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# Representation of solutions of neutral differential equations with delay and linear parts defined by pairwise permutable matrices

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# REPRESENTATION OF SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH DELAY AND WITH LINEAR PARTS DEFINED BY PAIRWISE PERMUTABLE MATRICES

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*Abstract.* This paper is devoted to the study of linear systems of neutral differential equations with delay. Assuming the linear parts to be given by pairwise permutable matrices, representation of a solution of a nonhomogeneous initial value problem using matrix polynomial of a degree depending on time is derived. Examples illustrating the obtained results are given.

2010 *Mathematics Subject Classification:* 34K40; 34K06; 39B12

*Keywords:* neutral differential equations, delayed equation, matrix polynomial

## 1. INTRODUCTION AND PRELIMINARIES

In [8], the authors considered the initial value problem for the system of delay differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-\tau) + f(t), \quad t \geq 0, \\ x(t) &= \varphi(t), \quad -\tau \leq t \leq 0 \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $A$, $B$ are $n \times n$ matrices, $\tau > 0$ is a delay, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$ is a given initial function, and $f \in C([0,\infty), \mathbb{R}^n)$ is a given function. They proved the following result.

**Theorem 1.** *Let $A$, $B$ be $n \times n$ permutable matrices, i.e. $AB = BA$, $f \in C([0,\infty), \mathbb{R}^n)$ and $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$ be given functions. Any solution of the initial value problem (1.1) has the form*

$$\begin{aligned} x(t) = \mathrm{e}^{A(t+\tau)} \mathrm{e}_\tau^{B_1 t} \varphi(-\tau) &+ \int_{-\tau}^{0} \mathrm{e}^{A(t-s)} \mathrm{e}_\tau^{B_1(t-\tau-s)} (\varphi'(s) - A\varphi(s)) \mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{A(t-s)} \mathrm{e}_\tau^{B_1(t-\tau-s)} f(s) \mathrm{d}s, \end{aligned}$$

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*where*

$$
e_\tau^{Bt} = \begin{cases} \Theta, & t < -\tau, \\ E, & -\tau \le t < 0, \\ E + Bt + B^2\frac{(t-\tau)^2}{2} + \cdots + B^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \le t < k\tau, k \in \mathbb{N}, \end{cases} \tag{1.2}
$$

*$\Theta$ and $E$ are the zero and the identity matrix, respectively, and $B_1 = e^{-A\tau}B$.*

Here $e_\tau^{Bt}$ is called delayed matrix exponential. Note that it is a matrix polynomial of time-dependent degree.

Theorem 1 led to stability and controllability results for differential equations with one or multiple delays, existence results for boundary value problems, representation of solutions of oscillating systems, as well as results in the theory of difference, partial differential, and functional differential equations (see e.g. [3–5, 10, 13–15] and references therein).

Note that in [14], a result, analogical to Theorem 1 with $\varphi \in C([-\tau,0],\mathbb{R}^n)$, was derived for differential equations with variable delay. Here we recall the result for a constant delay.

**Theorem 2.** *Let $A$, $B$ be $n \times n$ permutable matrices, i.e., $AB = BA$, $f \in C([0,\infty),\mathbb{R}^n)$ and $\varphi \in C([-\tau,0],\mathbb{R}^n)$ be given functions. Any solution of the initial value problem (1.1) has the form*

$$
x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ e^{At}e_\tau^{B_1(t-\tau)}\varphi(0) + \displaystyle\int_{-\tau}^{0} e^{A(t-s)}e_\tau^{B_1(t-2\tau-s)}B_1\varphi(s)\mathrm{d}s & \\ + \displaystyle\int_0^t e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}f(s)\mathrm{d}s, & 0 \le t. \end{cases} \tag{1.3}
$$

In the present paper, we derive representation of solutions of the initial value problem for neutral differential equation with a delay

$$
\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(t-\tau) + f(t), \quad t \ge 0, \tag{1.4}
$$

$$
x(t) = \varphi(t), \quad -\tau \le t \le 0 \tag{1.5}
$$

for $\varphi \in C^1([-\tau,0],\mathbb{R}^n)$, on suppose that the matrices $A$, $B$, $C$ are pairwise permutable, i.e., $AB = BA$, $BC = CB$, $CA = AC$. So at the end of Section 2, we obtain a result similar to Theorem 2 (see Theorem 6 below), which can be used when investigating stability, controllability, observability, boundary value problems, etc., for neutral differential equations. When investigating these problems one may find the papers [2,3,6,9–14] useful, where analogical problems for differential equations with delay are discussed.

