



Miskolc Mathematical Notes
Vol. 14 (2013), No 3, pp. 1059-1066

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.678

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Received 30 January, 2013

Abstract. In this paper, we study modules with the properties (CRE) and (CREE), which are adapted Zöschinger's modules with the properties (E) and (EE). It is shown that: (1) a module M has the property (CREE) if and only if every submodule of M has the property (CRE); (2) a ring R is *Rad*-supplemented if and only if every left R -module has the property (CRE); (3) over a commutative Von Neumann regular ring a module M has the property (CRE) if and only if M is cofinitely injective.

2010 Mathematics Subject Classification: 16D10; 16N80

Keywords: *Rad*-supplement, cofinite extension, Von Neumann regular ring, semiperfect ring

1. INTRODUCTION

In this note, R is an associative ring with identity and all modules are unital left R -modules. Let N be an R -module. The notation $M \subseteq N$ means that M is a submodule of N . Here, N is called *an extension* of M . A module M is called *injective* if M is a direct summand of every extension N , i.e. $N = M + K$ and $N \cap K = 0$ for some submodule $K \subseteq N$. If every R -module is injective, then R is semisimple. The converse is true.

A submodule S of a module M is said to be *small* if $S + L \neq M$ for every proper submodule L of M . For every module M , the trivial submodule 0 is a small submodule of M . As a generalization of direct summands, we say that a submodule V of a module M is supplement of a submodule U of M if $M = U + V$ and $U \cap V \ll V$ as in [11]. Zöschinger generalized injective modules to modules with the property (E). He said that a module M has the property (E) if M has a supplement in every extension. He also said that a module M has the property (EE) if M has ample supplements in every extension, i.e. for $M \subseteq N$, if $N = M + K$, K contains a supplement of M in N ([13]). Every left R -module has the property (E) if and only if R is left perfect [4].

In [1], a submodule M of a module N is called *cofinite* if the factor module $\frac{N}{M}$ is finitely generated. It is easy to see that every submodule of a finitely generated module is cofinite. Adapting Zöschinger's module with the properties (E) and (EE),

Çalışıcı and Türkmen say that a module M has the property (CE) ((CEE)) if M has a supplement (ample supplements) in every cofinite extension. Here N is called *cofinite extension* of M if M is a cofinite submodule of N . They give a characterization of semiperfect rings via these modules.

For a module M , $Rad(M)$ will indicate the Jacobson radical of M . A submodule V of a module M is called *Rad-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$ [4, Theorem 10.14]. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is *Rad-supplement*. A submodule U of M has ample *Rad-supplements* in M if every submodule V of M with $M = U + V$ contains a *Rad-supplement* in M . Now we have the following implications on submodules:

$$\text{direct summand} \implies \text{supplement} \implies \text{Rad-supplement}$$

In this paper, we investigate the properties (CRE) and (CREE) for modules as a generalization of the properties (CE) and (CEE), respectively. We show that a module has the property (CREE) if and only if every submodule has the property (CRE). We prove that every direct summand of a module M with the property (CRE) has this property. Using the property (CRE), we give a characterization of *Rad-supplemented rings*, which are a generalization of semiperfect rings.

2. MODULES WITH THE PROPERTIES (CRE) AND (CREE)

In this section, we define the concept of modules with the properties (CRE) and (CREE). We give various properties of these modules. We prove that every left R -module has the property (CRE) if and only if the ring R is *Rad-supplemented*.

Definition 1. Let M be an R -module. We say that M has *the property (CRE)* if M has a *Rad-supplement* in every cofinite extension, and M has *the property (CREE)* if M has ample *Rad-supplements* in every cofinite extension

It is easy to see that every module with property (CE) has property (CRE). It is natural to pose the question whether there exists similar result for the properties (CEE) and (CREE). Before the end of this section, we shall give an example of a module which has the property (CRE) but not (CE).

