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MODULES THAT HAVE A SUPPLEMENT IN EVERY COATOMIC EXTENSION

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Abstract. Let R be a ring and M be an R-module. M is said to be *an E^*-module (respectively, an EE^*-module)* if M has a supplement (respectively, ample supplements) in every coatomic extension N, i.e. $\frac{N}{M}$ is coatomic. We prove that if a module M is an EE^*-module, every submodule of M is an E^*-module, and then we show that a ring R is left perfect iff every left R-module is an E^*-module iff every left R-module is an EE^*-module. We also prove that the class of E^*-modules is closed under extension. In addition, we give a new characterization of left V-rings by cofinitely injective modules.

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1. INTRODUCTION

Throughout this paper, we assume that all rings are associative with identity and all modules are unital left modules. Let M be such a module. As usual, the notation $K \subseteq M$ means that K is a submodule of M. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap U \neq 0$ for every non-zero submodule $U \subseteq M$. Dually, a proper submodule K of M is said to be *small* in M and denoted by $K << M$, if $M \neq K + T$ for every proper submodule T of M. By radical of M, denoted by $Rad(M)$, we will indicate the sum of all small submodules, or, equivalently intersection of all maximal submodules of M (see [8]). If $M = Rad(M)$, i.e., M has no maximal submodules, M is called *radical*.

Let M be a module. M is said to be *coatomic* if $Rad(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$ for some submodule K of M in [9]. M is coatomic if and only if every proper submodule of M is contained in a maximal submodule of M. Semisimple modules are coatomic. In addition, every factor module of a coatomic module is again coatomic.

Let $U, V \subseteq M$ be modules. V is called *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V << V$. A submodule S of M has *ample supplements* in M if, whenever $M = S + L$, L contains a supplement K of S in M [8].

Let $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ be a short exact sequence of modules. Then, N is called *an extension* of M by K. To simplify the notation, we think of M as a submodule N. In [3], a module N is said to be a *cofinite extension* of M provided $M \subseteq N$ and $\frac{N}{M}$ is finitely generated. In light of this fact that finitely generated modules are coatomic, we call a module N *coatomic extension* of M if the factor module $\frac{N}{M}$ of N is coatomic.

It is well known that a module M is injective if and only if it is a direct summand of every extension N of M. Since every direct summand is a supplement, Zöschinger studied in [10] modules that have a supplement in every extension and termed these modules the property (E) as a generalization of injective modules. In particular, he proved in [10, Lemma 1.2] that every submodule of a module M has the property (E) if and only if M has ample supplements in every extension, namely the property (EE). It is obvious that the class of modules with the property (EE) contains properly artinian modules.

In [3], a module M is said to have *the property* (CE) (respectively, *the property* (CEE)) if M has a supplement (respectively, ample supplements) in every cofinite extension. It is shown in [3, Theorem 2.12] that R is semiperfect if and only if every left R-module has the property (CE).

Let M be a module. We call M *an* E^**-module* if M has a supplement in every coatomic extension, and M *an* EE^**-module* if it has ample supplements in every coatomic extension. The notation of E^*-modules lies between modules with (E) and modules with the property (CE). Some examples are given to show that these inclusions are proper.

In this paper, we study some basic properties of E^*-modules and EE^*-modules. We show that the class of E^*-modules closed under finite direct sums, extensions and direct summands. We prove that, over a left hereditary ring, every factor module of a coatomic E^*-module is an E^*-module. In Proposition 2, we show that if M is an EE^*-module, then every submodule of M is an E^*-module. This gives us that over semilocal rings every EE^*-module is strongly radical supplemented. Also, we prove in Theorem 2 that a ring R is left perfect if and only if every left R-module is an E^*-module if and only if every left R-module is an EE^*-module. In addition, we show that every simple left R-module is cofinitely injective if and only if, for every finitely generated left R-module M, $Rad(M) = 0$ if and only if R is a left V-ring (i.e., rings whose simple left modules are injective). Finally, we prove in Proposition 5 that every E^*-module over a left V-ring is injective.

