex functions.

Theorem 2. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$ *such that* $f' \in L_1[a,b]$. *If* $|f'|$ *is MT-convex on* $[a,b]$ *and* $|f'(x)| \le M$, $x \in [a,b]$, *then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{M\pi\left[(x-a)^2+(b-x)^2\right]}{4(b-a)} \tag{1.3}$$

for each $x \in [a,b]$.

Theorem 3. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$ *such that* $f' \in L_1[a,b]$. *If* $|f'|^q$ *is MT-convex on* $[a,b]$, $q > 1$, $p^{-1}+q^{-1} = 1$ *and* $|f'(x)| \le M$, $x \in [a,b]$, *then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{1.4}$$

for each $x \in [a,b]$.

Theorem 4. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$ *such that* $f' \in L_1[a,b]$. *If* $|f'|^q$ *is MT-convex on* $[a,b]$, $q \ge 1$ *and* $|f'(x)| \le M$, $x \in [a,b]$, *then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}} \pi^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{1.5}$$

for each $x \in [a,b]$.

Fractional calculus [7, 16, 19] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 2. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \;\; x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \;\; x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here, $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional integral inequalities, we refer the reader to the papers [2,3,6,9,21] and the reference cited therein.

Motivated by these results, in the present paper, we establish some Ostrowski type inequalities for MT-convex functions via Riemann-Liouville fractional integrals. So, new estimates on these types of Ostrowski inequalities via fractional integrals are provided and the results of [25] are generalized.

2. Ostrowski type fractional integral inequalities for MT-convex functions

In this section, we apply the following fractional integral identity from Set [22] to derive some new Ostrowski type fractional integral inequalities for MT-convex functions.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L_1[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$, one has*

$$\begin{aligned}&\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\\&=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)\,dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)\,dt.\end{aligned}\tag{2.1}$$

Using this lemma, we can obtain the following Ostrowski type fractional integral inequalities for MT-convex functions.

Theorem 5. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|$ is MT-convex on $[a,b]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$\begin{aligned}&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|\\&\le M\frac{\Gamma(\alpha+\frac12)\Gamma(\frac12)}{2\Gamma(\alpha+1)}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}.\end{aligned}\tag{2.2}$$

Proof. From (2.1) and since $|f'|$ is MT-convex, we have

$$\begin{aligned}&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|\\&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt\\&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt\end{aligned}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|\right]dt$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(b)\right|\right]dt$$
$$\leq \frac{M\,(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt$$
$$+\frac{M\,(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt$$
$$=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\int_0^1\left[t^{\alpha+\frac{1}{2}}(1-t)^{-1/2}+t^{\alpha-\frac{1}{2}}(1-t)^{1/2}\right]\mathrm{d}t$$
$$=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\left[\beta\left(\alpha+\frac{3}{2},\frac{1}{2}\right)+\beta\left(\alpha+\frac{1}{2},\frac{3}{2}\right)\right]$$
$$=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)},$$

where we have used the Beta function of Euler type, which is defined as

$$\beta\,(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt=\frac{\Gamma\,(x)\,\Gamma\,(y)}{\Gamma\,(x+y)},\quad \forall\; x,y>0.$$

The proof is completed. □

Remark 1. In Theorem 5, if we choose $\alpha=1$, we get the inequality in Theorem 2.

Theorem 6. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(b)\right]\right|$$
$$\leq\frac{M}{(1+p\alpha)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}. \tag{2.3}$$

Proof. From Lemma 1 and using the well-known Hölder's inequality, we have

$$\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)a)\right|dt$$

$$
\begin{aligned}
&+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^q$ is MT-convex and $|f'(x)| \leq M$, we get

$$
\begin{aligned}
\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt &\leq \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|^q\right]dt \\
&\leq M^q\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt=\frac{\pi}{2}M^q
\end{aligned}
$$

and similarly

$$
\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \leq \frac{\pi}{2}M^q.
$$

By simple computation, we have

$$
\int_0^1 t^{p\alpha}dt=\frac{1}{p\alpha+1}.
$$

Using these results, we complete the proof of (2.3). □

Remark 2. In Theorem 6, if we choose $\alpha=1$, we get the inequality in Theorem 3.

Theorem 7. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q\geq 1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$
\begin{aligned}
&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \\
&\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}. \qquad (2.4)
\end{aligned}
$$

Proof. From Lemma 1 and using the well-known power mean inequality, we have

$$
\begin{aligned}
&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt
\end{aligned}
$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right| dt$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is MT-convex on $[a,b]$ and $|f'(x)| \leq M$, we get

$$\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt$$
$$\leq \int_0^1 \left[t^\alpha \frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q + t^\alpha \frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|^q\right] dt$$
$$\leq M^q \int_0^1 \left[t^\alpha \frac{\sqrt{t}}{2\sqrt{1-t}} + t^\alpha \frac{\sqrt{1-t}}{2\sqrt{t}}\right] dt = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)} M^q$$

and similarly

$$\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt \leq \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)} M^q.$$

Using these inequalities, we complete the proof of (2.4). □

Remark 3. In Theorem 7, if we choose $\alpha = 1$, we get the inequality in Theorem 4.

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