3-uniform hypergraphs and linear cycles

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Abstract

Gyárfás, Győri and Simonovits [3] proved that if a 3-uniform hypergraph with n vertices has no linear cycles, then its independence number $\alpha \geq \frac{2n}{5}$. The hypergraph consisting of vertex disjoint copies of a complete hypergraph K_5^3 on five vertices shows that equality can hold. They asked whether this bound can be improved if we exclude K_5^3 as a subhypergraph and whether such a hypergraph is 2-colorable.

In this paper, we answer these questions affirmatively. Namely, we prove that if a 3-uniform linear-cycle-free hypergraph doesn't contain K_5^3 as a subhypergraph, then it is 2-colorable. This result clearly implies that its independence number $\alpha \geq \lceil \frac{n}{2} \rceil$. We show that this bound is sharp.

Gyárfás, Győri and Simonovits also proved that a linear-cycle-free 3-uniform hypergraph contains a vertex of strong degree at most 2. In this context, we show that a linear-cycle-free 3-uniform hypergraph has a vertex of degree at most $n - 2$ when $n \geq 10$.

1 Introduction

A 3-uniform hypergraph $H = (V, E)$ consists of a set of vertices V and a set of hyperedges E such that each hyperedge is a 3 element subset of V. H is k colorable if there is a coloring of the vertices of H with k colors such that there is no monochromatic hyperedge in H. Throughout the paper, we mostly use the terminology introduced in [3].

Definition 1. *A* linear tree *is a hypergraph obtained from a vertex by repeatedly adding hyperedges that intersect the previous hypergraph in exactly one vertex.*

A linear path $\mathcal{P}$ of length $k \geq 0$ is an alternating sequence $v_1, h_1, v_2, h_2, ..., h_k, v_{k+1}$ of distinct vertices and distinct hyperedges such that $h_i \cap h_{i+1} = \{v_{i+1}\}$ for each $i \in \{1, 2, \ldots, k-1\}$, $v_1 \in h_1, v_{k+1} \in h_k$ and $h_i \cap h_j = \emptyset$ if $|j - i| \geq 2$. The vertex set $V(\mathcal{P})$ of $\mathcal{P}$ is $\cup_{i=1}^{k} h_i$ or $\{v_1\}$ if $k = 0$.

We say that $\mathcal{P}$ is a linear path between/joining v_1 and v_{k+1} or in general, between vertex sets A and B if $v_1 \in A$, $v_{k+1} \in B$, $h_i \cap A = \emptyset$ for $2 \leq i \leq k$ and $h_i \cap B = \emptyset$ for $1 \leq i \leq k-1$. Typically A and B are (vertex sets of) hyperedges or one element sets.

A linear cycle of length $k \geq 3$ is an alternating sequence $v_1, h_1, v_2, h_2, ..., v_k, h_k$ of distinct vertices and distinct hyperedges such that $h_i \cap h_{i+1} = \{v_{i+1}\}$ for each $i \in \{1, 2, \ldots, k-1\}$, $h_1 \cap h_k = \{v_1\}$ and $h_i \cap h_j = \emptyset$ if $1 < |j - i| < k-1$.

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A skeleton T in H is a linear subtree of H which cannot be extended to a larger linear subtree by adding a hyperedge e of H for which $|e \cap V(T)| = 1$.

An independent set in H is a set of vertices containing no hyperedge of H. More precisely, if I is an independent set of H, then there is no $e \in E(H)$ such that $e \subseteq I$. Let $\alpha(H)$ denote the size of the largest independent set in H. Gyárfás, Győri and Simonovits [3] initiated the study of linear-cycle-free hypergraphs by showing:

Theorem 1. *(Gyárfás, Győri, Simonovits [3]) If H is a 3-uniform hypergraph on n vertices without linear cycles, then it is 3-colorable. Moreover, $\alpha(H) \geq \frac{2n}{5}$.*

If the hypergraph does not contain the complete 3-uniform hypergraph K_5^3 as a subhypergraph then a stronger theorem can be proved, answering a question of Gyárfás, Győri and Simonovits.

Theorem 2. *If a 3-uniform linear-cycle-free hypergraph H doesn't contain K_5^3 as a subhypergraph, then it is 2-colorable.*

Corollary 3. *If a 3-uniform linear-cycle-free hypergraph H on n vertices doesn't contain K_5^3 as a subhypergraph, then $\alpha(H) \geq \lceil \frac{n}{2} \rceil$ and the bound is sharp.*

Indeed, from Theorem 2, it trivially follows that $\alpha(H) \geq \lceil \frac{n}{2} \rceil$. The hypergraph H_n on n vertices obtained from the following construction shows that this inequality is sharp. Let H_3 be the hypergraph on 3 vertices v_1, v_2, v_3 such that $v_1 v_2 v_3 \in E(H_3)$ and let H_4 be the complete 3-uniform hypergraph K_4^3 on 4 vertices v_1, v_2, v_3, v_4. Now for each $3 \leq i \leq n-2$ let us define the hypergraph H_{i+2} such that $V(H_{i+2}) := V(H_i) \cup \{v_{i+1}, v_{i+2}\}$ and $E(H_{i+2}) := E(H_i) \cup \{v_{i+1} v_{i+2} v_j\}_{j=1}^i$. If n is even, we start this iterative process with the hypergraph H_4 and if n is odd, we start with H_3. Notice that $\alpha(H_{i+2}) = \alpha(H_i) + 1$ for each i, which implies that $\alpha(H_n) = \lceil \frac{n}{2} \rceil$.

It is another natural problem to bound the number of hyperedges or different types of degrees of vertices in hypergraphs with no linear cycles. The most plausible is the *degree* of a vertex $v \in V$ what is simply the number of hyperedges of H containing v. Given a 3-uniform hypergraph H and $v \in V(H)$, the *link* of v in H is the graph with vertex set $V(H)$ and edge set $\{xy : vxy \in E(H)\}$. The *strong degree* $d^+(v)$ of $v \in V(H)$ is the maximum number of independent edges in the link of v. It is interesting and known for many years that the maximal number of hyperedges in a 3-uniform hypergraph without linear cycles is $\binom{n-1}{2}$, which is the maximum number of hyperedges without a linear triangle [1, 2]. The relation to the strong degree is proved recently.

Theorem 4. *(Gyárfás, Győri, Simonovits [3]) Let H be a 3-uniform hypergraph without linear cycles. Then, it has a vertex v whose strong degree $d^+(v)$ is at most 2.*

In this paper, we show a similar and perhaps more natural theorem concerning the degree of a linear-cycle-free hypergraph.

Theorem 5. *Let H be a 3-uniform hypergraph on $n \geq 10$ vertices without linear cycles. Then, there is a vertex whose degree is at most $n - 2$.*

Remark 6. *There is a 3-uniform hypergraph on 9 vertices without linear cycles where the degree of every vertex is 8. This hypergraph H is defined by taking a copy of K_4^3 on vertices $\{u_1, u_2, v_1, v_2\}$ and a vertex disjoint copy of K_5^3 such that $u_1 u_2 x, v_1 v_2 x \in E(H)$ for each $x \in V(K_5^3)$ and there are no other hyperedges in H.*

Remark 7. *Theorem* 5 *cannot be improved because there is a* 3*-uniform hypergraph* H', *with* $E(H') := \{xab \mid a, b \in V(H') \setminus \{x\}\}$ *for a fixed vertex* $x \in V(H)$, *in which every vertex has degree at least* $n - 2$.

The paper is organized as follows: In section 2 we introduce some important definitions. In section 3 we prove Theorem 2 by means of our main lemma - Lemma 11 (which is proved in section 3.1). In section 4 we prove Theorem 5. Finally in section 5, we present some concluding remarks and open questions.

2 Definitions

The following notions of *association* are used throughout the paper.

Definition 2. *Given a vertex* $v \in V(H)$ *and a hyperedge* $abc \in E(H)$ *such that* $v \notin \{a, b, c\}$, *we say that* v *is* "strongly associated" *to* abc *if at least two of the three edges* vab, vbc, vca *are in* $E(H)$. *We say that* v *is* "weakly associated" *to* abc *if exactly one of the three edges* vab, vbc, vca *is in* $E(H)$. *We say that* v *is associated to* abc *if it is either strongly or weakly associated.*

The set of pairs $\{\{x, y\} \subset \{a, b, c\} \mid vxy \in E(H)\}$ *is called the* "support" *of* v *in* abc, *denoted* $s_{abc}(v)$ *and these hyperedges* vxy *are called* "supporting" *hyperedges of* v *in* abc.

Remark 8. *The main motivation for the above definition is the following fact. If* P *is a linear path ending in a hyperedge* abc *and* $v \notin V(P)$ *is a vertex strongly associated to* abc *then* P *can be extended by one of the supporting hyperedges of* v *in* abc *to a longer linear path.*

3 Proof of Theorem 2: 2-colorability of linear-cycle-free hypergraphs containing no K_5^3

Let H be a 3-uniform hypergraph without linear cycles.