Proposition 1. *Every radical module has the property (CRE).*

Proof. Let M be a module and N be any cofinite extension of M . Since $Rad(M) = M$, then $N = M + N$ and $M \cap N = M = Rad(M) \subseteq Rad(N)$. So N is a *Rad-supplement* of M in N . \square

By $P(M)$, we denote the sum of all radical submodules of the module M , that is, $P(M) = \sum \{U \subseteq M \mid Rad(U) = U\}$. A module M is *reduced* if $P(M) = 0$.

Since $P(M)$ is a radical submodule of M , we have the next result which is a direct consequence of Proposition 1.

Corollary 1. *For a module M , $P(M)$ has the property (CRE).*

Now we show that the property (CRE) is preserved by direct summands in the following proposition:

Proposition 2. *Every direct summand of a module with the property (CRE) has the property (CRE).*

Proof. Let N be a direct summand of M . Then there exists a submodule K of M such that $M = N \oplus K$. Let L be a cofinite extension of N , T be the external direct sum $L \oplus K$ and $\gamma : M \rightarrow T$ be the canonical embedding. Then $M \cong \gamma(M)$ has the property (CRE). We have

$$\frac{L}{N} \cong \frac{L \oplus K}{\gamma(M)}$$

is finitely generated. Since $\gamma(M)$ has the property (CRE), then there exists a submodule U of T such that $T = \gamma(M) + U$ and $\gamma(M) \cap U \subseteq \text{Rad}(U)$. Consider the projection $\pi : T \rightarrow L$. By this way, we have $N + \pi(U) = L$. Also $\ker(\pi) \subseteq \gamma(M)$, $\pi(\gamma(M) \cap U) \subseteq \pi(\gamma(M)) \cap \pi(U) = N \cap \pi(U) \subseteq \text{Rad}(\pi(U))$. Therefore $\pi(U)$ is a *Rad*-supplement of N in L . \square

In [3], Çalışıcı and Türkmen defined cofinitely injective modules, that is, a module M is called *cofinitely injective* if M is a direct summand of every cofinite extension.

Recall that a ring R is called *Von Neumann regular* if every element $a \in R$ can be written in the form axa , for some $x \in R$.

Proposition 3. *Let R be a commutative Von Neumann regular ring and M be an R -module. Then the following statements are equivalent:*

- (1) M has the property (CRE).
- (2) M is cofinitely injective.
- (3) M has the property (CE).

Proof. (1) \implies (2) Suppose that M is a module with the property (CRE). Let N be any cofinite extension of M . So, there exists a *Rad*-supplement V of M in N . Since R is a commutative Von Neumann regular ring, $\text{Rad}(V) = 0$ by [5, 3.73 and 3.75]. So $N = M \oplus V$.

(2) \implies (3) and (3) \implies (1) are clear. \square

Zöschinger proved in [13] that a module has the property (EE) if and only if every submodule has the property (E). We give an analogous characterization of our modules.

Theorem 1. *A module M has the property (CREE) if and only if every submodule of M has the property (CRE).*

Proof. (\implies) Let M be a module with the property (CREE) and T be any submodule of M . For a cofinite extension N of T , let $F = \frac{M \oplus N}{H}$, where the submodule H is the set of all elements $(a, -a)$ of F with $a \in T$ and let $\alpha : M \rightarrow F$ via $\alpha(m) = (m, 0) + H$, $\beta : N \rightarrow F$ via $\beta(n) = (0, n) + H$ for all $m \in M, n \in N$. It is clear that α and β are monomorphisms. Thus we have the following pushout:

$$\begin{array}{ccc} T & \xrightarrow{\mu_1} & N \\ \downarrow \mu_2 & & \downarrow \beta \\ M & \xrightarrow{\alpha} & F \end{array}$$

