Does Negative Mass Imply Superluminal Motion? An Investigation in Axiomatic Relativity Theory

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Abstract

Formalization of physical theories using mathematical logic allows us to discuss the assumptions on which they are based, and the extent to which those assumptions can be weakened. It also allows us to investigate hypothetical claims, and hence identify experimental consequences by which they can be tested. We illustrate the potential for these techniques by reviewing the remarkable growth in First Order Relativity Theory (FORT) over the past decade, and describe the current state of the art in this field. We take as a running case study the question "*Does negative mass imply superluminal motion?*", and show how a many-sorted first-order theory based on just a few intuitively obvious, but rigorously expressed, axioms allows us to formulate and answer this question in mathematically precise terms.

Key words: axiomatic physics, special relativity, dynamics, tachyon, negative mass, logic

PACS: 03.30.+p, 03.65.Ca

The authors would like to thank the anonymous referees for their helpful and insightful comments.

1 Axiomatization of Physical Theories

Relativity theory has been intrinsically axiomatic since its birth, since Einstein presented his 1905 theory of special relativity as a consequence of two informal postulates [12]. Since then several distinct formal axiomatizations of relativity theories (both special and general) have appeared in the literature (see, e.g., [6] and references therein). More recently, a number of researchers have started working on comparing and connecting these different axiomatizations, as well as developing and improving tools to make this possible [5, 7, 8, 28, 31]. In this paper, we work in the framework developed by the research team/school of Hajnal Andréka and István Németi [6, 1, 4], and illustrate the techniques involved by formulating and investigating the question "*does negative mass imply superluminal motion?*" within that framework.

We have chosen this question for our case study because it illustrates a particularly powerful application of the logical approach, viz. the ability to formulate and reason about concepts about which we do not yet have any experimental experience. In such circumstances the ability to write down formal definitions and make logical deductions is essential. If we can show that a concept leads inexorably to logical paradox, we thereby provide firm evidence that the concept is unphysical. Alternatively, we may discover physically feasible preconditions under which the concept is logically entailed, and this in turn gives the potential to devise relevant experimental tests. For example, we know that simple inelastic collisions between positive-mass slower-than-light particles cannot result in particles moving faster-than-light (tachyons). So any experiment in which tachyons are generated through a simple inelastic collision of slower-than-light particles must entail the existence of negative-mass particles. Conversely, as we show formally below, the possibility of simple inelastic collisions between negative-mass particles necessarily entails the existence of tachyons. As a result, those who wish to refute the possibility of negative mass need only refute the existence of tachyons — and the logical method can again be used to investigate this issue. For example, a common argument against tachyons (and hence against negative-mass particles) is that they would lead to causality violations, but the logical methods espoused in this paper can be used to show that this argument is itself logically flawed — tachyons can exist in relativity theories *without* introducing causality violations [2].

Even though negative mass has never been observed experimentally, physicists have speculated [13] about its existence since at least the 19th century and a considerable amount has been published on the subject. Even in the absence of physical evidence, there are situations where negative mass can be invoked as a useful simplifying concept. For example, negative mass can be used to simplify the dynamics of objects embedded in fluids [11], and similar classical situations where negative mass is a practical concept are discussed by Meyer [24] and Ziauddin [33]. However, there is confusion in the wider literature, because different authors deduce their findings from different background assumptions—this makes it unclear which results can sensibly be combined without accidentally generating logical inconsistencies.

One may contrast the possibility of negative mass to that of negative length. From an intuitive standpoint the length of an object 'ought' to be positive, but when lengths are used in computations it is convenient to use negative and positive values to take account of *orientation*. In contrast, the concept of negative mass is not just a convenient sign notation. It has real empirical, and hence physical, meaning. Something has negative inertial mass if it has negative resistance to changing its state of motion. So if we attempt to slow down a negative-mass body by pushing against it, its velocity will actually increase instead.

In Newtonian theory, mass refers to three distinct concepts. The inertial mass (m_i) of a particle determines how its acceleration is related to the forces acting upon it, its active mass (m_a) gives rise to gravitational fields, and its passive mass (m_p) determines how it is acted upon by gravity. Applying Newton's Third Law to gravitational forces tells us that m_a/m_p is the same for all particles, while the weak equivalence principle (that gravity and acceleration have identical effects) implies that m_i/m_p is a positive constant for any given particle. Choosing units such that $m_a/m_p = m_i/m_p = 1$ therefore allows us to declare that $m_a = m_p = m_i$, and since gravity is observed to be universally attractive one typically assumes that mass is positive.

Hohmann and Wohlfarth [17] note, however, that the experimental basis for these equalities applies only to observable matter, and discuss the possibility that negative mass particles might contribute, at least in part, to the 'dark matter' component of the Universe. For their purposes a particle has negative mass if $m_i/m_p = -1$, but since this violates the weak equivalence principle in relativity theory (which requires $m_i = m_p$) they use a modified version of Einstein gravity in which the geodesics followed by positive masses are defined by one space-time metric and those of negative masses by another, and assume that there is no non-gravitational coupling between the two types of particle (since we could otherwise have observed negative masses already). They note that 'bimetric' models of this kind, which generate asymmetric forces between positive and negative mass particles, are themselves considered by some to be inconsistent [26], and deduce a further constraint on their construction, viz. it is not possible to have gravitational forces of exactly equal strength and opposite direction acting on the two classes of test particle. However, even this result depends on background assumptions, and anti-gravity models are known to exist in which their theorem does not apply [18, 19].