Claim 9. *If* T *is a linear tree and* $v \in V(T)$ *such that* v *is strongly associated to a hyperedge* abc *of* T, *then* v *belongs to a hyperedge of* T *neighboring (not disjoint to)* abc. *If* $v \notin V(T)$, *and* v *is strongly associated to* $h_1, h_2 \in E(T)$ *then* h_1 *and* h_2 *are neighboring hyperedges.*

Proof. To prove the first statement of the claim, suppose that v is not in a neighboring hyperedge of abc. Then, take the linear path (of length at least 2) from v to abc in T and the appropriate supporting hyperedge of v in abc to produce a linear cycle, a contradiction. To prove the second statement, suppose that h_1 and h_2 are not neighboring hyperedges. Then, take the linear path (of length at least 1) in T joining h_1 and h_2 and an appropriate supporting hyperedge of v in h_1 and h_2 respectively to produce a linear cycle, a contradiction. $\square$

Definition 3 (thick pair, thick hyperedge)**.** *For any two vertices,* $a, b \in V(H)$, *we call the pair* $\{a, b\}$ "thick" *if there are at least two different hyperedges containing* $\{a, b\}$. *We call a hyperedge* abc "thick" *if all the pairs* $\{a, b\}$, $\{b, c\}$ *and* $\{c, a\}$ *are thick.*

Lemma 10. *If* $abc \in E(H)$ *is a thick hyperedge, then the set of vertices associated to it consists of one of the following*

1. *Two vertices that are strongly associated to abc (and no vertices that are weakly associated to abc).*

2. *One vertex that is strongly associated to abc and vertices $w_1, w_2, \ldots, w_m$ such that each w_i is weakly associated to abc and $|\cup_i s_{abc}(w_i)| = 1$. (It is possible that $m = 0$, i.e., no such w_i exists).*

Proof. If there is no vertex strongly associated to abc, then since abc is thick, we must have 3 distinct vertices v_1, v_2, v_3 such that $v_1ab, v_2bc, v_3ca \in E(H)$, a linear cycle, a contradiction. So there must be a vertex strongly associated to abc.

Now we show that if there are two vertices p, q strongly associated to a hyperedge $abc \in E(H)$, then there are no other vertices associated to abc. Suppose by contradiction that there are such vertices. Then, among these vertices there is a vertex r such that $|s_{abc}(p) \cup s_{abc}(q) \cup s_{abc}(r)| = 3$ since abc is thick. Now consider the bipartite graph whose two color classes are $\{p, q, r\}$ and $\{\{a, b\}, \{b, c\}, \{c, a\}\}$ where $v \in \{p, q, r\}$ is connected to $\{x, y\} \in \{\{a, b\}, \{b, c\}, \{c, a\}\}$ if $vxy \in E(H)$. It can be easily checked that Hall's condition holds for the color class $\{p, q, r\}$ and so there exists a matching between the two color classes, but this corresponds to a linear cycle (of size 3) in H, a contradiction.

So the only remaining possibility is that abc has exactly one vertex which is strongly associated to it and maybe some other vertices $w_1, w_2, \ldots, w_m$ that are weakly associated to it. We only have to show that $|\cup_i s_{abc}(w_i)| = 1$. Suppose by contradiction that there are vertices w_i and w_j such that their supports in abc are different. Let $s_{abc}(w_i) = \{\{a, b\}\}$ and $s_{abc}(w_j) = \{\{b, c\}\}$ without loss of generality. Then, since abc is thick, there is a vertex v such that $v \neq w_i$, $v \neq w_j$ and $acv \in E(H)$. Now, acv, abw_i, bcw_j is a linear cycle, a contradiction. $\square$

Given a set of vertices $S \subseteq V(H)$, the subhypergraph of H induced by S is defined as a hypergraph whose vertex set is S and edge set is $\{e \in E(H) \mid e \subseteq S\}$.

Lemma 11 (Main Lemma)**.** *Let T be a linear tree in H. Then there exists a 2-coloring $\gamma : V(T) \mapsto \{1, 2\}$, such that the following properties hold:*

1. *The subhypergraph induced by $V(T)$ is properly 2-colored.*

2. *For each vertex $v \in V(H) \setminus V(T)$ that is strongly associated to some hyperedge of T, v can be colored (by color 1 or 2) so that all hyperedges vab with $a, b \in V(T)$ are properly 2-colored.*

3. *For each remaining vertex $v \in V(H) \setminus V(T)$, all the hyperedges vab with $a, b \in V(T)$ satisfy the property $\gamma(a) \neq \gamma(b)$ (i.e., these hyperedges vab are properly 2-colored regardless of how we fix the color of v later).*

Before we prove this lemma, we will show how to prove Theorem 2 using it.

Observation 12. *Let $w \in V(T)$. Notice that the above lemma holds even if we add the extra condition that the color of w is given.*

Now we prove our main theorem using this lemma.

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Proof of Theorem 2. Let T_1 be any skeleton of H. Then there exists a 2-coloring of T_1 given by Lemma 11. Let $U_1 \subseteq V(H) \setminus V(T_1)$ be the set of all vertices such that each $u \in U_1$ is strongly associated to some hyperedge of T_1. If $|U_1| = 0$, then by Lemma 11 all the vertices of $V(H) \setminus V(T_1)$ can be 2-colored arbitrarily such that the hyperedges vab with $a, b \in V(T_1)$ are properly 2-colored. Also, since T_1 is a skeleton, there are no hyperedges vxy where $v \in V(T_1)$ and $x, y \in V(H) \setminus V(T_1)$. Therefore, the vertices of $V(H) \setminus V(T_1)$ can be 2-colored independently from vertices of $V(T_1)$ and so we have the same problem for the subhypergraph induced by $V(H) \setminus V(T_1)$. So we can assume that $|U_1| \neq 0$. Now let us define a sequence of linear trees $T_1, T_2, \ldots, T_i, T_{i+1}, \ldots, T_m$ recursively as follows: Let $U_i \subseteq V(H) \setminus \cup_{j=1}^{i} V(T_j)$ be the set of vertices where each $u \in U_i$ is strongly associated to some hyperedge of $\cup_{j=1}^{i} T_j$ and let T_{i+1} be a skeleton in the subhypergraph induced by $V(H) \setminus \cup_{j=1}^{i} V(T_j)$ so that T_{i+1} contains at least one vertex from U_i (we continue this procedure as long as $|U_i| \neq 0$; so $|U_m| = 0$). Notice that T_{i+1} might consist of just one vertex. In fact, we will show that $|V(T_{i+1}) \cap U_i| = 1$. Let H_i denote the subhypergraph of H induced by $\cup_{j=1}^{i} V(T_j)$.

Claim 13. *For each $1 \leq i \leq m - 1$, there is a linear path in H_i between any two vertices $u, v \in V(H_i)$. Moreover, $V(T_{i+1}) \cap U_i$ consists of only one vertex and this vertex can be strongly associated to hyperedge(s) of T_s for exactly one s, $1 \leq s \leq i$.*

Proof of Claim 13. We prove the claim by induction on i. For $i = 1$, the statement is trivial. Assume the statement is true for $i = k$. First we will show that there is a linear path between $u \in V(T_{k+1}) \cap U_k$ and any $v \in V(H_k)$. Let $abc \in E(T_s)$ (for some $1 \leq s \leq k$) be the hyperedge in $\cup_{j=1}^{k} T_j$ such that u is strongly associated to abc. Consider a linear path $\mathcal{P}_1$ in H_k between v and $\{a, b, c\}$ (in case, $v \in \{a, b, c\}$, $\mathcal{P}_1$ consists of just v). By adding an appropriate supporting hyperedge of u in abc, $\mathcal{P}_1$ is extended to a linear path between u and v. Notice that this path contains only one vertex from T_{k+1}. Since there is a linear path between every 2 vertices of T_{k+1} we have a linear path between any vertex of T_{k+1} and any vertex of H_k. By the induction hypothesis there is a linear path between any two vertices of H_k and so we have proved the first part of the claim.

Now assume by contradiction that there is a vertex $u' \neq u$, $u' \in V(T_{k+1}) \cap U_k$ which is strongly associated to a hyperedge $pqr \in \cup_{j=1}^{k} E(T_j)$. Take a linear path $\mathcal{P}_2$ in H_k between $\{a, b, c\}$ and $\{p, q, r\}$. Extend $\mathcal{P}_2$ on both ends by appropriate supporting hyperedges of u in abc and u' in pqr respectively. Then this path together with the linear path in T_{k+1} between u and u' is a linear cycle, a contradiction.

So $V(T_{k+1}) \cap U_k$ consists of only vertex, say u. If u is strongly associated to two hyperedges $h_1 \in T_r$ and $h_2 \in T_s$ (where $r \neq s$ and $r, s \leq k$), then take a linear path $\mathcal{P}$ in H_k between h_1 and h_2 and extend it by appropriate supporting hyperedges of u in h_1 and h_2 to a linear cycle, a contradiction. $\square$

We will show that for each $1 \leq k \leq m$, H_k is properly 2-colored such that each T_i, $i \leq k$ is 2-colored according to Lemma 11. For $k = 1$ the above statement is trivially true. Let us assume that the statement is true for k and show that it is true for $k + 1$.

By the above claim $V(T_{k+1}) \cap U_k$ consists of only one vertex u and this vertex is strongly associated to hyperedge(s) of T_s for exactly one $1 \leq s \leq k$. Also, it is easy to see that if $uab \in H_{k+1}$ and $a, b \in V(H_k)$ then $a, b \in V(T_i)$ for some $i \leq k$. If $i = s$ and $a, b \in V(T_s)$, then we know by Lemma 11 that there exists a color for u, say c such that hyperedges uab are properly 2-colored. Let us color u by c. If $i \neq s$, and $a, b \in V(T_i)$ then regardless of the color of u the hyperedges uab

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are 2-colored properly due to Lemma 11. Since the set of vertices that are strongly associated to hyperedges of T_{k+1} is disjoint from $V(H_k)$ (the already 2-colored part), we can apply Lemma 11 to color T_{k+1} such that u is still 2-colored with c by Observation 12. Therefore, we have shown that H_{k+1} is properly 2-colored such that each T_i, $i \leq k+1$ is 2-colored according to Lemma 11, as desired and so we have statement for H_m by induction.