where μ_1 and μ_2 are inclusion mappings. It is easy to prove that $F = \text{Im}(\alpha) + \text{Im}(\beta)$. Now we define $\gamma : F \rightarrow \frac{N}{T}$ by $\gamma((m, n) + H) = n + T$ for all $(m, n) + H \in F$. Then γ is an epimorphism. Note that $\ker(\gamma) = \text{Im}(\alpha)$ and so $\frac{N}{T} \cong \frac{F}{\text{Im}(\alpha)}$ is finitely generated. Since α is a monomorphism, we have $M \cong \text{Im}(\alpha)$. By the assumption, $\text{Im}(\alpha)$ has the property (CREE). Then it follows immediately that $\text{Im}(\alpha)$ has a *Rad*-supplement V in F with $V \subseteq \text{Im}(\alpha)$, i.e., $F = \text{Im}(\alpha) + V$ and $\text{Im}(\alpha) \cap V \subseteq \text{Rad}(V)$. Then $N = T + \beta^{-1}(V)$ and $T \cap \beta^{-1}(V) \subseteq \text{Rad}(\beta^{-1}(V))$. Hence $\beta^{-1}(V)$ is a *Rad*-supplement of T in N .

(\impliedby) Suppose that every submodule of M has the property (CRE). For a cofinite extension N of M , let $N = M + K$ for some submodule K of N . Then $\frac{N}{M} \cong \frac{K}{M \cap K}$ is finitely generated and so $M \cap K$ is a cofinite submodule of K . By the hypothesis, there exists a submodule V of K such that $K = (M \cap K) + V$ and $M \cap V \subseteq \text{Rad}(V)$. Note that $N = M + V$. It follows that V is a *Rad*-supplement of M in N . So M has the property (CREE). \square

Recall from [2] that a module M is (cofinitely) *Rad*-supplemented if every (cofinite) submodule has a *Rad*-supplement in M . It is clear that every finitely generated cofinitely *Rad*-supplemented module is *Rad*-supplemented.

Corollary 2. *Every submodule of a module with the property (CREE) has the property (CRE). Moreover, a module with the property (CREE) is cofinitely Rad-supplemented.*

Proposition 4. *Let M be a module and A be a submodule of B such that $\frac{B}{A}$ is noetherian. If A and $\frac{B}{A}$ have the property (CRE), then so is B .*

Proof. Let N be any cofinite extension of B . By the hypothesis, there is a *Rad*-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$. Note that $\frac{N}{B} \cong \frac{\frac{V}{A}}{\frac{B \cap V}{A}}$. Since $\frac{B}{A}$ is noetherian, then $\frac{V}{A}$ is finitely generated. Therefore V is a cofinite extension of A . Since A has the property (CRE), then A has a *Rad*-supplement in V , say K . We claim that K is a *Rad*-supplement of B in N . We have epimorphisms $f : K \rightarrow \frac{V}{A}$ and $g : \frac{V}{A} \rightarrow \frac{N}{B}$ such that $\ker(f) = K \cap A \subseteq \text{Rad}(K)$ and $\text{Ker}(g) = \frac{V}{A} \cap \frac{B}{A} \subseteq \text{Rad}(\frac{V}{A})$. Then

$g \circ f : K \rightarrow \frac{N}{B}$ is an epimorphism such that $K \cap B = \text{Ker}(g \circ f) \subseteq \text{Rad}(K)$ by [12, Lemma 1.1]. Finally, $N = K + B$. \square

Proposition 5. *Let M be a module and K be a radical submodule of M . If $\frac{M}{K}$ has the property (CRE), then M has the property (CRE).*

Proof. Let U be any cofinite extension of M . Then $\frac{U}{M}$ is finitely generated. So $\frac{\frac{U}{K}}{\frac{M}{K}} \cong \frac{U}{M}$ is finitely generated. Thus $\frac{U}{K}$ is a cofinite extension of $\frac{M}{K}$. By the hypothesis, there exists a submodule $\frac{V}{K}$ of $\frac{U}{K}$ such that $\frac{U}{K} = \frac{M}{K} + \frac{V}{K}$ and $\frac{M}{K} \cap \frac{V}{K} \subseteq \text{Rad}(\frac{V}{K})$. Then we have $U = M + V$. Since K is radical, $M \cap V \subseteq \text{Rad}(V)$. So V is a Rad-supplement of M in U . Therefore M has the property (CRE). \square

Now we prove that the class of modules with the property (CRE) is closed under extensions if we impose certain conditions.