In perhaps the best-known relativistic analysis of negative-mass particles, Bondi [10] successfully constructed a "world-wide nonsingular solution of Einstein's equations containing two oppositely accelerated pairs of bodies, each pair consisting of two bodies of opposite sign of mass". More recently, Belletête and Paranjape [9] have demonstrated in a general relativistic setting that Schwarzchild solutions exist representing matter distributions which are "perfectly physical", despite describing a negative mass geometry outside the matter distribution. Jammer [20] has discussed the historical and philosophical context of negative mass at length. While stressing the fact that no negative-mass particle has yet been observed experimentally, he notes that "no known physical law precludes the existence of negative masses". On the other hand, several unusual (and potentially unphysical) properties of negative mass bodies have been proven using various background theories ranging from Newtonian physics to string theory [27, 16].

However, all of this knowledge is based on assumptions and frameworks which differ from one author to the next, and it is consequently difficult to determine to what extent the various claims are consistent with one another or even exactly what basic assumptions are used in each framework. In the absence of experimental evidence, using a framework where the basic concepts and assumptions are crystal clear is essential, since any inadvertent combination of inconsistent results from the literature would allow us to confirm any claim, no matter how fanciful.

Here we introduce just such a framework to investigate of the consequences of having negative mass bodies. Our framework is delicate enough to formulate precise axioms with clear meanings and formally prove the connection between the existence of negative mass bodies and superluminal ones. At the same time, it is also simple enough to be grasped by a college physics student with only a basic understanding of mathematical logic.

Our results imply the existence of yet another constraint on the existence of negative mass particles. We show formally that if such particles exist, provided they can collide inelastically (i.e. fuse together) with 'normal' particles in collisions that conserve four-momentum, then faster-than-light (FTL) particles must also exist. We prove this by showing how, given any negative mass particle a with known 4-momentum, it is possible to specify a suitable positive mass particle b, such that the inelastic collision of a with b would generate an FTL body. We prove our claims within a general axiomatic logical framework, using axioms that are relevant in both Newtonian and relativistic dynamics. This ensures that we can be certain exactly what is assumed and what is not, and hence confirm the absence of unintended inconsistencies. Moreover, keeping things as general as possible ensures that our results have the widest possible applicability.

Another important feature of our approach is that we explicitly avoid using unstated and potentially unjustifiable assumptions in deriving our results. Avoiding such assumptions, and in particular the blanket assumption that negative-mass particles cannot exist, is important in this context, since it allows us to provide potentially educational explanations as to *why* such phenomena may or may not be physically feasible. In contrast, if we simply assert a priori that negative mass is unphysical, the only answer we can give to the question "why?", is "because we say so". For example, it might be argued informally that the entailed existence of FTL particles, proven in this paper, would itself entail the possibility of causality paradoxes, so that the consequences of negative mass particles are not 'reasonable'. But informal arguments of this nature can be flawed: using our formal approach, we and our colleagues have recently shown that spacetime (of any dimension $1+n$) can be populated with particles and observers in such a way that faster-than-light motion is possible, but this does *not* lead to the 'time travel' situations (so beloved of Star Trek fans) that give rise to causality problems [2]. Consequently, the fact that negative-mass particles entail the existence of FTL particles cannot, of itself, be used to argue logically against their existence.

Formal axiomatization also allows us to address consistency issues and what-if scenarios. It is possible to show, for example, that the consistency of relativistic dynamics with interacting particles having negative relativistic masses follows by a straightforward generalization of the model construction used by Madarász and Székely [23] to prove the consistency of relativistic dynamics and interacting FTL particles, see also [30]. The same approach allows us to derive and prove the validity of key relativistic formulae. For example, we can also show logically that all inertial observers of any particle must agree on the value of $m\sqrt{|1-v^2|}$, where m is the particle's relativistic mass and v its speed ($c=1$). This formally confirms the widely-held 'popular' belief that the observed relativistic mass and momentum of a positive-mass FTL particle must *decrease* as its relative speed increases [21].

We introduce our results in two stages. In Section 2, we show informally that there are several simple ways to create FTL particles using inelastic collisions between positive and negative relativistic mass particles. Then in Section 3, we reconstruct our informal arguments within an axiomatic framework so as to make explicit all the assumptions needed to prove our central claim, that the existence of particles with negative relativistic mass necessarily entails the existence of FTL particles.

In addition to its pedagogic advantages, actively restating and proving our statements formally has a further advantage over the informal approach. The mechanics of proof construction require us to identify all of the tacit assumptions underpinning our informal arguments, thereby revealing which assumptions are relevant and which are unwarranted or unnecessary. Identifying and avoiding those which are unnecessary is itself beneficial, since including different sets of conflicting, but unnecessary, hypotheses could potentially prevent us fusing different areas of physics – e.g., gravity and quantum theory – into a single coherent framework. This is, intriguingly, a task with which automated *interactive theorem provers* [32] are increasingly able to assist, both in terms of proof production and automatic checking of correctness. Indeed, this approach is already leading to the production and machine-verification of non-trivial relativistic theorems [15, 29].