In Section 3, we provide examples of systems covered by the discussed theory. We note that a family of permutable matrices can be constructed using e.g. [1].

Throughout the paper, we consider $\tau > 0$, $\mathbb{N}_0$ denotes the set of all nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\theta$ denotes the $n$-dimensional vector of zeros. Moreover, we assume the property of empty sum and empty product, i.e.,

$$\sum_{i \in \emptyset} f(i) = 0, \qquad \sum_{i \in \emptyset} F(i) = \Theta, \qquad \prod_{i \in \emptyset} f(i) = 1, \qquad \prod_{i \in \emptyset} F(i) = E$$

for any function $f$ and matrix function $F$, whether they are defined or not for indicated argument. Finally, $\lfloor \cdot \rfloor$ denotes the floor function.

## 2. MAIN RESULTS

In this section, we derive a representation of solutions of the initial value problem (1.4), (1.5) using piecewise defined matrix polynomial.

First, we consider the matrix equation of the form

$$\dot{X}(t) - C\dot{X}(t-\tau) = BX(t-\tau) \tag{2.1}$$

for $t \geq 0$ with initial condition

$$X(t) = \begin{cases} \Theta, & t < 0, \\ E, & t = 0. \end{cases} \tag{2.2}$$

The solution of (2.1), (2.2) is called the fundamental matrix solution. In fact, this solution solves equation (2.1) on

$$M := \bigcup_{k \in \mathbb{N}_0} (k\tau, (k+1)\tau),$$

and the discontinuity of the initial condition is transfered to each $k\tau$, $k \in \mathbb{N}$ (see [7]). So, at the points $k\tau$, $k \in \mathbb{N}$ we consider right-sided derivatives. Equivalently to (2.1), $X$ satisfies the corresponding integral equation

$$X(t) = E + CX(t-\tau) + B\int_0^t X(s-\tau)\mathrm{d}s, \quad t \geq 0 \tag{2.3}$$

with jumps at $k\tau$, $k \in \mathbb{N}$. In the following theorem, we state the solution $X(t)$.

**Theorem 3.** *Let $B$, $C$ be $n \times n$ permutable matrices. Then the fundamental solution $X(t)$ of (2.1), (2.2) has the form*

$$X(t) = \begin{cases} \Theta, & t < 0, \\ \displaystyle\sum_{j=0}^{k} \sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!}, & t \in [k\tau, (k+1)\tau), k \in \mathbb{N}_0. \end{cases} \tag{2.4}$$

*Proof.* Let $t \in [0, \tau)$. Then by (2.4), $X(t) = E$, $X(t-\tau) = \dot{X}(t) = \dot{X}(t-\tau) = \Theta$. So $X$ solves (2.1) on $[0, \tau)$.

Now, let $t \in [k\tau, (k+1)\tau)$ for some $k \in \mathbb{N}$. Then

$$X(t-\tau) = \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^j}{j!},$$

and differentiating (2.4), we get

$$\begin{aligned} \dot{X}(t) &= \sum_{j=1}^{k} \sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}, \\ \dot{X}(t-\tau) &= \sum_{j=1}^{k-1} \sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^{j-1}}{(j-1)!}. \end{aligned} \tag{2.5}$$

Hence,

$$\begin{aligned} &BX(t-\tau) + C\dot{X}(t-\tau) \\ &= \underbrace{\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} C^i B^{j+1} \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^j}{j!}}_{=:S_1} \\ &+ \underbrace{\sum_{j=1}^{k-1} \sum_{i=0}^{k-j-1} C^{i+1} B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^{j-1}}{(j-1)!}}_{=:S_2}. \end{aligned}$$

Changing the variable $j+1 \to j$ in $S_1$, and $i+1 \to i$ in $S_2$ results in

$$\begin{aligned} &BX(t-\tau) + C\dot{X}(t-\tau) \\ &= \sum_{j=1}^{k} \sum_{i=0}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!} \\ &+ \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} C^i B^j \binom{i+j-1}{i-1} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}. \end{aligned} \tag{2.6}$$

Now, we write the first term as

$$\sum_{j=1}^{k} \sum_{i=0}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

$$= \sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

using the empty sum property. So we can join the last sum in the above equality and the last sum in (2.6), and apply the combinatorial identity

$$\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$$

for any $a, b \in \mathbb{N}$, $a \geq b$, to obtain

$$BX(t-\tau) + C\dot{X}(t-\tau)$$
$$= \sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}. \tag{2.7}$$

Note that the first term on the right-hand side of the latter equality can be rewritten as

$$\sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} = \sum_{j=1}^{k} C^0 B^j \binom{0+j}{0} \frac{[t-(0+j)\tau]^{j-1}}{(j-1)!}.$$

Consequently, the terms on the right-hand side of (2.7) can be summed (the last sum is extended to $j = 1, 2, \ldots k$ by the empty sum property), which results in

$$BX(t-\tau) + C\dot{X}(t-\tau) = \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

where the right-hand side is exactly $\dot{X}(t)$ of (2.5). The statement is proved. $\square$

*Remark* 1. Formula (2.4) was obtained using step-by-step integration method with the aid of integral equation (2.3).