2. E^*-Modules and EE^*-Modules

It is clear that every module with the property (E) is an E^*-module, but the following example shows that an E^*-module doesn't have the property (E), in general.

Recall from [1] that a module M is called *strongly radical supplemented* if every submodule N of M containing $Rad(M)$ has a supplement in M. It is proven in [1, Corollary 2.1] that finite sums of strongly radical supplemented modules are strongly radical supplemented. Note that every radical module is strongly radical supplemented.

Example 1. For a non-complete local dedekind domain R, let M be the direct sum of left R-modules R^*, $K^{(I)}$ and R, where R^* is the completion of R, K is the quotient field of R and I is an index set, respectively. Since injective modules over a dedekind domain are strongly radical supplemented, it follows from [10, Lemma 3.3] that M is an E^*-module. On the other hand, M doesn't have the property (E) by [10, Theorem 3.5].

It is shown in [10, Lemma 1.3 (a)] that direct summands of modules with the property (E) have the property (E). Now we give an analogue of this fact for E^*-modules.

Proposition 1. *Every direct summand of an E^*-module is an E^*-module.*

Proof. Let M be an E^*-module and U be a direct summand of M. Then, we can write $M = U \oplus V$ for some submodule V of M. For any coatomic extension T of U, we consider the external direct product of these modules T and V. Put $W = T \oplus V$. Now we take the monomorphism $\Phi : M \longrightarrow W$ by $\Phi(m) = \Phi(u + v) = (u, v)$ for all $m = u + v \in U \oplus V = M$. It can be seen that $\Phi(M)$ is an E^*-module. Now

$$\frac{W}{\Phi(M)} = \frac{T \oplus V}{\Phi(M)} \cong \frac{T}{U}$$

is coatomic. It follows that $\Phi(M)$ has a supplement, say $U^{'}$, in W. Therefore, $T = U + \Psi(U^{'})$, where $\Psi : W \longrightarrow T$ is the projection. Since $ker(\Psi) \subseteq \Phi(M)$, we have $U \cap \Psi(U^{'}) << \Psi(U^{'})$ by [8, 19.3]. Hence, $\Psi(U^{'})$ is a supplement of U in T. □

A submodule of an E^*-module need not be an E^*-module, in general. To see this actuality, we shall consider the left $\mathbb{Z}$-modules $\mathbb{Z} \subseteq \mathbb{Q}$. But we have:

Proposition 2. *If M is an EE^*-module, then every submodule U of M is an E^*-module.*

Proof. Let N be a coatomic extension of U. We shall show that U has a supplement in N. By W, we denote the external direct product of M and N. Put $F = \frac{W}{W'}$, where the submodule $W^{'} = \{(u, -u) \in W \mid u \in U\} \subseteq W$. For these inclusion homomorphism $\iota_1 : U \longrightarrow N$ and $\iota_2 : U \longrightarrow M$, we can draw the pushout in the following:

$$\begin{array}{ccc} U & \xrightarrow{\iota_1} & N \\ \Big\downarrow{\scriptstyle \iota_2} & & \Big\downarrow{\scriptstyle \phi} \\ M & \xrightarrow{\xi} & F \end{array}$$

where ξ and ϕ are monomorphisms. Then $F = Im(\xi) + Im(\phi)$. Therefore $\frac{N}{U} \cong \frac{F}{Im(\xi)}$ is coatomic. Since $Im(\xi) \cong M$ is an EE^*-module, there exists a submodule L of $Im(\phi)$ such that L is a supplement of $Im(\xi)$ in F. Now

$$N = \phi^{-1}(F) = \phi^{-1}(Im(\xi)) + \phi^{-1}(L) = U + \phi^{-1}(L)$$

and

$$U \cap \phi^{-1}(L) << \phi^{-1}(L).$$

This means that $\phi^{-1}(L)$ is a supplement of U in N. □

A ring R is called *semilocal* if $\frac{R}{Rad(R)}$ is a semisimple artinian ring ([8]). The following corollary is an immediate consequence of Proposition 2.