In summary, an obvious didactic benefit of using a formal axiomatic framework for investigating questions such as the one investigated here is the elimination of tacit assumptions. In an axiomatic framework it is clear what is assumed and what is not, as well as where these assumptions are used. (For a more delicate discussion on the epistemological significance of the axiomatic framework used in this paper, see Friend's independent study [14] of this approach.)

2 Generating FTL particles from negative mass particles

Let us assume that particles do indeed exist with negative relativistic mass, and that it is possible for such particles to collide inelastically with 'normal' particles. As we now illustrate informally, the existence of FTL particles (tachyons) follows almost immediately, provided we assume that four-momentum is conserved in such collisions. For simplicity, we take $c = 1$. Throughout this paper, we will always understand 'mass' to mean 'relativistic mass'.

Recall first that the *four momentum* of a particle b is the four-dimensional vector $(m, \mathbf{p})$, where m is its relativistic mass and $\mathbf{p}$ its linear momentum (as measured by some inertial observer whose identity need not concern us, because switching to another observer may change the values of certain quantities but not the main phenomena). Notice also that the particle b is a tachyon if and only if $|m| < |\mathbf{p}|$ (i.e. its observed speed is greater than $c = 1$), and that all inertial observers agree as to this judgement (if one inertial observer considers b to be travelling faster than light, they all do — this is because all inertial observers consider each other to be travelling slower than light relative to one another. For a machine-verified proof of this assertion using our approach, see the work of Stannett and Németi [29]).

In this paper, we concentrate on three special types of collisions so as to emphasize how few background assumptions (e.g., about what kinds of positive mass particles exist) are needed to create FTL particles by inelastically colliding particles of positive and negative mass. We will assume the existence of two colliding particles a and b, where a has negative mass $m < 0$ and b has positive mass $M > 0$, which move along the same spatial line (though possibly in opposite directions). Taking the common line of travel to be the x-axis, positive in the direction of b's travel, the four-momenta of a and b can be written $(m, p, 0, 0)$ and $(M, P, 0, 0)$, respectively, for suitable values of p and P. Assuming that four-momentum is conserved during the collision, the four-momentum of the particle c generated by the fusion of a and b will be $(M + m, P + p, 0, 0)$, and this particle will be a tachyon provided

$$|M + m| < |P + p| \tag{1}$$

If this tachyon has negative mass and positive momentum, it moves in the negative x-direction (it is an unusual property of negative-mass particles that their velocity and momentum vectors point in opposite directions); if it has positive-mass and positive momentum it moves in the positive x-direction. By definition, $M > 0 > m$, and b has both positive mass ($M > 0$) and positive momentum ($P > 0$), since its motion defines the positive x-direction.

2.1 First thought experiment

Suppose a travels slower than light, while b moves at light-speed, so that the four-momenta of a and b can be written $(m, p, 0, 0)$ and $(M, M, 0, 0)$, respectively. According to (1), the particle created by their collision will be a tachyon provided

$$|M + m| < |M + p| \tag{2}$$

There are various ways in which this can happen, depending on the values of m and p (see Fig. 1 and Proposition 1). Notice that $|p| < |m|$ since a travels slower than light.

The case when $|m| = M$, i.e. $M = -m$, is ambiguous irrespective of the velocities of the colliding particles a and b. Since $M + m = 0$ and $|p| < |m| = M$, the linear momentum $M + p$ of the resulting particle c must be positive, even though it has zero relativistic mass. In terms of the space-time diagram (Fig. 2), this means that the particle's worldline is horizontal, i.e. it 'moves' with infinite speed. In these circumstances, the question whether c moves in the positive or negative x-direction is meaningless. However, like other observer-dependent concepts such as simultaneity or the temporal ordering of events, this indeterminacy does not lead to a logical contradiction [23].

2.2 Second thought experiment

Suppose b is stationary, i.e. $P = 0$. By arguments similar to those above, this will result in an FTL particle c whenever $|m| + |\mathbf{p}| > M > |m| - |\mathbf{p}|$, and its direction of travel will be determinate provided $M \neq -m$. See Fig. 3 and Proposition 2.

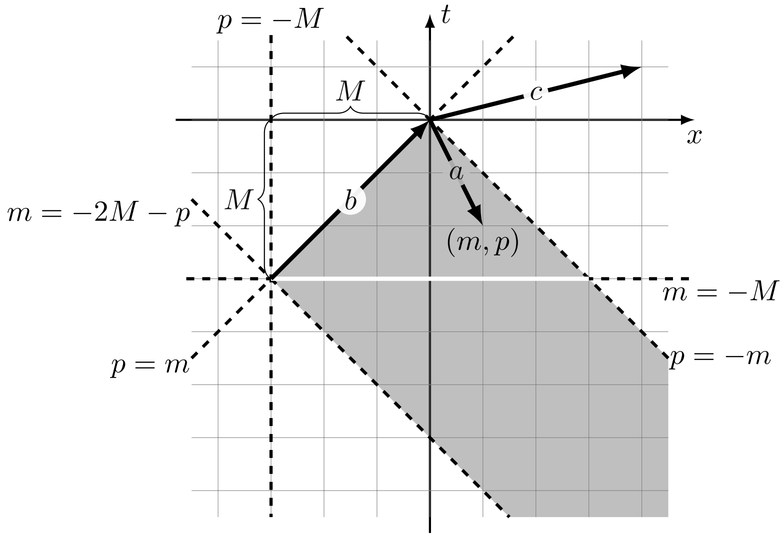

Figure 1: Illustration for generating an FTL particle by colliding a negative relativistic mass particle with a particle moving with the speed of light.