Some properties of the fundamental solution at the points $k\tau$, $k \in \mathbb{N}_0$ are concluded in the next lemma.

**Lemma 1.** *Let $X$ be defined by (2.4). For any $k \in \mathbb{N}_0$ the following holds true:*

(1) $\lim_{t \to k\tau+} X(t) = \lim_{t \to k\tau-} X(t) + C^k$,

(2) $\lim_{t \to k\tau+} \dot{X}(t) = \lim_{t \to k\tau-} \dot{X}(t) + kC^{k-1}B$.

*Proof.* If $k = 0$, the statements can be proved easily. For better clarity, we shall use the lower index $k \in \mathbb{N}_0$ to denote

$$X_k(t) := \sum_{j=0}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!}, \quad t \geq k\tau. \tag{2.8}$$

Note that using the empty sum property, this notation can be extended to negative integers as $X_k(t) = \Theta$ for $k < 0$, $t \in \mathbb{R}$.

Let $k \in \mathbb{N}$ be arbitrary and fixed. Then

$$\begin{aligned}\lim_{t\to k\tau+} X(t) = \lim_{t\to k\tau+} X_k(t) &= \lim_{t\to k\tau+} \sum_{j=0}^{k-1}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!} \\ &= \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[k\tau-(i+j)\tau]^j}{j!} \\ &\quad + \lim_{t\to k\tau+} \sum_{j=0}^{k-1} C^{k-j} B^j \binom{k}{k-j} \frac{[t-k\tau]^j}{j!} \\ &= X_{k-1}(k\tau) + C^k = \lim_{t\to k\tau-} X(t) + C^k.\end{aligned}$$

To prove the second statement, we differentiate (2.8) to get

$$\dot{X}_k(t) = \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}, \quad t \geq k\tau.$$

Consequently,

$$\begin{aligned}\lim_{t\to k\tau+} \dot{X}(t) = \lim_{t\to k\tau+} \dot{X}_k(t) &= \lim_{t\to k\tau+} \sum_{j=1}^{k-1}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!} \\ &= \sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[k\tau-(i+j)\tau]^{j-1}}{(j-1)!} \\ &\quad + \lim_{t\to k\tau+} \sum_{j=1}^{k-1} C^{k-j} B^j \binom{k}{k-j} \frac{[t-k\tau]^{j-1}}{(j-1)!} \\ &= \dot{X}_{k-1}(k\tau) + kC^{k-1}B = \lim_{t\to k\tau-} \dot{X}(t) + kC^{k-1}B.\end{aligned}$$

The proof is complete. □

Now we find a solution of the homogeneous initial value problem.

**Theorem 4.** *Let $B$, $C$ be $n \times n$ permutable matrices, and $\varphi \in C^1([-\tau,0],\mathbb{R}^n)$ be a given function satisfying*

$$\dot{\varphi}(0) = B\varphi(-\tau) + C\dot{\varphi}(-\tau). \tag{2.9}$$

*Any solution of the initial value problem consisting of the equation*

$$\dot{x}(t) - C\dot{x}(t-\tau) = Bx(t-\tau), \quad t \geq 0 \tag{2.10}$$

*and initial condition* (1.5)*, has the form*

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ X(t)(\varphi(0)-C\varphi(-\tau))+B\displaystyle\int_{-\tau}^{0}X(t-\tau-s)\varphi(s)\mathrm{d}s & \\ +C\displaystyle\int_{-\tau}^{0}\dot{X}(t-\tau-s)\varphi(s)\mathrm{d}s+\psi(t), & 0\le t\end{cases}\tag{2.11}$$

*where* $X(t)$ *is a fundamental matrix solution given by* (2.4)*,*

$$\psi(t):=C^{\lfloor\frac{t}{\tau}\rfloor+1}\varphi\left(t-\left(\left\lfloor\frac{t}{\tau}\right\rfloor+1\right)\tau\right).\tag{2.12}$$

*Proof.* Let $t\in(0,\tau)$. Then by (2.11),

$$x(t)=\varphi(0)-C\varphi(-\tau)+B\int_{-\tau}^{t-\tau}\varphi(s)\mathrm{d}s+C\varphi(t-\tau).$$