Theorem 2. *Let*

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

be a short exact sequence. Suppose that K is radical. If K and L have the property (CRE), so does M . If the sequence splits, the converse holds.

Proof. Without loss of generality; we assume that $K \subseteq M$. Since $\frac{M}{K} \cong L$ has the property (CRE) and K is radical, then we have M has the property (CRE) by Proposition 5. On the other hand, suppose that the sequence splits. Then $M \cong K \oplus L$. Let M be module with the property (CRE). So, K and L have the property (CRE) by Proposition 2. \square

Corollary 3. *Let M_i ($i \in I$) be any finite collection of radical modules and $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$. Then M has the property (CRE) iff M_i has the property (CRE) for each $i = 1, 2, \dots, n$.*

Proof. The necessity follows from Proposition 2. Conversely, to prove that M has the property (CRE) it is sufficient by induction on n to prove that this is the case when $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By using the following short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

Since M_1 is radical and M_1, M_2 have properties (CRE), then M has the property (CRE) by Theorem 2. \square

Let M be a module. A module N is called *maximal essential extension* of M if N is an essential extension of M , and whenever K is a proper extension of N , then K is not an essential extension of M ([8]).

Let M be a module. Then there exists a module N satisfying the following equivalent conditions:

- (1) N is an essential injective extension of M .
- (2) N is a maximal essential extension of M .

A module N satisfying above equivalent conditions is called *injective envelope* of M , denoted by $E(M)$ ([8, Theorem 2.21]).

Proposition 6. *Let M be a module. Then the following statements are equivalent.*

- (1) M has a *Rad-supplement* in every cofinite essential injective extension.
- (2) M has a *Rad-supplement* in every cofinite maximal essential extension.

Proof. (1) \implies (2) is clear.

(2) \implies (1) Let N be any cofinite essential extension of M , T be any cofinite maximal essential extension of M , $f : M \rightarrow N$, $g : M \rightarrow T$ be inclusion mappings and $I_T : T \rightarrow T$ be an identity mapping. Then we have the following commutative diagram with h necessarily monic:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow g & & \downarrow h \\ T & \xrightarrow{I_T} & T \end{array}$$

By the hypothesis, M has a *Rad-supplement* in T , say K , that is, $M + K = T$ and $M \cap K \subseteq \text{Rad}(K)$. Since $M \subseteq h(N)$, we obtain that $h(N) = h(N) \cap T = h(N) \cap (M + K) = M + (h(N) \cap K)$. Now, taking any $n \in N$, we have $h(N) = m + h(n_1) = h(m + n_1)$ where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since h is monic, and thus $M + h^{-1}(K) = N$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K) \subseteq \text{Rad}(h^{-1}(K))$. Since $h^{-1}(M) = M$ as h is monic. Hence $h^{-1}(K)$ is a *Rad-supplement* of M in N . \square

Proposition 7. *Let M be a module with the property (CRE). If M is noetherian, then M has the property (CE).*

Proof. Let N be any cofinite extension of M . Since M has the property (CRE), then there exists a submodule K of N such that $N = M + K$ and $M \cap K \subseteq \text{Rad}(K)$. Since M is noetherian, then $M \cap K$ is finitely generated. So it is easy to see that $M \cap K \ll K$. So M has the property (CE). \square

A ring R is called *semiperfect* if $\frac{R}{\text{Rad}(R)}$ is semisimple and idempotents can be lifted modulo $\text{Rad}(R)$. It is well known that a ring R is semiperfect if and only if every left (or right) ideal has a supplement in ${}_R R$ (or R_R). A ring R is called *Rad-supplemented* if every left (or right) ideal has a *Rad-supplement* in ${}_R R$ (or R_R) (see [2]). Clearly, every semiperfect ring is *Rad-supplemented*, but a *Rad-supplemented* ring need not be semiperfect in general (see [2]).

It is proved in [3, Proposition 2.11] that a ring R is semiperfect if and only if every left R -module has the property (CE). Now, we give the characterization of the rings over which every module has the property (CRE).