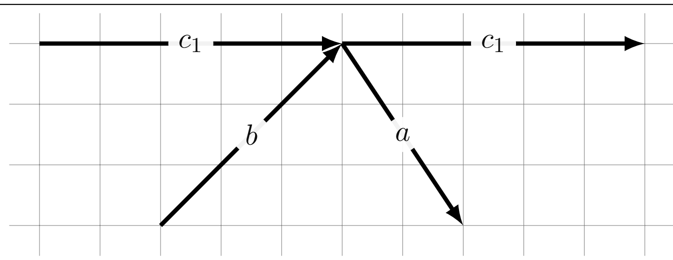

Figure 2: The "inelastic" collision of two particles having opposite relativistic masses is ambiguous in the sense that in this case we have two possible outcomes satisfying the conservation of four-momentum.

2.3 Third thought experiment

Suppose a and b have similar, but oppositely-signed, masses, and that they collide 'head-on' while travelling with equal speeds in opposite directions (relative to some observer, whose identity need not concern us). If the difference in the absolute values of their masses is small relative to their common speed, the resulting particle will be FTL because it will have a small mass relative to its large momentum (which is greater than those of the colliding particles as they have opposite masses); see Proposition 3 for more details.

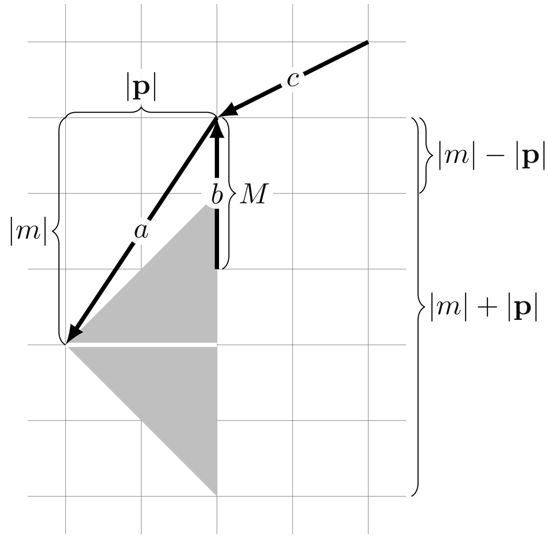

Figure 3: Illustration for generating an FTL particle by colliding a negative relativistic mass particle with a stationary particle of positive relativistic mass.

2.4 FTL particle creations requiring negative relativistic mass

We have seen above that the existence of negative-mass particles implies the existence of FTL particles. Conversely, it is easy to see that an inelastic collision between two slower-than-light particles having positive relativistic masses always leads to a slower-than-light particle. Consequently, the only way in which an inelastic collision between slower-than-light particles can create an FTL particle is if incoming particles can have negative relativistic masses.

In particular, if we impose the condition that such collisions are the only mechanism by which FTL particles can be created, then the existence of FTL particles implies the existence of negative-mass particles. While this suggests that tachyons and negative-mass particles are equally ‘exotic’, this is, of course, not the case, since the argument that FTL particles require the existence of negative-mass particles relies on the assumption that inelastic collisions are the only mechanism by which FTL particles can be created.

This is by no means a trivial assumption; indeed we have demonstrated elsewhere a consistent model of spacetime in which FTL particles exist, but in which no collisions are posited [2].

3 Axiomatic reconstruction

We have seen three thought experiments in which FTL particles are generated by colliding an arbitrary negative mass particle with an appropriate positive mass one. However, we have not explicitly identified the background assumptions needed to prove our claims concerning these thought experiments. In this section, we dig deeper by identifying these background assumptions; these will turn out to be so general that they are consistent with both relativistic and classical dynamics. To do so, we now reconstruct the above arguments in a precise axiomatic framework, in which each of the used background assumptions will be stated as an explicit axiom. Indeed, making all tacit assumptions explicit can be seen as one of the main advantages of the axiomatic method. Readers interested in the wider context are referred to [23, 29, 21].

3.1 Quantities and Vector Spaces

To formulate the intuitive image above, we need some structure of numbers describing physical quantities such as coordinates, relativistic masses and momenta. Traditional accounts of relativistic dynamics take for granted that the basic number system to be used for expressing measurements (lengths, masses, speeds, etc.) is the field $\mathbb{R}$ of real numbers, but this assumption is far more restrictive than necessary.[1] Instead, we will only assume that the number system is a linearly ordered field Q equipped with the usual constants, zero (0) and one (1); the usual field operations, addition (+), multiplication ($\cdot$) and their inverses; and the usual ordering ($\leq$) and its inverse; we also assume that the field is *Euclidean*, i.e. positive quantities have square roots. Formally, this is declared as an axiom:

<u>AxEField</u> The structure $\langle Q, 0, 1, +, \cdot, \leq \rangle$ of quantities is a linearly ordered field (in the algebraic sense) in which all non-negative numbers have square roots, i.e. $(\forall x \in Q)((0 \leq x) \Rightarrow (\exists y \in Q)(x = y^2))$.