In the remaining vertices, namely $V(H) \backslash V(H_m)$, since there are no strongly associated vertices, by Lemma 11 they can be 2-colored independently from H_m and we now have a smaller vertex set: $V(H) \backslash V(H_m)$ to color. Therefore, by induction on number of vertices we may 2-color H properly. □

3.1 Proof of Lemma 11 (Main Lemma)

We identify some sets of vertices of size 5 which play an important role in the forthcoming proof.

Definition 4. *Let $h_1 = abc$, $h_2 = bde$ where $h_1, h_2 \in E(T)$. If there is no hyperedge $h \in H$ such that $|h \cap (h_1 \cup h_2)| = 2$, then the set of vertices $\{a, b, c, d, e\}$ is called a* special block *of T.*

Claim 14. *Let $h_1 = abc$, $h_2 = bde$ be thick hyperedges of T. If $abe, cbd \in E(H)$ or $abd, cbe \in E(H)$, then $\{a, b, c, d, e\}$ is a special block.*

Proof of Claim 14. Suppose $xyz \in E(H)$ such that $\{x, y, z\} \cap \{a, b, c, d, e\} = \{x, y\}$. It is easy to see that if $x, y \in \{a, c, d, e\}$ then xyz forms a linear triangle with either h_1, h_2 or with abe, cbd or with abd, cbe. So the only cases that are left to be considered are $\{x, y\} = \{d, b\}$ or $\{x, y\} = \{e, b\}$. Since $\{d, e\}$ is a thick pair either dea or dec is a hyperedge in H. W.l.o.g. let us say $dec \in E(H)$. Then in either of the two remaining cases, xyz along with abc and dec will create a linear cycle, a contradiction. □

Claim 15. *Let h_1, h_2 be thick hyperedges of T. If there are two vertices of h_2 which are strongly associated to h_1, then $h_1 \cup h_2$ is a special block.*

Proof of Claim 15. We will show that $|h_1 \cap h_2| = 1$. Assume by contradiction that $|h_1 \cap h_2| = \emptyset$ and $u, v \in h_2$ are strongly associated to h_1. Then it is easy to see that we can choose appropriate supporting hyperedges h_3, h_4 of u and v, respectively, in h_2 such that the hyperedges h_2, h_3, h_4 form a linear triangle, a contradiction.

Let $h_1 = abc$ and $h_2 = bde$, i.e., d and e are strongly associated to h_1. Assume by contradiction that there exists a hyperedge $xyz \in H$ such that $\{x, y\} \subset \{a, b, c, d, e\}$ and $z \notin \{a, b, c, d, e\}$. First let us observe that $\{x, y\} \not\subset \{a, b, c\}$ because the hyperedge abc already has two vertices d, e strongly associated to it and hence cannot have any other vertex associated to it due to Lemma 10. So if we consider the bipartite graph whose color classes are $\{d, e\}$ and $\{\{a, b\}, \{b, c\}\}$ where $v \in \{d, e\}$ is connected to $\{x, y\} \in \{\{a, b\}, \{b, c\}\}$ if $vxy \in E(H)$, We claim that Hall's condition holds for this bipartite graph. Since the hyperedge abc is thick, using Lemma 10, $s_{abc}(d) \cup s_{abc}(e) = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. So the union of the neighborhood of d and e in this bipartite graph is $\{\{a, b\}, \{b, c\}\}$. Since d and e are strongly associated to abc, they each have at least one neighbor in $\{\{a, b\}, \{b, c\}\}$. So there is a matching by Hall's theorem.

So either $abe, cbd \in E(H)$ or $abd, cbe \in E(H)$. Now, by applying Claim 14, we can conclude that $\{a, b, c, d, e\}$ is a special block. □

Since the hypergraph induced on $\{a, b, c, d, e\}$ is not K_5^3, it is easy to see that there is a proper coloring $\gamma : \{a, b, c, d, e\} \mapsto \{1, 2\}$.

Claim 16. *Assume that $h_1 = abc$, $h_2 = bde$ and $\{a, b, c, d, e\}$ is a special block of T. Let T_a, T_b, T_c, T_d, T_e be maximal linear subtrees of T such that $V(T_x) \cap \{a, b, c, d, e\} = \{x\}$ where $x \in \{a, b, c, d, e\}$. Then, if Lemma 11 holds for each T_x, where $x \in \{a, b, c, d, e\}$ and coloring $\gamma : \{a, b, c, d, e\} \mapsto \{1, 2\}$ is given, then it holds for T as well.*

Observation 17. *It is easy to see that $V(T_x) \cap V(T_y) = \emptyset$ for any distinct $x, y \in \{a, b, c, d, e\}$ and $\cup_{x \in \{a,b,c,d,e\}} E(T_x) \cup \{h_1, h_2\} = E(T)$.*

Proof of Claim 16. Take the 2-colorings of T_x's ($x \in \{a, b, c, d, e\}$) guaranteed by Lemma 11 and Observation 12 such that the color of x is $\gamma(x)$.

First we show that the hypergraph induced on $V(T)$ is properly 2-colored. Clearly there is no hyperedge with its vertices in three different T_x's unless it is contained in $\{a, b, c, d, e\}$ because $\{a, b, c, d, e\}$ is a special block and there is no linear cycle in H.

Now we will prove that $w \in V(T_y)$ is not strongly associated to any hyperedge of T_x (for any $y \neq x$). Suppose w is strongly associated to a hyperedge h of T_x. Since w is in T, by Claim 9, there is a hyperedge h' of T which contains w such that $|h \cap h'| = 1$. So h' is a hyperedge of T that has a common vertex with both T_x and T_y. Therefore, h' must be either h_1 or h_2. Moreover, $w = y$ and $h \cap h' = \{x\}$ must hold. Let $h = xpq$. Since $w = y$ is strongly associated to h, either $xyp \in E(H)$ or $xyq \in E(H)$, a contradiction to the assumption that x and y belong to a special block.

So by applying Lemma 11 to T_x, for each hyperedge uvw with $u, v \in V(T_x)$ and $w \in V(T) \backslash V(T_x)$ the color of u and the color of v are different and so uvw is properly 2-colored. Since there is no hyperedge with its vertices in three different T_x's, the hypergraph induced by $V(T)$ is properly 2-colored.

Let $v \in V(H) \backslash V(T)$. First assume that v is not strongly associated to any hyperedge of T and let $p, q \in V(T)$ be arbitrary. We have to show that if $vpq \in E(H)$ then the colors of p and q are different. If $p, q \in T_x$ for some $x \in \{a, b, c, d, e\}$ then we are done because we assumed Lemma 11 holds for T_x. So, let $p \in T_x$ and $q \in T_y$ for some distinct $x, y \in \{a, b, c, d, e\}$. Since both p and q can't be in $\{a, b, c, d, e\}$ (by definition of special block), the linear path between p and q in T has at least 2 hyperedges. This linear path, together with vpq forms a linear cycle, a contradiction.

Now assume that v is strongly associated to a hyperedge of T. If v is strongly associated to hyperedges h_x, h_y of T such that $h_x \in E(T_x)$ and $h_y \in E(T_y)$, then as before we can extend a linear path in T between h_x and h_y to a linear cycle by adding appropriate supporting hyperedges of v in h_x and h_y. This implies that there is a unique $x \in \{a, b, c, d, e\}$ such that v is strongly associated to hyperedge(s) of T_x. Now we show that v can be colored so that all the hyperedges vpq are properly 2-colored.

By the argument in the previous paragraph if $vpq \in E(H)$ then both p and q are in T_y for some $y \in \{a, b, c, d, e\}$. If $y = x$, then by applying Lemma 11 to T_y, there is a coloring of v such that hyperedges vpq are properly 2-colored. If $y \neq x$, then v is not strongly associated to any hyperedge of T_y. So by applying Lemma 11 to T_y again, the colors of p and q are different. Therefore, the hyperedges vpq are properly 2-colored as desired. $\square$

So applying Claim 16 recursively, it suffices to prove Lemma 11 for a linear subtree T of H which has no special block. So from now on, we may assume that there is no special block in T.

We will now construct an auxillary (simple) graph G_T by following the steps in the *Construction* below, one after another. This graph is connected, and its vertex set and edge set satisfy: $V(G_T) = V(T)$ and if $ab \in E(G_T)$ then there exists a vertex $x \in V(T)$ such that $abx \in E(T)$. We show

later that this graph G_T is actually a tree and that a proper 2-coloring of G_T will give us a proper 2-coloring of the hypergraph induced on $V(T)$ as demanded by Lemma 11.

Construction. *We perform the steps as follows. First Step 1 as long as we can, then Step 2 as long as we can, and so on. Naturally, edges added earlier will not be added again.*

Step 1. If $abc, ebd \in E(T)$, abc is a thick hyperedge and e is strongly associated to abc then,

> *(a) add eb to $E(G_T)$.*
>
> *(b) and if $ace \in E(H)$ also holds, then add ac to $E(G_T)$ as well.*

Step 2. If $abc \in E(T)$, $vab \in E(H)$ and v is weakly associated to abc, then add ab to $E(G_T)$.

Step 3. If $abc, ebd \in E(T)$, $v \in V(H) \setminus V(T)$ is strongly associated to abc and ebd, and if acv (respectively edv) is a hyperedge of H, then add ac (respectively ed) to $E(G_T)$.