Corollary 1. *Let M be an EE^*-module over a semilocal ring R. Then, M is strongly radical supplemented.*

Proof. Let $Rad(M) \subseteq U \subseteq M$. Then $\frac{M}{U}$ is a factor module of $\frac{M}{Rad(M)}$. Since R is a semilocal ring, $\frac{M}{Rad(M)}$ is semisimple as a $\frac{R}{Rad(R)}$-module. Therefore, $\frac{M}{U}$ is a coatomic R-module. By the hypothesis, U has a supplement in M. This means that M is strongly radical supplemented. □

Let Γ be a class of modules. Then, Γ is called *closed under extension* if $M, \frac{N}{M} \in \Gamma$ implies $N \in \Gamma$. The following crucial lemma is used to show that the class of E^*-modules is closed under extensions.

Lemma 1. *Let M be a module and K be a small submodule of M. Then, M is coatomic if and only if the factor module $\frac{M}{K}$ is coatomic.*

Proof. ($\Longrightarrow$) It is clear.
($\Longleftarrow$) Let U be a proper submodule of M. Since $K \ll M$, then $\frac{U+K}{K}$ is a proper submodule of $\frac{M}{K}$. Since $\frac{M}{K}$ is coatomic, $\frac{U+K}{K}$ is contained in a maximal submodule of $\frac{M}{K}$, say $\frac{V}{K}$. Therefore, V is a maximal submodule of M. Hence, M is coatomic as required. □

Recall from [9, Lemma 1.5 (a)] that the class of coatomic modules is closed under extensions.

Theorem 1. *Let $M \subseteq N$ be modules. If M and $\frac{N}{M}$ are E^*-modules, then N is an E^*-module.*

Proof. Let K be a coatomic extension of N. For $M \subseteq N \subseteq K$,

$$\frac{K}{N} \cong \frac{\frac{K}{M}}{\frac{N}{M}}$$

is coatomic, and thus $\frac{K}{M}$ is a coatomic extension of $\frac{N}{M}$. By the hypothesis, the submodule $\frac{N}{M}$ has a supplement, say $\frac{L}{M}$, in $\frac{K}{M}$. So we can write $\frac{N}{M} + \frac{L}{M} = \frac{K}{M}$ and $\frac{N}{M} \cap \frac{L}{M} = \frac{N \cap L}{M} \ll \frac{L}{M}$. Therefore, $K = N + L$. Now

$$\frac{\frac{L}{M}}{\frac{N\cap L}{M}} \cong \frac{L}{N \cap L} \cong \frac{N+L}{N} = \frac{K}{N}$$

is coatomic. Applying Lemma 1, we obtain that $\frac{L}{M}$ is coatomic. Since M is an E^*-module, there exists a submodule $M^{'}$ of L such that $M + M^{'} = L$ and $M \cap M^{'} \ll M^{'}$. Then, $K = N + L = N + (M + M^{'}) = N + M^{'}$. Assume that $N + M^{''} = K$ for some submodule $M^{''} \subseteq M^{'}$. Then, $M + M^{''} \subseteq L$. Since $\frac{L}{M}$ is a supplement of $\frac{N}{M}$ in $\frac{K}{M}$, it follows that $L = M + M^{''}$. By the minimality of $M^{'}$, we have $M^{''} = M^{'}$. Therefore M is an E^*-module. □

Note that, by Theorem 1, a finitely generated semisimple module is an E^*-module.