Hence, $\dot{x}(t)=B\varphi(t-\tau)+C\dot{\varphi}(t-\tau)$, $x(t-\tau)=\varphi(t-\tau)$ and $\dot{x}(t-\tau)=\dot{\varphi}(t-\tau)$. So it is easy to see that, $x$ solves (2.10) on $(0,\tau)$. Moreover,

$$\lim_{t\to0^+}x(t)=\varphi(0)-C\varphi(-\tau)+C\varphi(-\tau)=\varphi(0)=\lim_{t\to0^-}x(t)$$

and

$$\lim_{t\to0^+}\dot{x}(t)=B\varphi(-\tau)+C\dot{\varphi}(-\tau)=\dot{\varphi}(0)=\lim_{t\to0^-}\dot{x}(t)$$

by (2.9). Thus $x$ and $\dot{x}$ are both continuous at 0.

Now, let $t\in(k\tau,(k+1)\tau)$ for some $k\in\mathbb{N}$. Using the notation (2.8) and definition (2.12) of function $\psi$, we can write

$$\begin{aligned}x(t)&=X_k(t)(\varphi(0)-C\varphi(-\tau))\\&+\int_{-\tau}^{t-(k+1)\tau}\left(BX_k(t-\tau-s)+C\dot{X}_k(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\&+\int_{t-(k+1)\tau}^{0}\left(BX_{k-1}(t-\tau-s)+C\dot{X}_{k-1}(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\&+C^{k+1}\varphi(t-(k+1)\tau).\end{aligned}\tag{2.13}$$

Hence,

$$\begin{aligned}x(t-\tau)&=X_{k-1}(t-\tau)(\varphi(0)-C\varphi(-\tau))\\&+\int_{-\tau}^{t-(k+1)\tau}\left(BX_{k-1}(t-2\tau-s)+C\dot{X}_{k-1}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\&+\int_{t-(k+1)\tau}^{0}\left(BX_{k-2}(t-2\tau-s)+C\dot{X}_{k-2}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\&+C^{k}\varphi(t-(k+1)\tau).\end{aligned}\tag{2.14}$$

Differentiating (2.13) and applying Lemma 1 leads to

$$\begin{aligned}\dot{x}(t) &= \dot{X}_k(t)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{t-(k+1)\tau}\big(B\dot{X}_k(t-\tau-s)+C\ddot{X}_k(t-\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+\int_{t-(k+1)\tau}^{0}\big(B\dot{X}_{k-1}(t-\tau-s)+C\ddot{X}_{k-1}(t-\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+(k+1)C^kB\varphi(t-(k+1)\tau)+C^{k+1}\dot{\varphi}(t-(k+1)\tau).\end{aligned}\tag{2.15}$$

On the other side, differentiating (2.14) and applying Lemma 1 gives

$$\begin{aligned}\dot{x}(t-\tau) &= \dot{X}_{k-1}(t-\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{t-(k+1)\tau}\big(B\dot{X}_{k-1}(t-2\tau-s)+C\ddot{X}_{k-1}(t-2\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+\int_{t-(k+1)\tau}^{0}\big(B\dot{X}_{k-2}(t-2\tau-s)+C\ddot{X}_{k-2}(t-2\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+kC^{k-1}B\varphi(t-(k+1)\tau)+C^{k}\dot{\varphi}(t-(k+1)\tau).\end{aligned}\tag{2.16}$$

Note that by Theorem 3,

$$\dot{X}_k(t) = BX_{k-1}(t-\tau)+C\dot{X}_{k-1}(t-\tau), \quad t\in[k\tau,(k+1)\tau)$$

for any $k\in\mathbb{N}_0$. Clearly, it is also valid for $k=-1$ provided that $X_{-1}(t)=\Theta=X_{-2}(t)$ for any $t\in\mathbb{R}$.

Now we can use formulas (2.14), (2.15) and (2.16) to directly derive equation (2.10) on $(k\tau,(k+1)\tau)$.