Theorem 3. *The following statements are equivalent for a ring R :*

- (1) R is *Rad-supplemented*.
- (2) ${}_R R$ has ample *Rad-supplements* in every cofinite extension.
- (3) ${}_R R$ has ample *Rad-supplements* in every finitely generated extension.
- (4) Every left R -module has the property (CRE).
- (5) Every left R -module has the property (CREE).

Proof. (1) \implies (4) Let M be an R -module and N be a cofinite extension of M . Then there exists a finitely generated submodule K of N such that $N = M + K$. Since R is a *Rad-supplemented* ring, then $M \cap K$ has a *Rad-supplement* in K , say T , i.e. $K = (M \cap K) + T$ and $(M \cap K) \cap T \subseteq \text{Rad}(T)$ by [10, Theorem 3.7]. It follows that $N = M + T$ and $M \cap T \subseteq \text{Rad}(T)$. This completes the proof.

(4) \implies (1) Suppose that every left R -module has the property (CRE). Then every left ideal of R has a *Rad-supplement* in R as a left R -module. Hence ${}_R R$ is *Rad-supplemented*.

(2) \implies (1) If ${}_R R$ has the property (CREE), it is *Rad-supplemented* by Corollary 2.

(5) \implies (4) and (5) \implies (3) \implies (2) are clear. □

A module M over a ring R is called *uniserial* if the lattice of submodules of M is a chain and M is called *serial* if M is a direct sum of uniserial modules. A ring R is called *left (right) uniserial* if the module ${}_R R$ (R_R) is uniserial. R is called a *uniserial ring* if R is both right and left uniserial. A uniserial domain R is called *nearly simple* if $\text{Rad}(R)$ is the unique nonzero two sided ideal of R and $\text{Rad}^2(R) \neq 0$ [7].

The following example shows that a module with the property (CRE) need not be (CE).

Example 1. (See[2]) Let $G = \{f : \mathbb{Q} \rightarrow \mathbb{Q} \mid f(t) = at + b \text{ for } a, b \in \mathbb{Q} \text{ and } a > 0\}$ be the group of affine linear functions on the field of rational numbers \mathbb{Q} . Choose any irrational number $\epsilon \in \mathbb{R}$ and set $P = \{f \in G \mid \epsilon \leq f(\epsilon)\}$ and $P^+ = \{f \in G \mid \epsilon < f(\epsilon)\}$. Note that P , resp. P^+ , defines a left order on G . Take an arbitrary field F and consider the semigroup group ring $F[P]$ in which the right ideal $M = \sum_{g \in P^+} gF[P]$ is maximal. The set $F[P] \setminus M$ is a left and right ore set and the corresponding localization R is a nearly simple uniserial domain. Taking any nonzero element $r \in R$, $S = \text{End}(\frac{R}{rR})$ is a *Rad-supplemented* ring which is not semiperfect. Since S is a *Rad-supplemented* ring, then ${}_S S$ has the property (CRE) by Theorem 3. It is clear that ${}_S S$ has not the property (CE) by [3, Proposition 2.11]

The ring R is called *left hereditary* if and only if all left ideals of R are projective. It is well known that for a left hereditary ring R , every submodule of a free R -module is isomorphic to a direct sum of left ideals of R and hence is projective.

Corollary 4. *For a left hereditary ring R , the following statements are equivalent.*

- (1) R is semiperfect.
- (2) Every left R -module has the property (CRE).

Proof. (1) \implies (2) Let M be any R -module and N be a cofinite extension of M . Then there exists a submodule K of N such that $N = M + K$. Since K is finitely generated, then so is $\frac{N}{M} \cong \frac{K}{M \cap K}$. Therefore $\frac{N}{M}$ has a projective cover because R is semiperfect. By [6, Lemma 4.40], there exists a submodule L of N such that $N = M + L$ and $M \cap L \ll L$. So $M \cap L \subseteq \text{Rad}(L)$. Thus M has the property (CRE).

(2) \implies (1) By Theorem 3, R is Rad -supplemented. Applying [9, Corollary 2.22], it is semiperfect. \square

Acknowledgement. We would like to thank the referee for the valuable suggestions and comments.

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