Step 4. After completing the above steps, for every hyperedge $abc \in E(T)$ we do the following. If abc is thick, and less than two of the three pairs ab, bc, ca are in $E(G_T)$ we add some more pairs arbitrarily so that $E(G_T)$ has exactly two pairs from ab, bc, ca. If abc is not thick and less than two of the three pairs ab, bc, ca are in $E(G_T)$, we add pairs from ab, bc, ca such that only one pair remains outside $E(G_T)$ and it is not a thick pair.

Remark 18. *Notice that all edges xy added in Steps 1, 2, 3 satisfy that $\{x, y\}$ is a thick pair.*

Now we claim the following.

Claim 19. G_T *is a tree (so it can be properly 2-colored).*

Before we prove the above claim, we will show that it implies Lemma 11.

First let us prove that a proper 2-coloring of G_T gives us a proper 2-coloring of the subhypergraph induced by $V(T)$. Since $V(G_T) = V(T)$, a proper 2-coloring of G_T gives us a proper 2-coloring of the hyperedges of T. Therefore, it suffices to prove that for every hyperedge $abc \in E(T)$, the hyperedges xyv where $x, y \in \{a, b, c\}$ and $v \in V(T) \setminus \{a, b, c\}$ are properly 2-colored. If abc is not thick, then it is easy to see that xy (which has to be a thick pair) must be in G_T (due to Step 4 of Construction of G_T) which means that x and y have different colors and so the hyperedge xyv is properly 2-colored, as desired. If abc is thick, then v must be associated to abc. If v is weakly associated to abc, then by the construction of G_T (Step 2 of Construction), xy must be in G_T and so xyv is properly 2-colored again. If v is strongly associated to abc, then by Claim 9, v belongs to a hyperedge h neighboring abc in T (i.e., $|h \cap abc| = 1$). W.l.o.g we may assume that $h \cap abc = \{b\}$, and let $h := vbw$. By Construction Step 1a and 1b of G_T, we have $bv, ac \in E(G_T)$ if $acv \in E(H)$. So b and v have different colors and a and c have different colors. Therefore, all the hyperedges vxy are properly 2-colored. So the subhypergraph induced by $V(T)$ is properly 2-colored.

Now let $v \in V(H) \setminus V(T)$. Note that for any $xyv \in E(H)$ where $x, y \in V(T)$, x, y must belong to a hyperedge of T. We will show that v can be colored as required in Lemma 11. If v is not strongly associated to any hyperedge of T, then for every $xyv \in E(H)$, $xy \in E(G_T)$ and so xyv is properly 2-colored regardless of the color of v. So assume that v is strongly associated to hyperedges $h_1, h_2, \ldots, h_k$ of T. We consider two cases. If $k \geq 2$, then by Claim 9, $|h_i \cap h_j| \neq \emptyset$ for every $i, j \in \{1, 2, \ldots, k\}$. Since h_i are hyperedges of a linear tree, and every two of them have a

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common vertex, there is a vertex o such that $\cap_i h_i = \{o\}$. Let us show that choosing the color of v to be different from the color of o guarentees that all the hyperedges $xyv \in E(H)$ for $x, y \in V(T)$ are properly 2-colored, as required by Lemma 11. If $\{x, y\} \not\subseteq h_i$ for any i, then as we saw before xyv is properly 2-colored independent of the color of v. So $xy \in h_i$ for some i. If $o \in \{x, y\}$, then since o and v are colored differently, xyv is 2-colored properly. If $o \notin \{x, y\}$, then by the construction of G_T (see Construction Step 3), xy is in G_T and so xyv is properly 2-colored, as desired. So the only remaining case is if $k = 1$. In this case, the hyperedge h_1 has two vertices of the same color and if we color v differently from this color, hyperedges vxy where $x, y \in V(T)$ are properly 2-colored. This completes the proof of Lemma 11.

Proof of Claim 19. Notice that G_T is connected as guaranteed by Construction Step 4. Assume by contradiction that G_T has a cycle. Since T is a linear tree, this cycle has to be a triangle abc where $abc \in E(T)$ is a thick hyperedge. First observe that none of the pairs ab, bc, ca were added during Step 4 of the construction of G_T. We now consider different cases for how abc could be formed.

Case 1. *One of the pairs ab, bc, ca was added by Construction Step 1b.*

W.l.o.g let the pair added by Construction Step 1b was ac. Then, there exists a hyperedge $ebd \in E(T)$ such that e is strongly associated to abc and $ace \in E(H)$. So either abe or bce is in $E(H)$. Clearly, there is no $w \notin \{a, b, c, d, e\}$ such that wab or wbc is a hyperedge of H for otherwise we have a linear cycle. Since abc is thick, ab, bc are thick pairs. If either bcd or abd is in $E(H)$, then the conditions of Hall's theorem hold for the bipartite graph whose color classes are $\{ab, bc\}$ and $\{d, e\}$ where $xy \in \{ab, bc\}$ is connected to $z \in \{d, e\}$ if and only if $xyz \in E(H)$. So there is a matching and by Claim 14, we have a contradiction since we assumed there is no special block of T. So assume that $bcd, abd \notin E(H)$. So the only hyperedges (besides abc) containing ab and bc are abe and bce which implies that ab and bc were not added by Construction Steps 1b, 2 and 3. So both ab and bc were added by Construction Step 1a. Assume that bc was added because either b or c was strongly associated to a hyperedge h'. This means that h' is thick and $h' = bde$ because otherwise we have $wbc \in E(H)$ for some $w \notin \{a, b, c, d, e\}$, a contradiction. So c is strongly associated to bde. Similarly, a is strongly associated to bde. So by Claim 15, $\{a, b, c, d, e\}$ is a special block, a contradiction.

So from now on, we can assume that Construction Step 1b was never used to add the pairs ab, bc, ca.

Case 2. *One of the pairs ab, bc, ca was added by Construction Step 3.*

W.l.o.g let us say ac was added by Construction Step 3. Then, there is a hyperedge $bde \in E(T)$ and $v \in V(H) \backslash V(T)$ such that v is strongly associated to both hyperedges abc, bed and $acv \in E(H)$. Since ab is a thick-pair, there is a vertex $w \notin \{a, b, c\}$ such that $abw \in E(H)$. If $w \notin \{a, b, c, d, e, v\}$ then since $acv, wab \in E(H)$ and one of $bev, bdv \in E(H)$, they form a linear cycle, a contradiction. If $w = e$, then since $abe, acv \in E(H)$ and one of $bdv, dev \in E(H)$, we have a linear cycle again, a contradiction. Similarly $w \neq d$. Therefore, $w = v$. So the only hyperedge besides abc which contains ab, is abv. Similarly, the only hyperedge besides abc which contains bc is bcv. This implies that ab and bc were not added by Construction Step 1, 2 and 4. Also, it's easy to see that they were not added by Construction Step 3, otherwise v would have been strongly associated to a hyperedge of T which is not a neighbor of ebd, which is a contradiction.

So the only remaining case is when ab, bc, ca are added by Construction Step 1a or 2.

Case 3. *ab, bc, ca were added by Construction Step* 1*a or* 2.

Two of the pairs ab, bc, ca cannot be added by Construction Step 2 due to Lemma 10. Therefore, we have two subcases: Either exactly one of ab, bc, ca was added by Construction Step 2 and the other two were added by Construction Step 1a or all of them were added by Construction Step 1a.

Assume that all of the pairs ab, bc, ca were added by Construction Step 1a. Let $xy \in \{ab, bc, ca\}$. Let us say xy was added because there is a thick hyperedge $h_{xy} \in E(T)$ which is strongly associated to either x or y. If any two of the there hyperedges h_{ab}, h_{bc}, h_{ca} are the same, then by Claim 15, we have a special block in T, a contradiction. Therefore, $h_{ab} \neq h_{bc} \neq h_{ca}$. But then, we have hyperedges $abv_1, acv_2, bcv_3 \in E(H)$ where $v_1 \in h_{ab}, v_2 \in h_{bc}, v_3 \in h_{ac}$ which form a linear cycle, a contradiction.

Now assume that one of the pairs ab, bc, ca was added by Construction Step 2 and the other two were added by Construction Step 1a. W.l.o.g assume that ab and bc were added by Construction Step 1a and ca by Construction Step 2. Let us say ab (respectively bc) was added because there is a thick hyperedge $h_{ab} \in E(T)$ (respectively $h_{bc} \in E(T)$) which is strongly associated to either a or b (respectively b or c). So there are vertices $v_1 \in h_{ab}$ and $v_2 \in h_{bc}$ such that $abv_1, bcv_2 \in E(H)$. If $h_{ab} = h_{bc}$, then by Claim 15 we have a special block in T, a contradiction. So $h_{ab} \neq h_{bc}$ as before. Let us say ac was added because there is a vertex w weakly associated to abc such that $wac \in E(H)$. If $w \neq v_1$ and $w \neq v_2$, then we have a linear cycle, namely acw, abv_1, bcv_2, a contradiction. So let us assume w.l.o.g that $w = v_1$. Let $h_{ab} = v_1ex$ where x is either a or b. If $x = b$, then h_{ab}, v_1ac, bcv_2 is a linear cycle, a contradiction. If $x = a$, then clearly b is strongly associated to $h_{ab} = v_1xe$. So either the hyperedge $abe \in E(H)$ or $bev_1 \in E(H)$. This hyperedge together with acv_1 and bcv_2 gives us a linear cycle, a contradiction. □

4 Proof of Theorem 5: A degree condition for linear-cycle-free hypergraphs

Let H be a 3-uniform hypergraph without any linear cycles. The following is our main lemma.