Using (2.13) and Lemma 1 we get

$$\begin{aligned}\lim_{t\to k\tau+} x(t) &= X_k(k\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{0}\big(BX_{k-1}((k-1)\tau-s)+C\dot{X}_{k-1}((k-1)\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+C^{k+1}\varphi(-\tau) = X_{k-1}(k\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{0}\big(BX_{k-1}((k-1)\tau-s)+C\dot{X}_{k-1}((k-1)\tau-s)\big)\varphi(s)\mathrm{d}s\\ &+C^{k}\varphi(0) = \lim_{t\to k\tau-} x(t)\end{aligned}$$

where the last equality follows again, by (2.13) with $k-1$ instead of $k$. Finally, the continuity of $\dot{x}$ at the point $k\tau$, $k\in\mathbb{N}$ is a consequence of continuity of $x$ and $\dot{x}$ at $(k-1)\tau$ due to the form of (2.10), e.g. for $k=1$ it holds

$$\lim_{t\to\tau+}\dot{x}(t) = \lim_{t\to\tau+} Bx(t-\tau)+C\dot{x}(t-\tau) = \lim_{t\to 0+} Bx(t)+C\dot{x}(t)$$

$$= \lim_{t\to 0^-} Bx(t) + C\dot{x}(t) = \lim_{t\to \tau^-} \dot{x}(t).$$

This completes the proof. □

*Remark* 2. Condition (2.9) is necessary and sufficient for the solution to have continuous derivative at the points $k\tau$, $k \in \mathbb{N}_0$ (cf. [7]). Later, we will have analogical conditions (see (2.17), (2.22)) for the case of nonhomogeneous equation without and with a linear non-delayed term.

In the rest of the paper, we denote $x_h(t)$ solution $x(t)$ of (2.10), (1.5), given by (2.11) to emphasize the fact that, it is a solution of a homogeneous equation.

The following theorem provides a result on a solution of a nonhomogeneous neutral differential equation with a given initial condition.

**Theorem 5.** *Let $B$, $C$ be $n\times n$ permutable matrices, $f \in C([0,\infty),\mathbb{R}^n)$ and $\varphi \in C^1([-\tau,0],\mathbb{R}^n)$ be given functions satisfying*

$$\dot{\varphi}(0) = B\varphi(-\tau) + C\dot{\varphi}(-\tau) + f(0). \tag{2.17}$$

*Any solution of the initial value problem consisting of the equation*

$$\dot{x}(t) - C\dot{x}(t-\tau) = Bx(t-\tau) + f(t), \quad t \ge 0 \tag{2.18}$$

*and initial condition (1.5), has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ X(t)(\varphi(0) - C\varphi(-\tau)) + B\displaystyle\int_{-\tau}^{0} X(t-\tau-s)\varphi(s)\mathrm{d}s & \\ +C\displaystyle\int_{-\tau}^{0} \dot{X}(t-\tau-s)\varphi(s)\mathrm{d}s + \psi(t) + \int_0^t X(t-s)f(s)\mathrm{d}s, & \\ & 0 \le t \end{cases} \tag{2.19}$$

*where $X(t)$ is a fundamental matrix solution given by (2.4), and $\psi(t)$ is given by (2.12).*

*Proof.* Note that $x(t) = x_h(t) + \int_0^t X(t-s)f(s)\mathrm{d}s$ where $x_h(t)$ is a solution of homogeneous equation (2.10). Let $t \in (0,\tau)$. Then

$$x(t) = \varphi(0) - C\varphi(-\tau) + B\int_{-\tau}^{t-\tau} \varphi(s)\mathrm{d}s + C\varphi(t-\tau) + \int_0^t f(s)\mathrm{d}s,$$

$x(t-\tau) = \varphi(t-\tau)$, $\dot{x}(t) = B\varphi(t-\tau) + C\dot{\varphi}(t-\tau) + f(t)$. Hence (2.18) is verified. Moreover,

$$\lim_{t\to 0^+} x(t) = \lim_{t\to 0^+} x_h(t) = \varphi(0) = \lim_{t\to 0^-} x(t)$$

and

$$\lim_{t\to 0^+} \dot{x}(t) = B\varphi(-\tau) + C\dot{\varphi}(-\tau) + f(0) = \dot{\varphi}(0) = \lim_{t\to 0^-} \dot{x}(t)$$

by (2.17).

Next, let $t \in (k\tau, (k+1)\tau)$ for some $k \in \mathbb{N}$. Since $X(t)$ of (2.4) is piecewise defined, we have to split the integral $\int_0^t X(t-s)f(s)\mathrm{d}s$ to be able to differentiate it later. So we write

$$x(t) = x_h(t) + \int_0^{t-k\tau} X_k(t-s)f(s)\mathrm{d}s + \sum_{j=0}^{k-1} \int_{t-(j+1)\tau}^{t-j\tau} X_j(t-s)f(s)\mathrm{d}s.$$

Similarly, we have

$$\begin{aligned} x(t-\tau) = x_h(t-\tau) &+ \int_0^{t-k\tau} X_{k-1}(t-\tau-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-2} \int_{t-(j+2)\tau}^{t-(j+1)\tau} X_j(t-\tau-s)f(s)\mathrm{d}s. \end{aligned}$$