We write $\sqrt{x}$ for this root, which can be assumed without loss of generality to be both unique and non-negative (regarding machine-verified proofs of this and other relevant claims concerning Euclidean fields, see [29]).

[1]The assumption that $\mathbb{R}$ is the correct number system for expressing lengths (say) is experimentally untestable. Given that we only have access to finitely many measurements, and many of these are (necessarily computable) approximations to 'true' (possibly uncomputable) values, it is not experimentally possible to decide if all non-empty bounded sets of lengths have a supremum, as would be the case if the use of $\mathbb{R}$ were physically necessary.

We choose to use Euclidean fields, as this allows us to refer to 'lengths of vectors' and considerably simplifies the proofs. In practice proofs can generally be modified (by referring instead to 'squared length') to work over any arbitrary ordered field, such as the field of rational numbers. However, that also makes them more complicated. For a paper discussing special relativity in this framework, see [22].

3.2 Inertial particles and observers

We denote the set of physical *bodies* (things that can move) by B. This includes the sets $\mathsf{IOb} \subseteq B$ of **inertial observers**, $\mathsf{Ip} \subseteq B$ of **inertial particles**. Given any inertial observer $k \in \mathsf{IOb}$ and inertial particle $b \in \mathsf{Ip}$, we write $w\ell_k(b) \subseteq Q^4$ for the **worldline** of particle b as observed by k. The coordinates of $\bar{x} \in Q^n$ are denoted by $x_1, x_2, \ldots, x_n$.

The following axiom asserts that the motion of inertial particles are uniform and rectilinear according to inertial observers.

<u>AxIp</u> For all $k \in \mathsf{IOb}$ and $b \in \mathsf{Ip}$, the worldline $w\ell_k(b)$ is either a line, a half-line or a line segment[2].

Suppose observer $k \in \mathsf{IOb}$ sees particle $b \in \mathsf{Ip}$ at the distinct locations $\bar{x}, \bar{y} \in Q^4$. Then its **velocity** according to k is the associated change in spatial component divided by the change in time component,

$$\mathbf{v}_k(b) := \begin{cases} \frac{\mathsf{space}(\bar{x},\bar{y})}{\mathsf{time}(\bar{x},\bar{y})} & \text{if } \mathsf{time}(\bar{x},\bar{y}) \neq 0 \\ \mathsf{undefined} & \text{otherwise} \end{cases}$$

where $\mathsf{space}(\bar{x}, \bar{y}) := (x_2 - y_2, x_3 - y_3, x_4 - y_4)$ and $\mathsf{time}(\bar{x}, \bar{y}) := x_1 - y_1$. The length[3] of the velocity vector (if it is defined) is the particle's **speed**,

$$v_k(b) := |\mathbf{v}_k(b)|.$$

By AxIp, these concepts are well-defined because $w\ell_k(b)$ lies in a straight line. So the velocities of the considered particles are constants.

If $\mathbf{v}_k(b)$ is defined, we say that b is observed by k to have **finite speed**, and write $v_k(b) < \infty$. The anomalous case $\mathsf{time}(\bar{x}, \bar{y}) = 0$ corresponds to a situation where all

[2]Taking $\bar{x}$ and $\bar{y}$ to be of sort Q^4, and λ to be of sort Q, these concepts are defined formally as follows. A *line* is a set of the form $\{\bar{z} \mid (\exists \bar{x}, \bar{y}, \lambda)(\bar{z} = \lambda\bar{x} + (1-\lambda)\bar{y})\}$. A *half-line* is a set of the form $\{\bar{z} \mid (\exists \bar{x}, \bar{y}, \lambda)((0 \leq \lambda) \& (\bar{z} = \lambda\bar{x} + (1-\lambda)\bar{y}))\}$. A *line segment* is a set of the form $\{\bar{z} | (\exists \bar{x}, \bar{y}, \lambda)((0 \leq \lambda \leq 1) \& (\bar{z} = \lambda\bar{x} + (1-\lambda)\bar{y}))\}$.

[3]The **Euclidean length**, $|\bar{x}|$, of a vector $\bar{x}$ is the non-negative quantity $|\bar{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$.

points in $w\ell_k(b)$ are simultaneous from k's point of view, so that k considers the particle to require no time at all to travel from one spatial location to another.

3.3 Collision axioms

In this subsection, we introduce some very simple axioms concerning the dynamics of collisions, and show that the existence of negative relativistic mass implies the existence of faster-than-light (FTL) inertial particles.