Lemma 20. *If there are no vertices $u, v \in V(H)$ such that $uvx \in E(H)$ for all $x \in V(H) \setminus \{u, v\}$ then there is a vertex of degree at most $|V(H)| - 2$ whenever $|V(H)| \geq 6$.*

We prove Lemma 20 in Section 4.1. Using this lemma, we will prove Theorem 5 in Section 4.2.

4.1 Proof of Lemma 20

First let us prove some preliminary lemmas.

Preparatory lemmas

The length of a linear path is defined as the number of hyperedges in it. Let k be the length of a longest linear path in H. Among all skeletons that contain a linear path of length k, let T be a skeleton of maximum possible size. Below we prove some lemmas about such a skeleton.

Lemma 21. *Any hyperedge $abc \in E(T)$ is strongly associated to at most one vertex of $V(H) \setminus V(T)$.*

Proof. Suppose by contradiction that $abc \in E(T)$ is strongly associated to two vertices $v_1, v_2 \in V(H) \setminus V(T)$. Consider the bipartite graph whose color classes are $\{v_1, v_2\}$ and $\{ab, bc, ca\}$ where $v \in \{v_1, v_2\}$ and $xy \in \{ab, bc, ca\}$ are adjacent iff $vxy \in E(H)$. Then it can be easily seen that there is a matching saturating $\{v_1, v_2\}$ between the two color classes. If we replace abc by the two hyperedges corresponding to this matching we will get a skeleton of bigger size and it is easy to see that the length of the longest linear path in it does not decrease, a contradiction. $\qquad\square$

In the remainder of this paper the degree of a vertex $v \in V(T)$ in the subhypergraph of H induced by $V(T)$ is denoted by $d_T(v)$. We have the following corollary of the above lemma.

Corollary 22. *Let $|V(H) \setminus V(T)| = t$. Then the degree of any vertex $v \in V(T)$ which is in exactly one hyperedge of T, is at most $d_T(v) + t + 1$.*

Proof. Let uvw be the hyperedge of T containing v. The total number of hyperedges incident on v is $d_T(v)$ plus the number of hyperedges incident on v that contain a vertex from $V(H) \setminus V(T)$.

It is easy to check that if $x \in V(H) \setminus V(T)$, then at most two hyperedges of H contain both v and x: namely vxu and vxw. Moreover, if both of them are in H then x is strongly associated to uvw and there is at most one such x by Lemma 21. Therefore, for all $x \in V(H) \setminus V(T)$ except at most one, there is at most one hyperedge containing v and x. Thus the corollary follows. $\qquad\square$

Definition 5 (star)**.** Star *of the skeleton T at $v \in V(T)$ is defined as the subtree of T consisting of the hyperedges of T incident to v. The vertex v is called the* center *of this star.*

Definition 6 (opposite pair)**.** *Let us define a graph $G(T)$ consisting of all the pairs covered by the hyperedges of the skeleton T. For a vertex $v \in V(T)$ and a vertex pair $\{x, y\}$ such that $xy \in E(G(T))$, we say $\{x, y\}$ is* opposite *to v if x and y are at equal distance from v in $G(T)$. This equal distance is also called the distance between v and the opposite pair $\{x, y\}$.*

Note that every hyperedge of T contains exactly one pair opposite to v.

In the next lemma, by means of opposite pairs, we can describe all the hyperedges intersecting a given star exactly in its center.

Lemma 23. *Let $v \in V(T)$ and $vab \in E(H)$ be such that a, b are not contained in the star at $v \in V(T)$. Then $\{a, b\}$ is a pair opposite to v in T.*

Proof. Since T is a skeleton of maximum size (among those skeletons containing a linear path of length k), clearly it is impossible that $a, b \in V(H) \setminus V(T)$. Moreover, if exactly one of a, b is in $V(T)$, then since $\{a, b\}$ does not intersect the star at $v \in V(T)$, it is easy to find a linear cycle, a contradiction. Therefore, both a, b are in $V(T)$. Now assume for a contradiction that $\{a, b\}$ is a pair which is not opposite to v in T. Without loss of generality let us assume that distance from v to b in $G(T)$ is strictly smaller than the distance from v to a. Then it is easy to see that the linear path between v and b in T does not contain a, so the hyperedge vba together with this linear path forms a linear cycle, a contradiction. $\qquad\square$

Lemma 24. *Let $\{p_0 q_0 p_1, p_1 q_1 p_2, p_2 q_2 p_3, \ldots, p_{k-2} q_{k-2} p_{k-1}, p_{k-1} q_{k-1} p_k\}$ be a linear path in T. Let $p_0 q_0 x \in E(H)$ for some $x \in V(T)$ and let us consider the linear path between x and p_0. Let $\mathcal{P}'$ be the subpath of this linear path without the starting and ending hyperedges (i.e., not including the two hyperedges which contain p_0 and x). Then, for any $y, z \in V(\mathcal{P}') \setminus \{p_1\}$, $p_0 y z \notin E(H)$.*

Proof. Suppose for a contradiction that $p_0yz \in E(H)$ for some $y, z \in V(\mathcal{P}') \setminus \{p_1\}$. Since yz does not intersect the star at p_0, by Lemma 23, yz is a pair opposite to p_0 in T. Then it is easy to see that p_0yz, p_0q_0x and the linear path between the pair $\{y, z\}$ and x in T form a linear cycle, a contradiction. □

We are now ready to prove Lemma 20. We divide its proof into two cases depending on whether the length of a longest linear path in H is at least 3 or at most 2 (in Section 4.1.1 and Section 4.1.2 respectively).

4.1.1   The length of a longest linear path in H is at least 3

Let k be the length of the longest linear path in H.

Definition 7 (windmills). *Given a linear path $\{p_0q_0p_1, p_1q_1p_2, p_2q_2p_3, \ldots, p_{k-2}q_{k-2}p_{k-1}, p_{k-1}q_{k-1}p_k\}$ of length k in H and a skeleton containing it, the set of hyperedges of this skeleton which contain p_1 (respectively p_{k-1}) except $p_1q_1p_2$ (respectively except $p_{k-2}q_{k-2}p_{k-1}$) is called as a windmill at p_1 (respectively p_{k-1}) and the size of this set is called the size of the windmill. In other words, windmill at p_1 is a star at p_1 minus the hyperedge $p_1q_1p_2$ (and similarly, windmill at p_{k-1} is a star at p_{k-1} minus the hyperedge $p_{k-2}q_{k-2}p_{k-1}$).*

So there are two windmills corresponding to a linear path of length k and a skeleton containing it. The windmill of smaller size among the two is referred as the smaller windmill. If they are of same size, then either one can be considered as the smaller windmill.

Note that as we assumed $k \geq 3$, the two windmills do not have any hyperedges in common.

Among all skeletons that contain a linear path of length k, let us consider skeletons that are of maximum possible size (so preparatory lemmas of the previous section can still be applied). Now among these skeletons let us choose a skeleton T and a linear path $\mathcal{P}$ of length k in T such that the size of the smaller windmill corresponding to T and $\mathcal{P}$ is minimum. Let $\mathcal{P} = \{p_0q_0p_1, p_1q_1p_2, p_2q_2p_3, \ldots, p_{k-1}q_{k-1}p_k\}$. Without loss of generality, we may assume that the smaller windmill is at p_1.

We distinguish two cases depending on the size of the smaller windmill corresponding to T and $\mathcal{P}$.

Case 1. *The size of the smaller windmill (corresponding to T and $\mathcal{P}$) is at least 2.*

We will show that the degree of p_0 is at most $|V(H)| - 2 = n - 2$.

If x is in $V(T) \setminus \{p_1, p_0, q_0\}$, then we claim that $p_0q_0x \notin E(H)$ because if x is in the windmill around p_1 then the linear path $\mathcal{P}$ can be extended. If x is not in the windmill around p_1 then by replacing the hyperedge $p_0q_0p_1$ with p_0q_0x will decrease the size of the smaller windmill (and the length of the longest linear path in the skeleton does not decrease) contradicting the assumption that the size of the smaller windmill is minimum.

So the hyperedges in $V(T)$ containing p_0 are of the type p_0p_1x where $x \in V(T) \setminus \{q_0\}$ or of the type p_0xy where $x, y \in V(T) \setminus \{p_1, q_0\}$ plus the hyperedge $p_0q_0p_1$. Below we will count the number of hyperedges of these two types separately.

First, let us count the number of hyperedges of the type p_0p_1x where $x \in V(T) \setminus \{q_0\}$. Since p_0p_1 can't be opposite to any $x \in V(T) \setminus \{q_0\}$, by Lemma 23, p_0p_1 must intersect the star at x. This means that x should be contained in the star at p_1. So the number of hyperedges of the type p_0p_1x

where $x \in V(T) \setminus \{q_0\}$ is $2w_1$ where w_1 is the size of the windmill at p_1 (note that by definition, windmill at p_1 does not contain the edge $p_1q_1p_2$). Let w_2 be the size of the windmill at p_{k-1} (So $w_1 \leq w_2$ by our assumption).

Now, let us count the number of hyperedges of the type p_0xy where $x, y \in V(T) \setminus \{p_1, q_0\}$. Since xy doesn't intersect the star at p_0, by Lemma 23, xy must be opposite to p_0. If the pair xy is contained in a hyperedge of either windmill (at p_1 or p_{k-1}) then we can extend $\mathcal{P}$ by p_0xy, a contradiction.