Corollary 2. *Let M be a module and K be a maximal submodule of M. If K is an E^*-module, then M is an E^*-module. In particular, modules containing a simple maximal submodule are E^*-modules.*

Proof. Let K be an E^*-module. Since simple modules are E^*-modules, the factor module $\frac{M}{K}$ is an E^*-module. Applying Theorem 1, we get M is an E^*-module. □

Now we can prove every finite direct sum of E^*-modules in the following Proposition.

Proposition 3. *Let M_i $(i \in I)$ be any finite collection of E^*-modules and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Then, M is an E^*-module.*

Proof. In order to show that M is an E^*-module, we use induction on n. Suppose that $n = 2$. Other case can prove by a similar way. Let $M = M_1 \oplus M_2$. Then, $M_2 \cong \frac{M}{M_1}$. By the hypothesis and Theorem 1, we obtain that M is an E^*-module. □

Recall that over a left hereditary ring every factor module of an injective module is injective. In the following, we show that every factor module of a coatomic E^*-module over a left hereditary ring is an E^*-module.

Proposition 4. *Let R be a left hereditary ring and M be a coatomic E^*-module. Then every factor module of M is an E^*-module.*

Proof. For any submodule U of M, let N be a coatomic extension of $\frac{M}{U}$. Then, N is coatomic. By $E(M)$, we denote the injective hull of M. Since R is left hereditary, $\frac{E(M)}{U}$ is injective, and so there exists a commutative diagram with exact rows in the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\iota_1} & M & \xrightarrow{\pi} & \frac{M}{U} & \longrightarrow & 0 \\ & & \downarrow{id} & & \downarrow{\Psi} & & \downarrow{\iota_2} & & \\ 0 & \longrightarrow & U & \xrightarrow{f} & K & \xrightarrow{\Phi} & N & \longrightarrow & 0 \end{array}$$

i.e., $f\,id = \Psi\iota_1$ and $\Phi\Psi = \iota_2\pi$, where $\Psi : M \longrightarrow K$ is a monomorphism by [7, Lemma 2.16] . It follows that $N \cong \frac{K}{\Psi(M)}$. Since M is an E^*-module, $\Psi(M)$ has a supplement, say V, in K. By the last part proof of Proposition 2, we obtain that $\Phi(V)$ is a supplement of $\frac{M}{U}$ in N. Hence, $\frac{M}{U}$ is an E^*-module. □

Recall from [6] that an epimorphism $f : P \longrightarrow M$ is called *a small cover* if $Ker(f) << P$, and a projective module P together with a small cover $f : P \longrightarrow M$ is called *a projective cover* of M. A ring R is called *semiperfect* if every finitely generated left (or right) R-module has a projective cover, and it is called *left perfect* if every left R-module has a projective cover.

It is known that a ring R is semiperfect if and only if R is semilocal and idempotents can be lifted modulo $Rad(R)$, and it is left perfect if and only if R is semilocal and $Rad(R)$ is a left t-nilpotent ideal. Local rings are semiperfect ([6]).

Now we give a characterization of left perfect rings via E^*-modules. Firstly, we have the following lemma.

Lemma 2. *The following statements are equivalent over an arbitrary ring.*

(1) *Every left module is an E^*-module.*
(2) *Every left module is an EE^*-module.*

Proof. Suppose that every left module is an E^*-module. Let M be any module. For a coatomic extension N of M, let $N = M + S$ for some submodule S of N. Then $\frac{N}{M} \cong \frac{S}{M \cap S}$ is coatomic. By the hypothesis, $M \cap S$ has a supplement in S, say W. So we can write $S = (M \cap S) + W$ and $(M \cap S) \cap W = M \cap W \ll W$. Then we have $N = M + W$ and $N \cap W \ll W$. Therefore, M is an EE^*-module. The converse is clear by definitions. □

Theorem 2. *Let R be a ring. The following three statements are equivalent.*

(1) *R is left perfect.*
(2) *Every left R-module is an E^*-module.*
(3) *Every left R-module is an EE^*-module.*

Proof. (1) ⇒ (2) By [4, 39.9], over a left perfect ring every left module has the property (E). This completes the proof of (2).