When computing the derivatives, we apply Lemma 1 to obtain

$$\begin{aligned} \dot{x}(t) &= \dot{x}_h(t) + \int_0^{t-k\tau} \dot{X}_k(t-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-1} \int_{t-(j+1)\tau}^{t-j\tau} \dot{X}_j(t-s)f(s)\mathrm{d}s + X_k(k\tau)f(t-k\tau) \\ &+ \sum_{j=0}^{k-1} \Big( X_j(j\tau)f(t-j\tau) - X_j((j+1)\tau)f(t-(j+1)\tau) \Big) \\ &= \dot{x}_h(t) + \int_0^{t-k\tau} \dot{X}_k(t-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-1} \int_{t-(j+1)\tau}^{t-j\tau} \dot{X}_j(t-s)f(s)\mathrm{d}s + \sum_{j=1}^{k} C^j f(t-j\tau) + f(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x}(t-\tau) &= \dot{x}_h(t-\tau) + \int_0^{t-k\tau} \dot{X}_{k-1}(t-\tau-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-2} \int_{t-(j+2)\tau}^{t-(j+1)\tau} \dot{X}_j(t-\tau-s)f(s)\mathrm{d}s + X_{k-1}((k-1)\tau)f(t-k\tau) \\ &+ \sum_{j=0}^{k-2} \Big( X_j(j\tau)f(t-(j+1)\tau) - X_j((j+1)\tau)f(t-(j+2)\tau) \Big) \\ &= \dot{x}_h(t-\tau) + \int_0^{t-k\tau} \dot{X}_{k-1}(t-\tau-s)f(s)\mathrm{d}s \end{aligned}$$

$$+\sum_{j=0}^{k-2}\int_{t-(j+2)\tau}^{t-(j+1)\tau}\dot{X}_j(t-\tau-s)f(s)\mathrm{d}s+\sum_{j=1}^{k-1}C^j f(t-(j+1)\tau)+f(t-\tau).$$

Using the above formulas, one can easily verify equation (2.10) on $(k\tau,(k+1)\tau)$.

To prove the continuity at $k\tau$, we use the continuity of $x_h(t)$,

$$\lim_{t\to k\tau+} x(t)=\lim_{t\to k\tau+} x_h(t)+\sum_{j=0}^{k-1}\int_{(k-j-1)\tau}^{(k-j)\tau} X_j(k\tau-s)f(s)\mathrm{d}s$$
$$=\lim_{t\to k\tau-} x_h(t)+\sum_{j=0}^{k-1}\int_{(k-j-1)\tau}^{(k-j)\tau} X_j(k\tau-s)f(s)\mathrm{d}s=\lim_{t\to k\tau-} x(t).$$

Continuity of $\dot{x}$ at $k\tau$, $k\in\mathbb{N}$ can be proved inductively as in the proof of Theorem 4. □

*Remark* 3. Putting $C=\Theta$ in the latter theorem states that,

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ \mathrm{e}_\tau^{B(t-\tau)}\varphi(0)+B\displaystyle\int_{-\tau}^{0}\mathrm{e}_\tau^{B(t-2\tau-s)}\varphi(s)\mathrm{d}s & \\ +\displaystyle\int_0^t \mathrm{e}_\tau^{B(t-\tau-s)}f(s)\mathrm{d}s, & 0\le t\end{cases}$$

for $\mathrm{e}_\tau^{Bt}$ defined by (1.2), is a solution of equation

$$\dot{x}(t)=Bx(t-\tau)+f(t),\quad t\ge 0 \tag{2.20}$$

along with initial condition (1.5) assuming that

$$\dot{\varphi}(0)=B\varphi(-\tau)+f(0). \tag{2.21}$$

As $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$, one can make per partes leading to

$$x(t)=\mathrm{e}_\tau^{Bt}\varphi(-\tau)+\int_{-\tau}^{0}\mathrm{e}_\tau^{B(t-\tau-s)}\dot{\varphi}(s)\mathrm{d}s+\int_0^t \mathrm{e}_\tau^{B(t-\tau-s)}f(s)\mathrm{d}s$$

for any $t\ge 0$. Note that for $t\in[-\tau,0)$ this formula becomes $x(t)=\varphi(t)$. This corresponds to a solution of (2.20), (1.5) from [9, Theorems 1, 2]. The assumption (2.21) means the continuity of $\dot{x}(t)$ at 0.