So the number of such xy pairs is at most

$$\frac{V(T) - (2w_1 + 1) - 2w_2}{2} = \frac{(n - t) - (2w_1 + 1) - 2w_2}{2}.$$

Therefore, the total degree of p_0 in the subhypergraph induced by $V(T)$,

$$d_T(p_0) \leq 1 + 2w_1 + \frac{(n - t) - (2w_1 + 1) - 2w_2}{2}.$$

Thus by Corollary 22, the degree of p_0 is at most

$$1 + 2w_1 + \frac{(n - t) - (2w_1 + 1) - 2w_2}{2} + t + 1 = \frac{n + t + 2w_1 - 2w_2 + 3}{2} \leq \frac{n + t + 3}{2}.$$

So we are done unless $\frac{n+t+3}{2} \geq n - 1$, which simplifies to $n - t = |V(T)| \leq 5$, so T contains at most 2 hyperedges. Therefore the length of $\mathcal{P}$ is at most 2 (recall that $\mathcal{P}$ is contained in T). However, as $\mathcal{P}$ is a longest linear path in H, this contradicts the assumption of Section 4.1.1.

Case 2. *The size of the smaller windmill (corresponding to T and $\mathcal{P}$) is 1.*

There are three types of hyperedges in H that contain p_0: hyperedges of the type p_0q_0x where $x \in V(H) \setminus \{p_0, q_0\}$, hyperedges of the type p_0yz or of the type p_0p_1w where $y, z, w \in V(H) \setminus \{p_0, q_0, p_1\}$. (Note that we consider the hyperedge $p_0q_0p_1$ as of the type p_0q_0x.)

Let r be the number of hyperedges in H of the type p_0q_0x where $x \in V(H) \setminus \{p_0, q_0\}$ and let s be the number of hyperedges in H of the type p_0yz where $y, z \in V(H) \setminus \{p_0, q_0, p_1\}$. Below we upper bound the number of hyperedges of these two types together.

Claim 25. $r + s \leq n - 2$ *and if equality holds then* $p_0p_kq_{k-1} \in E(H)$.

Proof. First we claim that $r + s \leq n - s$. Consider a hyperedge of the type p_0yz where $y, z \in V(H) \setminus \{p_0, q_0, p_1\}$. Since $\{y, z\}$ doesn't intersect the star at p_0, by Lemma 23, the pair $\{y, z\}$ is opposite to p_0. We claim that if $p_0yz \in E(H)$ then the pair $\{y, z\}$ must be contained in the linear path $\mathcal{P}$. It is easy to see that since $\{y, z\}$ is opposite to p_0, either both y and z are contained in $\mathcal{P}$ or both of them are not in $\mathcal{P}$. In the latter case, $\mathcal{P}$ can be extended by adding the hyperedge p_0yz, contradicting the maximality of $\mathcal{P}$. So y and z are contained in $\mathcal{P}$. Now consider the opposite pair $\{y_1, z_1\}$ closest (in the sense of distance defined in Definition 6) to p_0 in $\mathcal{P}$ such that $p_0y_1z_1 \in E(H)$. By Lemma 24, the farthest $x \in \mathcal{P}$ from p_0 such that $p_0q_0x \in E(H)$ can be either y_1 or z_1. This means that a vertex in $V(H) \setminus \{p_0, q_0\}$ can not be contained in both a hyperedge of type p_0q_0x and a hyperedge of type p_0yz except for the vertices y_1, z_1. Since the hyperedges of the type p_0yz cover $2s$ vertices from $V(H) \setminus \{p_0, q_0\}$ and hyperedges of the type p_0q_0x cover r vertices from $V(H) \setminus \{p_0, q_0\}$, we have $r + 2s \leq n - 2 + 2 = n$, proving that $r + s \leq n - s$.

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Since $r + s \leq n - s$, Claim 25 is proved if $s \geq 2$ and so we can assume $s \leq 1$. Recalling the assumption of Lemma 20, there are no vertices $u, v \in V(H)$ such that $uvx \in E(H)$ for every $x \in V(H) \setminus \{u, v\}$, so we have $r \leq n - 3$. Thus,

$$r + s \leq n - 3 + 1 = n - 2,$$

as desired.

Now let us observe what happens when $r + s = n - 2$. Then we must have $s \geq 1$ (as $r \leq n - 3$). That is, there exists a hyperedge of the type $p_0 y z$ where $y, z \in V(H) \setminus \{p_0, q_0, p_1\}$. The pair $\{y, z\}$ must be opposite to p_0 and is contained in $\mathcal{P}$ as before. So if $\{y, z\} \neq \{p_k, q_{k-1}\}$ then by Lemma 24, $p_0 q_0 p_k, p_0 q_0 q_{k-1} \notin E(H)$ (here we used the existence of an edge of the type $p_0 y z$ and that p_k and q_{k-1} are further than y, z). So the vertices p_k, q_{k-1} do not belong to any hyperedge of type $p_0 q_0 x$ or $p_0 y z$. So, by a similar argument as in the previous paragraph, $r + 2s \leq n - 4 + 2 = n - 2$ which is a contradiction since we assumed $r + s = n - 2$ and $s \geq 1$. $\quad\square$

We distinguish two subcases based on the existence of a hyperedge of certain type.

Case 2.1. *There is a hyperedge of type $p_0 q_0 x \in E(H)$ where $x \in V(T) \setminus \{p_0, p_1, p_2, q_0, q_1\}$.*

In this case, we claim that number of hyperedges of the type $p_0 p_1 y$ in H where $y \in V(H) \setminus \{p_0, q_0, p_1\}$ is at most 1 and if such a hyperedge exists then y is either p_2 or q_1. Assume that $p_0 p_1 y' \in E(H)$ where $y' \in V(H) \setminus \{p_0, q_0, p_1\}$. Let $\mathcal{P}_1$ be a linear path in T between (and including) x and p_1. If $y' \notin \mathcal{P}_1$, then $p_0 q_0 x$, $p_0 p_1 y'$ and $\mathcal{P}_1$ form a linear cycle. So $y' \in \mathcal{P}_1$. Since $\{p_0, p_1\}$ cannot be an opposite pair of any vertex on $\mathcal{P}_1$ except q_0, by Lemma 23, $\{p_0, p_1\}$ must intersect the star at y'. So y' is either p_2 or q_1. If both hyperedges $p_0 p_1 p_2$ and $p_0 p_1 q_1$ are in H then $p_0 q_0 x$, $\mathcal{P}_1 \setminus \{p_1 p_2 q_1\}$ and either $p_0 p_1 p_2$ (in case p_2 is on the path $\mathcal{P}_1 \setminus \{p_1 p_2 q_1\}$) or $p_0 p_1 q_1$ (in case q_1 is on the path $\mathcal{P}_1 \setminus \{p_1 p_2 q_1\}$) form a linear cycle. Therefore the desired claim follows.

If neither of the hyperedges $p_0 p_1 p_2$, $p_0 p_1 q_1$ are in H, then the degree of p_0 is $r + s$ and by Claim 25, $r + s \leq n - 2$ and so Lemma 20 holds. Therefore, from now on, we may assume that exactly one of the two hyperedges $p_0 p_1 p_2$, $p_0 p_1 q_1$ is in H. If $r + s$ is strictly less than $n - 2$ then degree of p_0 is at most $n - 2$ and Lemma 20 holds again. So we also assume that $r + s = n - 2$. By Claim 25 if $r + s = n - 2$, then $p_0 p_k q_{k-1} \in E(H)$. It follows that the size of the windmill at p_{k-1} is 1 because if it is more than 1, then the linear path consisting of $p_0 p_k q_{k-1}$, $\mathcal{P} \setminus p_{k-1} p_k q_{k-1}$ and one of the hyperedges of the windmill at p_{k-1} (different from $p_{k-1} p_k q_{k-1}$) form a linear path longer than $\mathcal{P}$, a contradiction. Therefore the size of the windmills at p_{k-1} and p_1 are both 1. By symmetry, if we define r' and s' for p_k analogous to how we defined r and s for p_0, Claim 25 holds for them. Since a hyperedge of the type $p_k q_{k-1} x$ exists where $x \in V(T) \setminus \{p_k, q_{k-1}, p_{k-1}, q_{k-2}, p_{k-2}\}$ (namely $p_k q_{k-1} p_0$), by repeating the same argument as before we can assume that $r' + s' = n - 2$ and so $p_0 q_0 p_k \in E(H)$. Using Lemma 24 for p_k (instead of p_0), it is easy to see that $s' \leq 1$. So $r' \geq n - 3$. We know that $p_0 p_1 y \in E(H)$ where y is either p_2 or q_1. Now $p_0 p_1 y$, $p_0 q_0 p_k$ and either $p_k q_{k-1} y$ or $p_k q_{k-1} p_1$ (one of them exists because $r' \geq n - 3$) form a linear cycle, a contradiction.