Suppose an inertial observer k sees two inertial bodies travelling at finite speed fuse to form a third one at some point $\bar{x}$. In this case, the worldlines of the two incoming particles terminate at $\bar{x}$, while that of the outgoing particle originates there. Formally, we say that an inertial particle b is **incoming** at $\bar{x}$ (according to k) provided $\bar{x} \in w\ell_k(b)$ and $\bar{x}$ occurs strictly later (according to k) than any other point on $w\ell_k(b)$, i.e. $\bar{y} \in w\ell_k(b)$ & $\bar{y} \neq \bar{x} \;\Rightarrow\; y_1 < x_1$. **Outgoing** bodies are defined analogously. An **inelastic collision** between two inertial particles a and b (according to observer k) is then a scenario in which there is a unique additional particle $c \in \mathsf{Ip}$ and a point $\bar{x}$ such that a and b are incoming at $\bar{x}$, c is outgoing at $\bar{x}$. We write $\mathsf{inecoll}_k(ab : c)$ to denote that the distinct inertial particles a and b **collide inelastically**, thereby generating inertial particle c (according to observer k). The **relativistic mass** of inertial particle b according to observer k is denoted by $m_k(b)$.

ConsFourMomentum Four-momentum is conserved in inelastic collisions of inertial particles according to inertial observers, i.e.

$$\begin{aligned}&\mathsf{inecoll}_k(ab : c) \Rightarrow\\ &\quad m_k(c) = m_k(a) + m_k(b) \quad \&\\ &\quad\quad m_k(c)\mathbf{v}_k(c) = m_k(a)\mathbf{v}_k(a) + m_k(b)\mathbf{v}_k(b)\end{aligned}$$

The next axiom, AxInecoll, states that inertial particles moving with finite speeds can be made to collide inelastically in any frame in which their relativistic masses are not equal-but-opposite. Since a collision of particles having equal but opposite relativistic masses does not lead to an inelastic collision according to our formal definition, we do not include this case in this axiom (this does not mean that such particles cannot collide, just that such a collision will not comply with our definition of inelasticity in the associated frame because the third participating particle has infinite speed).

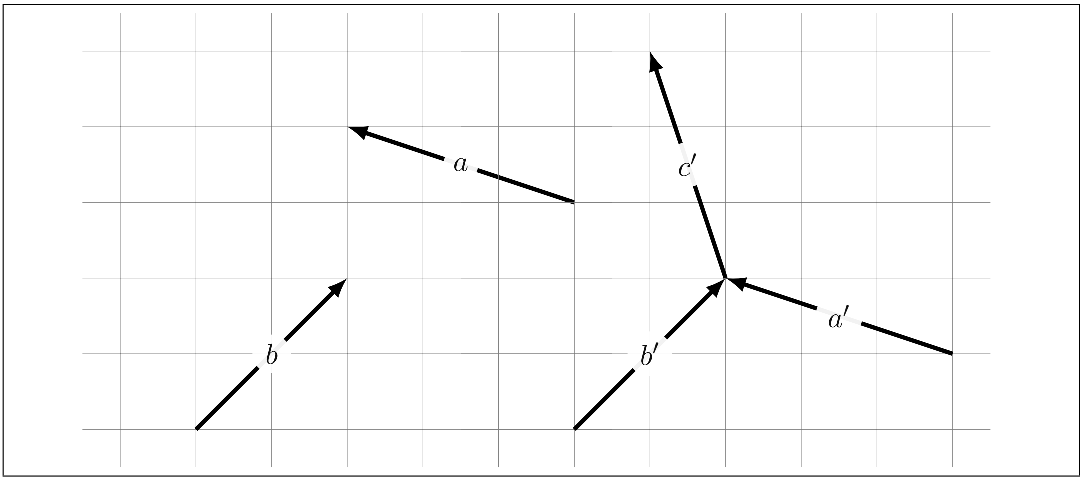

Figure 4: Illustration for axiom AxInecoll

AxInecoll If $k \in \mathsf{IOb}$ and $a, b \in \mathsf{Ip}$ such that $v_k(a) < \infty$, $v_k(b) < \infty$ and $m_k(a) + m_k(b) \neq 0$, then there are $a', b' \in \mathsf{Ip}$ such that a' and b' collide inelastically, with $m_k(a') = m_k(a)$, $\mathbf{v}_k(a') = \mathbf{v}_k(a)$, $m_k(b') = m_k(b)$ and $\mathbf{v}_k(b') = \mathbf{v}_k(b)$.[4] See Fig. 4.

4 Formulating the thought experiments

Here we are going to formalize and prove the thought experiments of Subsections 2.1, 2.2 and 2.3.

Formula ∃NegMass below says that there is at least one inertial particle of finite speed and negative relativistic mass.

∃NegMass There are $k \in \mathsf{IOb}$ and $a \in \mathsf{Ip}$ such that $m_k(a) < 0$ and $v_k(a) < \infty$.

Formula ∃FTLIp below says that there is at least one faster than light inertial particle.

∃FTLIp There are $k \in \mathsf{IOb}$ and $b \in \mathsf{Ip}$ such that $1 < v_k(b) < \infty$.

[4]Because here we use the framework of [3], we express possible worldlines of particles using existential quantifiers as is usual in frameworks of classical logic. See [25] for an axiomatic framework where this possibility is expressed instead by a modal logic operator.

4.1 First thought experiment

Axiom $\mathsf{AxThExp}_1$ below says that the thought experiment described in Subsection 2.1 can be done by asserting that inertial observers can send out particles moving with the speed of light 1 in any direction and having arbitrary positive relativistic mass.