(2) ⇒ (3) It follows from Lemma 2.

(3) ⇒ (1) Since EE^*-modules have the property (CEE), by the hypothesis, every R-module has the property (CEE). It follows from [3, Theorem 2.12] that R is semiperfect. Therefore, R is semilocal.

Now it is enough to show that every left R-module is strongly radical supplemented by [2, Theorem 1]. Let M be any left R-module. By the hypothesis, M is an EE^*-module. Applying Corollary 1, we deduce that M is strongly radical supplemented. □

Now we give an example of a module, which has the property (CE), but not an E^*-module.

Example 2. Let p be a prime integer in $\mathbb{Z}$. Consider the local dedekind domain $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } p \nmid b\}$. Let N be the left R-module $R^{(\mathbb{N})}$. Put $M = Rad(N)$. Since R is a local ring, the factor module $\frac{N}{M}$ is semisimple as a $\frac{R}{Rad(R)}$−module. Therefore, $\frac{N}{M}$ is a semisimple R-module and so N is a coatomic extension of M. It follows from [3, Theorem 2.12] that M has the property (CE). On the other hand, M is not an E^*-module by [2, Theorem 1].

In [5], a ring R is said to be *a left V-ring* if every simple left R-module is injective. It is well known that a ring R is a left V-ring if and only if $Rad(M) = 0$ for every left R-module M. Recall from [3] that a module M is called *cofinitely injective* if M is a direct summand of every cofinite extension N of M. Clearly, injective modules are cofinitely injective, and a cofinitely injective module has the property (CE). Now we have the next result:

Proposition 5. *Let R be a left V-ring and M be an E^*-module over the ring. Then, M is injective.*

Proof. Let N be an extension of M. Since every module over a left V-ring is coatomic, $\frac{N}{M}$ is coatomic. Therefore, N is a coatomic extension of M. By assumption, we can write $N = M + K$ and $M \cap K << K$ for some submodule $K \subseteq N$. Since R is a left V-ring, we obtain that $M \cap K \subseteq Rad(K) \subseteq Rad(N) = 0$. This means that M is a direct summand of N. Hence, M is injective. □

Theorem 3. *The following statements are equivalent for a ring R.*

(1) *Every simple left R-module is cofinitely injective.*
(2) *If M is a finitely generated left R-module, $Rad(M) = 0$.*
(3) *Every proper left ideal I of R is an intersection of maximal left ideals.*

Proof. (1) $\Longrightarrow$ (2) Let M be an arbitrary finitely generated left R-module and let $m \in Rad(M)$. We claim that $m = 0$. Put $K = Rm$. Then, K has a maximal submodule L. Therefore, $\frac{K}{L}$ is a simple left R-module and by assumption $\frac{K}{L}$ is cofinitely injective. Now, for $L \subseteq K \subseteq M$ modules,

$$\frac{\frac{M}{L}}{\frac{K}{L}} \cong \frac{M}{K}$$

is coatomic since M is finitely generated. So there exists the decomposition $\frac{M}{L} = \frac{K}{L} \oplus \frac{T}{L}$ for some submodule $\frac{T}{L} \subseteq \frac{M}{L}$. Note that

$$\frac{K}{L} \cong \frac{\frac{M}{L}}{\frac{T}{L}} \cong \frac{M}{T}$$

is simple. Thus, T is a maximal submodule of M. Therefore, $m \in K \cap T \subseteq L$. It follows that $m = 0$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) follow from [5, Theorem 6.1]. □

As a consequence of the above, we have the following.

Corollary 3. *Let R be a ring. Then, R is a left V-ring if and only if every simple left R-module is cofinitely injective.*

Proof. The proof follows from Theorem 3 and [5, Theorem 6.1]. □

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