Case 2.2. *There is no hyperedge of type $p_0 q_0 x \in E(H)$ where $x \in V(T) \setminus \{p_0, p_1, p_2, q_0, q_1\}$.*

Let d_0 be the degree of p_0 in the subhypergraph of H induced by $\{p_0, p_1, p_2, q_0, q_1\}$. Clearly $d_0 \leq 6$. If $p_k q_{k-1} p_0$ or $p_k q_{k-1} q_0$ are in H, then the size of the windmill at p_{k-1} is 1, otherwise the linear path consisting of $p_k q_{k-1} p_0$, $\mathcal{P} \setminus p_{k-1} p_k q_{k-1}$ and one of the hyperedges of the windmill at p_{k-1} form a linear path longer than $\mathcal{P}$, a contradiction. So by symmetry (renaming p_i to p_{k-i} for

each $0 \leq i \leq k$ and q_i to q_{k-1-i} for each $0 \leq i \leq k-1$) we are done by Case 2.1. Thus we can assume

$$p_k q_{k-1} p_0, p_k q_{k-1} q_0 \notin E(H). \tag{1}$$

If there is a vertex $v \in V(H) \setminus V(T)$ which is strongly associated to $p_0 q_0 p_1$, then we claim that $d_0 \leq 4$ because if either $p_0 q_0 p_2$ or $p_0 q_0 q_1$ is in H, then it is easy to check that we have a linear cycle. Let $|V(H) \setminus V(T)| = t$. So the degree of p_0 in the subhypergraph of H induced by T, $d_T(p_0) \leq d_0 + \frac{n-t-7}{2}$ (here we used $p_k q_{k-1} p_0 \notin E(H)$). By Corollary 22, degree of p_0 is at most

$$d_0 + \frac{n-t-7}{2} + t + 1 \leq \frac{n+t+3}{2}.$$

Then, Lemma 20 holds unless $\frac{n+t+3}{2} \geq n-1$ which simplifies to $n-t = |V(T)| \leq 5$, so T contains at most 2 hyperedges. Therefore the length of $\mathcal{P}$ - a longest linear path of H- is also at most 2 contradicting the assumption of Section 4.1.1.

If there is no vertex $v \in V(H) \setminus V(T)$ which is strongly associated to $p_0 q_0 p_1$, then degree of p_0 is at most $d_T(p_0) + t$. And, since $d_T(p_0) \leq d_0 + \frac{n-t-7}{2}$, the degree of p_0 is at most

$$d_0 + \frac{n-t-7}{2} + t \leq d_0 + \frac{n+t-7}{2},$$

and Lemma 20 holds unless $d_0 + \frac{n+t-7}{2} \geq n-1$ which simplifies to

$$d_0 \geq \frac{n-t+5}{2}. \tag{2}$$

If $n-t > 7$ then $d_0 > 6$ which is impossible. So we may assume $n-t \leq 7$. If $n-t = |V(T)| \leq 5$ then T contains at most two hyperedges, so the length of $\mathcal{P}$ is also at most 2 contradicting the assumption of Section 4.1.1. Since $n-t$ is odd (as the number of vertices in a skeleton is always odd) the only remaining case is when $n-t = 7$. In this case the size of the skeleton T is 3 and since T contains a linear path of length at least 3 (as we are in Section 4.1.1), T is a linear path of length exactly 3 (i.e., T and $\mathcal{P}$ contain the same set of hyperedges). Thus $k = 3$. Moreover, by (2), $d_0 \geq 6$. However, since $d_0 \leq 6$, we have $d_0 = 6$. By symmetry, the degree of q_0 in the subhypergraph induced by $\{p_0, p_1, p_2, q_0, q_1\}$ is also 6. This implies that

$$q_0 p_1 q_1, p_0 p_1 p_2, p_0 p_2 q_1 \in E(H). \tag{3}$$

By (1), we can assume $p_3 q_2 p_0, p_3 q_2 q_0 \notin E(H)$. Recall that $p_3 p_0 q_0 \notin E(H)$. Therefore, any hyperedge containing p_3 in the subhypergraph induced by $V(T)$ is contained in $\{p_3, q_2, p_2, q_1, p_1\}$. However, by (3) the two hyperedges $p_3 p_2 p_1, p_3 p_2 q_1 \notin E(H)$, since otherwise we can find a linear cycle in H. Therefore, the degree of p_3 in the sybhypergraph induced by $V(T)$ is $d_T(p_3) \leq 6-2 = 4$. Thus, by Corollary 22, degree of p_3 is at most $d_T(p_3) + |V(H) \setminus V(T)| + 1 \leq 5 + |V(H) \setminus V(T)| = |V(H)| - 2$, as desired, finishing the proof of this case.

4.1.2 The length of a longest linear path in H is at most 2

Let k be the length of the longest linear path in H. So $k \leq 2$. Among all skeletons that contain a linear path of length k, let T be a skeleton of maximum possible size.

As the length of a longest linear path in H is at most 2, it is easy to see that all of the hyperedges in T share a common vertex, b. We consider the following three cases depending on the number of hyperedges in T.

Case 1. *T consists of at least 3 hyperedges.*

Let $E(T) = \{v_1v_2b, v_3v_4b, \ldots, v_{2s-1}v_{2s}b\}$.

Claim 26. $v_iv_jv_k \notin E(H)$ *for any* $i, j, k \in \{1, 2, \ldots, 2s\}$. *Thus, every hyperedge in the subhypergraph induced by* $V(T)$ *must contain* b.

Proof. Indeed, if v_i, v_j, v_k belong to three different hyperedges of T, then it is easy to find a linear cycle, so suppose two of them belong to the same hyperedge. Without loss of generality, let $\{v_i, v_j\} = \{v_1, v_2\}$. Then replacing v_1v_2b with $v_iv_jv_k$ we can produce a linear path of length 3 in H (here we used that T contains at least 3 hyperedges), a contradiction to the assumption of Section 4.1.2; proving the claim. □

Now we consider two subcases.

Case 1.1. *There exist* $i', j' \in \{1, 2, 3, \ldots, 2s\}$ *such that* $v_{i'}bv_{j'} \notin E(H)$.

By Claim 26, every hyperedge in the subhypergraph induced by $V(T)$ must be of the form v_ibv_j for some $i, j \in \{1, 2, 3, \ldots, 2s\}$. Therefore, as $v_{i'}bv_{j'} \notin E(H)$, the degree $d_T(v_{i'})$ of $v_{i'}$ in the subhypergraph induced by $V(T)$ is at most $|V(T)| - 3$. Thus by Corollary 22, the degree of $v_{i'}$ is at most $|V(T)| - 3 + |V(H) \setminus V(T)| + 1 = |V(H)| - 2$, and we are done.

Case 1.2. $v_ibv_j \in E(H)$ *for every* $i, j \in \{1, 2, 3, \ldots, 2s\}$.

Consider a vertex v_i with $i \in \{1, 2, 3, \ldots, 2s\}$. By Claim 26, degree of v_i in the subhypergraph induced by $V(T)$ is $|V(T)| - 2$.

Note that there is no hyperedge of the form v_ixy where $x, y \in V(H) \setminus V(T)$ because of the maximality of T. Moreover, there is no hyperedge of the form v_iv_jx where $x \in V(H) \setminus V(T)$ and $j \in \{1, 2, 3, \ldots, 2s\}$. Indeed, the hyperedges v_iv_jx, $v_ibv_{i'}$, $v_jbv_{j'}$ for any two distinct vertices i', j' with $i', j' \in \{1, 2, 3, \ldots, 2s\} \setminus \{i, j\}$ form a linear cycle. Therefore, any hyperedge containing v_i, and a vertex $x \in V(H) \setminus V(T)$, must be of the form v_ixb.

Therefore the total degree of v_i is at most $(|V(T)| - 2) + |V(H) \setminus V(T)| = |V(H)| - 2$, as desired.

Case 2. *T consists of exactly 2 hyperedges.*

Let $E(T) = \{a_1a_2b, c_1c_2b\}$. Since $|V(H)| \geq 6$, $|V(H) \setminus V(T)| \neq \emptyset$. We consider the following two subcases.

Case 2.1. *There is no vertex in* $V(H) \setminus V(T)$ *which is strongly associated to any hyperedge of* T.

First suppose $|V(H)| \leq 7$. Consider a vertex $v \in V(H) \setminus V(T)$. It is easy to see that if vxy is a hyperedge, then $x, y \in V(T)$. Moreover, x, y are contained in a hyperedge of T, so v is associated to a hyperedge of T. Since v is not strongly associated to any hyperedge of T and T has only 2 hyperedges, it follows that v has degree at most $2 \leq |V(H)| - 2$ (since $|V(H)| \geq 6$) as required.

So we can assume $|V(H)| \geq 8$. Suppose there is no hyperedge of the form vxy with $v \in V(H) \setminus V(T)$ and $x, y \in V(T)$. Then it is easy to see that any hyperedge of H which contains $a_1 \in V(T)$ is contained in $V(T)$, so degree of a_1 is at most $6 \leq |V(H)| - 2$, as desired. So we can assume that there exists a hyperedge $vx'y'$ with $v \in V(H) \setminus V(T)$ and $x', y' \in V(T)$. It follows that x', y' are contained in a hyperedge of T. By assumption v is not strongly associated to any

hyperedge of T. So the number of hyperedges vxy such that $x, y \in V(T)$ is at most 2 (as there are only two hyperedges in T).

Now we upper bound the number of hyperedges vxy such that $x, y \in V(H) \setminus V(T)$. Let us define a (trace) graph G_v on the vertex set $V(G_v) := V(H) \setminus (V(T) \cup \{v\})$ where $ab \in E(G_v)$ if and only if $abv \in E(H)$. Now notice that if there are two edges $pq, rs \in E(G_v)$ that are disjoint then $vpq, vrs, vx'y' \in E(H)$ form a skeleton with 3 hyperedges which contradicts the assumption of Case 2. So every two edges of G_v have a common vertex and so $E(G_v)$ is either a triangle or a star. In either case, $|E(G_v)| \leq |V(G_v)|$.

Therefore, the total degree of v is at most $2 + |E(G_v)| \leq 2 + |V(G_v)| = 2 + n - 6 = n - 4$, as desired.