$\underline{\mathsf{AxThExp}_1}$ For $k \in \mathsf{IOb}$, $m \in Q$ and $\mathbf{v} \in Q^3$ for which $m > 0$ and $|\mathbf{v}| = 1$, there is $b \in \mathsf{Ip}$ such that $\mathbf{v}_k(b) = \mathbf{v}$ and $m_k(b) = m$.

Proposition 1. *Assume* $\mathsf{ConsFourMomentum}$, $\mathsf{AxEField}$, AxIp, $\mathsf{AxInecoll}$, $\mathsf{AxThExp}_1$. *Then*

$$\exists\mathsf{NegMass} \;\Rightarrow\; \exists\mathsf{FTLIp}. \tag{3}$$

Proof. By axiom $\exists\mathsf{NegMass}$, there is an inertial observer k and inertial particle a such that $m_k(a) < 0$ and $v_k(a) < \infty$. Let $\mathbf{v} \in Q^3$ for which $|\mathbf{v}| = 1$. Then by axiom $\mathsf{AxThExp}_1$, there is an inertial particle b such that $m_k(b) = -2m_k(a)$ and

$$\mathbf{v}_k(b) = \begin{cases} \mathbf{v} & \text{if } v_k(a) = 0, \\ \frac{-\mathbf{v}_k(a)}{v_k(a)} & \text{if } v_k(a) \neq 0. \end{cases}$$

By axiom $\mathsf{AxInecoll}$, there are inelastically colliding inertial particles a', b' and c' such that $\mathsf{inecoll}_k(a'b' : c')$, $m_k(a') = m_k(a)$, $\mathbf{v}_k(a') = \mathbf{v}_k(a)$, $m_k(b') = m_k(b)$ and $\mathbf{v}_k(b') = \mathbf{v}_k(b)$. By $\mathsf{ConsFourMomentum}$,

$$\begin{aligned} m_k(c') &= m_k(a') + m_k(b') \\ &= m_k(a) + m_k(b) = -m_k(a) \end{aligned} \tag{4}$$

and

$$m_k(c')\mathbf{v}_k(c') = \begin{cases} -2m_k(a)\mathbf{v} & \text{if } v_k(a) = 0, \\ m_k(a)\mathbf{v}_k(a) + 2m_k(a)\frac{\mathbf{v}_k(a)}{v_k(a)} & \text{if } v_k(a) \neq 0. \end{cases} \tag{5}$$

Hence

$$\mathbf{v}_k(c') = \begin{cases} 2\mathbf{v} & \text{if } v_k(a) = 0, \\ -(v_k(a) + 2)\frac{\mathbf{v}_k(a)}{v_k(a)} & \text{if } v_k(a) \neq 0. \end{cases} \tag{6}$$

Therefore, $v_k(c') = |\mathbf{v}_k(c')| > 1$ and $v_k(c') < \infty$; and this is what we wanted to prove. □

4.2 Second thought experiment

Axiom $\mathsf{AxThExp_2}$ below ensures the existence of the particle having positive relativistic mass used in the thought experiment described in Subsection 2.2.

$\underline{\mathsf{AxThExp_2}}$ For every $k \in \mathsf{IOb}$ and $m > 0$, there is $b \in \mathsf{Ip}$ such that $v_k(b) = 0$ and $m_k(b) = m$.

Formula $\exists\mathsf{MovNegMass}$ below asserts that there is at least one moving inertial particle of finite speed and negative relativistic mass.

$\underline{\exists\mathsf{MovNegMass}}$ There are $k \in \mathsf{IOb}$ and $b \in \mathsf{Ip}$ such that $m_k(b) < 0$ and $0 < v_k(b) < \infty$.

For the sake of economy, we use axiom $\exists\mathsf{MovNegMass}$ instead of $\exists\mathsf{NegMass}$ because in this case we do not have to assume anything about the possible motions of inertial observers or the transformations between their worldviews. We note, however, that these two axioms are clearly equivalent in both Newtonian and relativistic kinematics (assuming that inertial observers can move with respect to each other).

Proposition 2. *Assume* $\mathsf{ConsFourMomentum}$, $\mathsf{AxEField}$, AxIp, $\mathsf{AxInecoll}$, $\mathsf{AxThExp_2}$. *Then*

$$\exists\mathsf{MovNegMass} \;\Rightarrow\; \exists\mathsf{FTLIp}. \tag{7}$$

Proof. By axiom $\exists\mathsf{MovNegMass}$, there is an inertial observer k and inertial particle a such that $m_k(a) < 0$ and $0 < v_k(a) < \infty$. By axiom $\mathsf{AxThExp_2}$, there is an inertial particle b such that $m_k(b) = -m_k(a)\,(1 + v_k(a)/2)$ and $v_k(b) = 0$. By axiom $\mathsf{AxInecoll}$, there are inelastically colliding inertial particles a', b' and c' such that $\mathsf{inecoll}_k(a'b'\!:\!c')$, $m_k(a') = m_k(a)$, $\mathbf{v}_k(a') = \mathbf{v}_k(a)$, $m_k(b') = m_k(b)$ and $\mathbf{v}_k(b') = \mathbf{v}_k(b)$. By $\mathsf{ConsFourMomentum}$,

$$\begin{aligned} m_k(c') &= m_k(a') + m_k(b') \\ &= m_k(a) + m_k(b) = \frac{-m_k(a)v_k(a)}{2} \end{aligned} \tag{8}$$

and

$$m_k(c')\mathbf{v}_k(c') = m_k(a)\mathbf{v}_k(a). \tag{9}$$

It follows that

$$\mathbf{v}_k(c') = -2\frac{\mathbf{v}_k(a)}{v_k(a)},$$

and hence that $v_k(c') = 2 > 1$, which is what we wanted to prove. □

4.3 Third thought experiment

Finally let us introduce the following axiom ensuring the existence of the particles having positive relativistic mass needed in the thought experiment of Subsection 2.3.