Finally, we give a result on a representation of a solution of initial value problem (1.4), (1.5).

**Theorem 6.** *Let $A$, $B$, $C$ be $n\times n$ pairwise permutable matrices, $f\in C([0,\infty),\mathbb{R}^n)$ and $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$ be given functions satisfying*

$$\dot{\varphi}(0)=A\varphi(0)+B\varphi(-\tau)+C\dot{\varphi}(-\tau)+f(0). \tag{2.22}$$

*Any solution of the initial value problem* (1.4), (1.5) *has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ e^{At}Y(t)(\varphi(0) - C\varphi(-\tau)) + \widetilde{B}\int_{-\tau}^{0} e^{A(t-s)}Y(t-\tau-s)\varphi(s)ds \\ +\widetilde{C}\int_{-\tau}^{0} e^{A(t-s)}\dot{Y}(t-\tau-s)\varphi(s)ds + \psi(t) \\ +\int_{0}^{t} e^{A(t-s)}Y(t-s)f(s)ds, & 0 \le t \end{cases} \tag{2.23}$$

*where* $Y(t)$ *is given by*

$$Y(t) = \begin{cases} \Theta, & t < 0, \\ \sum_{j=0}^{k}\sum_{i=0}^{k-j} \widetilde{C}^i \widetilde{B}^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!}, & t \in [k\tau, (k+1)\tau), k \in \mathbb{N}_0, \end{cases} \tag{2.24}$$

$\widetilde{B} = (B + AC)e^{-A\tau}$, $\widetilde{C} = Ce^{-A\tau}$, *and* $\psi$ *is given by* (2.12).

*Proof.* Let us introduce the substitution $x(t) = e^{At}y(t)$. Then $y(t)$ fulfills

$$\begin{aligned} \dot{y}(t) - \widetilde{C}\dot{y}(t-\tau) &= \widetilde{B}y(t-\tau) + \widetilde{f}(t), \quad t \ge 0, \\ y(t) &= \widetilde{\varphi}(t), \quad -\tau \le t \le 0 \end{aligned}$$

with $\widetilde{f}(t) = e^{-At}f(t)$, $\widetilde{\varphi}(t) = e^{-At}\varphi(t)$. By application of Theorem 5, we obtain the solution

$$y(t) = \begin{cases} \widetilde{\varphi}(t), & -\tau \le t < 0, \\ Y(t)(\widetilde{\varphi}(0) - \widetilde{C}\widetilde{\varphi}(-\tau)) + \widetilde{B}\int_{-\tau}^{0} Y(t-\tau-s)\widetilde{\varphi}(s)ds \\ +\widetilde{C}\int_{-\tau}^{0} \dot{Y}(t-\tau-s)\widetilde{\varphi}(s)ds + \widetilde{\psi}(t) \\ +\int_{0}^{t} Y(t-s)\widetilde{f}(s)ds, & 0 \le t \end{cases}$$

where

$$\widetilde{\psi}(t) := \widetilde{C}^{\lfloor \frac{t}{\tau} \rfloor + 1}\widetilde{\varphi}\left(t - \left(\left\lfloor \frac{t}{\tau} \right\rfloor + 1\right)\tau\right).$$

Note that to get a $C^1$–solution we need to change condition (2.17) to (2.22). It is easy to see that $\widetilde{\varphi}(0) = \varphi(0)$, $\widetilde{C}\widetilde{\varphi}(-\tau) = C\varphi(-\tau)$ and $\widetilde{\psi}(t) = e^{-At}\psi(t)$. Therefore, $e^{At}y(t)$ gives the solution $x(t)$ of (2.23). □

*Remark* 4. Let us put $C = \Theta$ in Theorem 6. Then $\widetilde{C} = \Theta$, $\widetilde{B} = B_1 = \mathrm{e}^{-A\tau}B$ and

$$Y(t) = \begin{cases} \Theta, & t < 0, \\ \sum_{j=0}^{k} B_1^j \dfrac{(t-j\tau)^j}{j!}, & t \in [k\tau,(k+1)\tau), k \in \mathbb{N}_0. \end{cases}$$

Thus it is easy to see that $Y(t) = \mathrm{e}_\tau^{B_1(t-\tau)}$ for $\mathrm{e}_\tau^{Bt}$ defined by (1.2). Moreover, $\psi(t) = \theta$ and the solution $x(t)$ given by (2.23) coincides with the one stated by (1.3) in Theorem 2, or, after per partes, with the one from Theorem 1. Note that condition (2.22), in this case

$$\dot{\varphi}(0) = A\varphi(0) + B\varphi(-\tau) + f(0),$$

implies the continuity of $\dot{x}(t)$ at 0, and the solution gets smoother at $k\tau$ for each $k \in \mathbb{N}$ as was proved in [14].