Case 2.2. *There is a vertex $v \in V(H) \setminus V(T)$ which is strongly associated to a hyperedge of T.*

As before, notice that if vxy is a hyperedge, then either $x, y \in V(T)$ or $x, y \in V(H) \setminus V(T)$. Assume without loss of generality that v is strongly associated to $a_1 a_2 b$. So $vba_i \in E(H)$ for some $i \in \{1, 2\}$ which implies that there is no hyperedge vxy with $x, y \in V(H) \setminus V(T)$ because otherwise vxy, vba_i, bc_1c_2 form a linear path of length 3, a contradiction. Therefore, any hyperedge incident on v is of the form vxy with $x, y \in V(T)$. Moreover, the pair xy is contained in a hyperedge of T.

If v is not strongly associated to bc_1c_2, then the degree of v is at most $1 + 3 = 4$ and we are done since we assumed $|V(H)| \geq 6$. Therefore, we may assume v is strongly associated to bc_1c_2 as well and so $vbc_j \in E(H)$ for some $j \in \{1, 2\}$. Now it is easy to see that $a_1 a_2 c_k \notin E(H)$ and $c_1 c_2 a_k \notin E(H)$ for any $k \in \{1, 2\}$ because otherwise we have a linear cycle. If there is a vertex among $\{a_1, a_2, c_1, c_2\}$ with degree at most 2 in the subhypergraph induced by $V(T)$, then by Corollary 22, the degree of this vertex in H is at most $2 + t + 1 = t + 3$ where $|V(H) \setminus V(T)| = t$ but then we are done because $|V(H)| = t + 5$. So we may assume that all of the vertices $\{a_1, a_2, c_1, c_2\}$ have degree at least 3 in the subhypergraph induced by $V(T)$. It is easy to see that the only way this degree condition is met for the vertex a_i is if $a_i b c_1, a_i b c_2 \in E(H)$ for each $i \in \{1, 2\}$. This implies that $a_1 a_2 v, c_1 c_2 v \notin E(H)$ because otherwise we have a linear cycle. So the degree of v is at most 4 and we are done because $|V(H)| \geq 6$.

Case 3. *T consists of only one hyperedge.*

Let $E(T) = \{abc\}$. Consider the trace graph G_a where $\{x, y\} \in E(G_a)$ if and only if $axy \in E(H)$. Now notice that if there are two edges $pq, rs \in E(G_a)$ that are disjoint then $apq, ars \in E(H)$ form a skeleton with two hyperedges, a contradiction. So every two edges of G_a have a common vertex. It is easy to see that the set of edges of such a graph is either a star (a graph where all the edges have a common vertex) or a triangle. Notice that there may be some isolated vertices in the graph. Since $|V(G_a)| = |V(H)| - 1 \geq 5$, it is easy to see that $|E(G_a)| \leq |V(G_a)| - 1$ holds. So the degree of a in H is $|E(G_a)| \leq |V(G_a)| - 1 = |V(H)| - 2$ as desired.

4.2 Proof of Theorem 5

Now we prove Theorem 5 using Lemma 20. If we assume for a contradiction that Theorem 5 does not hold, then by Lemma 20 we know that there are vertices $u, v \in V(H)$ such that $uvx \in E(H)$ for every $x \in V(H)$ whenever $|V(H)| \geq 6$.

Lemma 27. *Let H be a 3-uniform linear-cycle-free hypergraph. Let $s \geq -1$ be an integer. If the degree of every vertex in H is at least $|V(H)| + s$ and $|V(H)| \geq 6$, then it has a subhypergraph H_0 such that $|V(H_0)| = |V(H)| - 4$ and degree of every vertex in H_0 is at least $|V(H_0)| + s + 2$.*

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Proof. Let $|V(H)| = n$. Since $s \geq -1$, the degree of every vertex in H is at least $|V(H)| - 1$, so by Lemma 20, there exist vertices $u, v \in V(H)$ such that $uvw \in E(H)$ for every $w \in V(H)$.

Claim 28. *Suppose $xyu \in E(H)$ where $x, y \in V(H) \setminus \{u, v\}$. If $xab \in E(H)$ where $a, b \in V(H) \setminus \{u, v, x\}$, then $y \in \{a, b\}$.*

Proof. Suppose by contradiction that $y \notin \{a, b\}$. Then the hyperedges, uva, xab and xyu form a linear cycle, a contradiction. $\square$

Since degree of u is at least $n + s \geq n - 1$, there exists a hyperedge $x_1 y_1 u$ where $x_1, y_1 \in V(H) \setminus \{u, v\}$. Consider the trace graph $G_{u,v}$ where $\{p, q\} \in E(G_{u,v})$ if and only if either $pqu \in E(H)$ or $pqv \in E(H)$. Let the degree of x_1 in $G_{u,v}$ be d and let the corresponding edges be $x_1 y_1, x_1 y_2, \ldots, x_1 y_d$.

First let us assume $d \geq 2$. If $x_1 y_i u, x_1 y_j v \in E(H)$ where $i \neq j$, then $x_1 y_i u, x_1 y_j v$ and uva where $a \notin \{u, v, y_i, y_j, x_1\}$ form a linear cycle. Therefore, either for every $1 \leq i \leq d$, $x_1 y_i u \in E(H), x_1 y_i v \notin E(H)$ or for every $1 \leq i \leq d$, $x_1 y_i v \in E(H), x_1 y_i u \notin E(H)$. W.l.o.g assume the former. So degree of x_1 in H is $d + 1$ plus the number of hyperedges $x_1 ab$ such that $a, b \in V(H) \setminus \{u, v, x_1\}$. By assumption x_1 has degree at least $n + s \geq n - 1$. Since $d + 1 \leq n - 3 + 1 = n - 2$, there exists a hyperedge $x_1 a_1 b_1$ where $a_1, b_1 \in V(H) \setminus \{u, v, x_1\}$. By Claim 28, it follows that $y_1, y_2, \ldots, y_d \in \{a_1, b_1\}$, so $d \leq 2$. Thus, $d = 2$ and so if $x_1 ab \in E(H)$ where $a, b \in V(H) \setminus \{u, v, x_1\}$, then $\{y_1, y_2\} = \{a, b\}$. So the degree of x_1 is at most $d + 1 + 1 = 4$ a contradiction since $n \geq 6$.

Now we are left with the case when $d = 1$. By Claim 28, if $x_1 ab \in E(H)$ where $a, b \in V(H) \setminus \{u, v, x_1\}$, then $y_1 \in \{a, b\}$, so every hyperedge containing x_1, except $x_1 uv$, is of the form $x_1 y_1 a \in E(H)$ where $a \in V(H) \setminus \{x_1, y_1\}$. So, $x_1 y_1 a \in E(H)$ for every $a \in V(H) \setminus \{x_1, y_1\}$ because otherwise degree of x_1 is at most $n - 2 < n - s$, a contradiction. Let the subhypergraph induced by $V(H) \setminus \{u, v, x_1, y_1\}$ be H_0.

Consider an arbitrary vertex $a \in V(H_0)$. It is easy to see that if $abu \in E(H)$ for some $b \in V(H_0) \setminus \{a\}$ then the hyperedges $abu, uvx_1, x_1 y_1 a$ form a linear cycle, a contradiction. Similarly, $abv, abx_1, aby_1 \notin E(H)$ for any $b \in V(H_0) \setminus \{a\}$. Moreover, there exists no hyperedge in H that contains one vertex from uv, one vertex from $x_1 y_1$ and the vertex a, since this hyperedge together with $uvw, x_1 y_1 w$ for any $w \in V(H) \setminus \{u, v, x_1, y_1, a\}$ forms a linear cycle, a contradiction. Therefore, the degree of a in H_0 is exactly 2 less than its degree in H. So degree of a in H_0 is at least $n + s - 2 = |V(H)| + s - 2 = |V(H_0)| + s + 2$. Since the vertex a was chosen arbitrarily, the desired lemma follows. $\square$

We will use the following simple corollary obtained by repeated applications of Lemma 27.

Corollary 29. *If H_l is a subhypergraph of H where degree of each vertex in $V(H_l)$ is at least $|V(H_l)| + n_l$, where $n_l \geq -1$, then it has a subhypergraph H_{l+1} such that the degree of every vertex in $V(H_{l+1})$ is at least $|V(H_{l+1})| + n_{l+1}$, where $n_{l+1} = n_l + 2$ and $|V(H_{l+1})| = |V(H_l)| - 4$.*

Assume by contradiction that Theorem 5 does not hold. That is, there is a linear-cycle-free hypergraph $H := H_1$ where degree of every vertex is at least $|V(H)| - 1$ and $|V(H)| \geq 10$. Then by using Corollary 29, there is an l such that $|V(H_l)| \leq 5$ and the degree of every vertex in H_l is at least $|V(H_l)| + 3$ (notice that since $|V(H)| = |V(H_1)| \geq 10$, we must have $l \geq 3$), which is impossible.

5 Concluding Remarks

The following problems asked by Gyárfás, Győri and Simonovits remain open.

Problem 30. *Can one describe the structure of 3-uniform hypergraphs with no linear cycles?*

It is conceivable that one might construct a linear-cycle-free hypergraph by repeatedly adding hyperedges in a certain fashion. For example, if H is a linear-cycle-free hypergraph, then adding two new vertices u, v to $V(H)$ and adding all the hyperedges of the type uvx for $x \in V(H)$ to $E(H)$, will give us another linear-cycle-free hypergraph.

Problem 31. *Which results extend to r-uniform hypergraphs?*

For $r = 4$ the structure of the "skeleton" seems to be more complicated. It is, however, conceivable that the current methods are useful for this case. In general, the approach of using skeletons seems to be very effective in proving results about linear-cycle-free hypergraphs. It would be interesting to discover more applications of this approach.

Acknowledgment

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