$\underline{\mathsf{AxThExp_3}}$ For all $\varepsilon > 0$, $k \in \mathsf{IOb}$ and $a \in \mathsf{Ip}$, there is $b \in \mathsf{Ip}$ such that $(1+\varepsilon)|m_k(a)| < m_k(b) < (1+2\varepsilon)|m_k(a)|$ and $\mathbf{v}_k(a) = -\mathbf{v}_k(b)$.

Proposition 3. *Assume* $\mathsf{ConsFourMomentum}$, $\mathsf{AxEField}$, AxIp, $\mathsf{AxInecoll}$, $\mathsf{AxThExp_3}$. *Then*

$$\exists\mathsf{MovNegMass} \;\Rightarrow\; \exists\mathsf{FTLIp}. \tag{10}$$

Proof. By axiom $\exists\mathsf{MovNegMass}$, there is an inertial observer k and inertial particle a such that $m_k(a) < 0$ and $0 < v_k(a) < \infty$. Let $0 < \varepsilon < v_k(a)$. Then by axiom $\mathsf{AxThExp_3}$, there is an inertial particle b such that $(1+\varepsilon)|m_k(a)| < m_k(b) < (1+2\varepsilon)|m_k(a)|$ and $\mathbf{v}_k(b) = -\mathbf{v}_k(a)$.

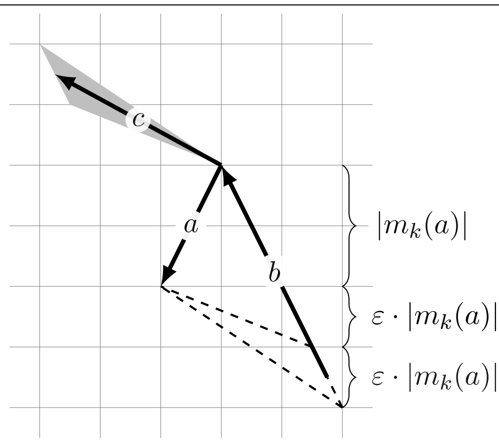

Figure 5: Illustration for the proof of Proposition 3

By axiom $\mathsf{AxInecoll}$, there are inelastically colliding inertial particles a', b' and c' such that $\mathsf{inecoll}_k(a'b' : c')$, $m_k(a') = m_k(a)$, $\mathbf{v}_k(a') = \mathbf{v}_k(a)$, $m_k(b') = m_k(b)$ and $\mathbf{v}_k(b') = \mathbf{v}_k(b)$. By $\mathsf{ConsFourMomentum}$,

$$\varepsilon|m_k(a)| < |m_k(c')| < 2\varepsilon|m_k(a)| \tag{11}$$

and

$$2|m_k(a)|v_k(a) < (2+\varepsilon)|m_k(a)|v_k(a) < |m_k(c')\mathbf{v}_k(c')|. \tag{12}$$

Hence

$$v_k(c') = |\mathbf{v}_k(c')| > \frac{2|m_k(a)|v_k(a)}{2\varepsilon|m_k(a)|} > \frac{v_k(a)}{\varepsilon}. \tag{13}$$

Therefore, $1 < v_k(c') < \infty$; and this is what we wanted to prove. □

5 Concluding remarks

Using only basic postulates concerning the conservation of four-momentum, we have shown axiomatically that the existence of particles having negative relativistic masses implies the existence of FTL particles. The following are the two most straightforward applications of this result.

- If an experiment eventually shows the existence of particles having negative masses, then we will know that FTL particles must also exist. If evidence exists suggesting otherwise, our approach would then imply that one or more of the natural assumptions encoded in our axioms must be false. This in turn would provide information suitable for guiding further experimentation.

- Similarly, if we can prove that FTL particles cannot exist, and *no* evidence can be found suggesting that the natural physical assumptions encoded by our axioms are invalid, then this can be used to prove the non-existence of particles having negative masses.

It is also worth noting that we have made no restrictions on the worldview transformations between inertial observers. Hence our axioms are so general that they are compatible with both Newtonian and relativistic kinematics. In addition to making our axioms relatively easy for students to understand, and hence our results more believable, the benefit of being so parsimonious with the basic assumptions is that it makes results obtained using our axiomatic method that much more difficult to challenge, because so few basic assumptions have been made concerning physical behaviours in the "real world".

Acknowledgements

This research was partially supported by the Royal Society [grant number IE110369]; London Mathematical Society [grant number 41508]; and by the Hungarian Scientific Research Fund [grant numbers T81188, PD84093].

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Received 13 June 2017