## 3. Examples

In this section, we provide two examples illustrating the proved results. On the first one, the behavior of solutions of a scalar equation can be seen from a numerically computed graph. The second one provides an example of permutable matrices of a higher dimension.

*Example* 1. Let us consider the scalar neutral differential equation with a delay

$$\begin{aligned} \dot{x}(t) - \dot{x}(t-1) &= ax(t) + 2x(t-1) + 2\cos t, \quad t \geq 0 \\ x(t) &= t, \quad -1 \leq t \leq 0 \end{aligned} \tag{3.1}$$

for $a \in \mathbb{R}$.

One does not have to verify the condition of permutability of matrices, since $n = 1$. Moreover, condition (2.22) is satisfied for any $a \in \mathbb{R}$. So the solution can be immediately obtained by Theorem 6. Figure 1 depicts different solutions corresponding to particular values of parameter $a$.

*Example* 2. Let us consider the system

$$\begin{aligned} \dot{x}(t) - \dot{x}(t-\pi) + 0.2\dot{y}(t-\pi) &= -0.4x(t) - 0.02y(t) - 0.02z(t) \\ &\quad - 0.5x(t-\pi) + 0.1y(t-\pi) \\ &\quad + 0.1z(t-\pi) + f_1(t) \\ \dot{y}(t) - \dot{y}(t-\pi) &= -0.4y(t) - 0.5y(t-\pi) + f_2(t) \\ \dot{z}(t) - \dot{z}(t-\pi) &= -0.02y(t) - 0.5z(t) + 0.1y(t-\pi) + f_3(t) \end{aligned}$$

for $t \geq 0$ along with initial condition

$$(x(t), y(t), z(t)) = (\sin(t), \sin(2t), \cos(t)), \quad -\pi \leq t \leq 0,$$

where $f \in C([0,\infty), \mathbb{R}^3)$ is a given function such that $f(0) = (2.52, 0, 0.5)^T$.

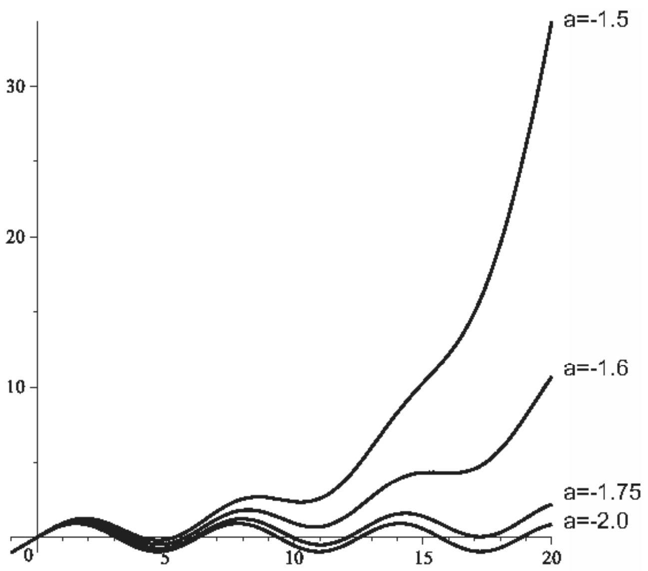


FIGURE 1. Solutions of (3.1) corresponding to different values of parameter $a$.

The latter example can be considered as a system modeling the sizes of changes in three given populations. So, $x > 0$ ($x < 0$) means that the first population is growing (dying). Here it is assumed by the matrix

$$C = \begin{pmatrix} 1 & -0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

that the speed of changes in each population is similar to the one before time $\pi$. The number $-0.2$ above the diagonal means that the speed of growth of the second population affects negatively the speed of growth of the first one. Matrices

$$A = \begin{pmatrix} -0.4 & -0.02 & -0.02 \\ 0 & -0.4 & 0 \\ 0 & -0.02 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 & 0.1 & 0.1 \\ 0 & -0.5 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}$$

reflect the effort of populations before the time $\pi$ and now to stabilize the growth or diminution of the populations, i.e., to prevent hardly predictable changes in the number of members of each population; but also influences coming from changes in other populations.

Now, it is easy to verify that all the assumptions of Theorem 6 are satisfied, and to obtain the solution.

*Remark* 5. The space of matrices commuting with the matrix $C$ of (3.2) has dimension 5, and these have the form

$$\Xi = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix}$$

with real parameters $a, b, c, d, e$, as can be derived from [1].

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