TWO-DIMENSIONAL MODULI SPACES OF RANK 2 HIGGS BUNDLES OVER $\mathbb{C}P^1$ WITH ONE IRREGULAR SINGULAR POINT

PÉTER IVANICS, ANDRÁS STIPSICZ, AND SZILÁRD SZABÓ

ABSTRACT. We give a complete description of the two-dimensional moduli spaces of stable Higgs bundles of rank 2 over $\mathbb{C}P^1$ with one irregular singular point, having a regular leading-order term, and endowed with a generic compatible parabolic structure such that the parabolic degree of the Higgs bundle is 0. Our method relies on elliptic fibrations of the rational elliptic surface, an equivalence of categories between irregular Higgs bundles and some sheaves on a ruled surface, and an analysis of stability conditions.

1. INTRODUCTION

In this article we consider 2 complex dimensional moduli spaces of singular Higgs bundles over $\mathbb{C}P^1$ with irregular singularities. It is known [5] that if one fixes finitely many points on a curve C and suitable polar parts for a Higgs bundle near those points, then one gets a holomorphic symplectic moduli space of Higgs bundles over C with the given irregular part and residues at the singularities. In some cases these spaces turn out to be of complex dimension 2. Our aim in this article is to give a complete description of the two-dimensional holomorphic symplectic moduli spaces of rank 2 Higgs bundles over $\mathbb{C}P^1$ having a unique pole of order 4 as singularity, and regular leading-order term. One needs to distinguish two cases, depending on whether the leading-order term is a regular semi-simple endomorphism (untwisted case), or has non-vanishing nilpotent part (twisted case). As we will see, the corresponding fiber at infinity of the Hitchin fibration is $\widetilde{E}_7$ in the untwisted case and $\widetilde{E}_8$ in the twisted case. The corresponding de Rham moduli spaces of irregular connections are related to the Painlevé II (untwisted case) and Painlevé I (twisted case) equations. The polar part of an irregular Higgs bundle depends on some complex parameters

(U) $$a_{\pm}, b_{\pm}, c_{\pm}, \lambda_{\pm} \in \mathbb{C}, \quad a_+ \neq a_-$$

in the untwisted case (referred to as (U)) and

(T) $$b_{-8}, \dots, b_{-3} \in \mathbb{C}, \quad b_{-7} \neq 0$$

in the twisted case (referred to as (T)), see Subsection 2.3.

In the following statements we let $\mathcal{M}$ be a moduli space of rank 2, parabolic degree 0 stable parabolic irregular Higgs bundles over $\mathbb{C}P^1$ with a unique pole of order 4 with a regular leading-order term and fixed parameters (U) or (T). For details and definitions see Subsection 2.3. If the parabolic structure is generic, the degree of the underlying vector bundle is necessarily equal to -1. It is expected that moduli spaces $\mathcal{M}^{ss}$ of semi-stable irregular Higgs bundles with fixed polar parts underlie completely integrable systems with Abelian varieties as generic fibers. If $\dim_{\mathbb{C}}(\mathcal{M}^{ss}) = 2$ this would then imply that $\mathcal{M}^{ss}$ is an elliptic fibration over a curve. For generic weights $\mathcal{M}^{ss} = \mathcal{M}^s$, where $\mathcal{M}^s$ is the moduli space of stable irregular Higgs bundles. Our results below will confirm this expectation, with one singular fiber of type $\widetilde{E}_7$ (untwisted case) or $\widetilde{E}_8$ (twisted case). On the other hand, there are several possibilities for the other singular fibers [15, 17, 20].

Corresponding author: Szilárd Szabó.

In [21], a general equivalence of categories between irregular Higgs bundles and some pure 1-dimensional rank one sheaves on a ruled surface was shown to hold, assuming that the leading order term of the Higgs field is semi-simple. We will use this equivalence to prove our first result, giving a complete description of these further singular fibers in the untwisted case in terms of the parameters of (U). (For the definition of various types of singular fibers see [14] or Section 3.)

Theorem 1.1. *Assume that the polar part of the Higgs bundle is untwisted. Then the moduli space $\mathcal{M}^s$ is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the Hitchin fibration is:*

(1) a type III fiber if $\Delta = 0$ and $\lambda_+ = 0$;
(2) a type II and an I_1 fiber if $\Delta = 0$ and $\lambda_+ \neq 0$;
(3) an I_2 and an I_1 fiber if $\Delta \neq 0$ and $\lambda_+ = 0$;
(4) and three I_1 fibers otherwise,

where $\Delta = \left((b_- - b_+)^2 - 4(a_- - a_+)(c_- - c_+)\right)^3 - 432(a_- - a_+)^4 \lambda_+^2$.

Remark 1.2. *Since by Equation (24) in Subsection 2.3 we have that $\lambda_+ + \lambda_- = 0$, the above conditions could be phrased in terms of λ_- as well.*

Notice that according to [20, Proposition 4.2] this is a complete list of the possible singular fibers of elliptic fibrations on the rational elliptic surface without multiple fibers and having a singular fiber of type $\widetilde{E}_7$. The proof of Theorem 1.1 is given in Sections 4 and 5, where an explicit description of the Hitchin fibers corresponding to the reducible singular curves in the fibration is given. In Section 5 we also work out the stability analysis in the case of rank 2 irregular Higgs bundles in the degree 0 case; strictly speaking we do not need this analysis to prove the theorem, nevertheless we found it interesting enough to include it.

Similarly to Theorem 1.1, the next theorem provides a complete description of the singular fibers of the fibration in the twisted case, in terms of the parameters (T).

Theorem 1.3. *Assume that the polar part of the Higgs bundle is twisted. Then the moduli space $\mathcal{M}^s$ is biregular to the complement of the fiber at infinity (of type $\widetilde{E}_8$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the Hitchin fibration is:*

(1) a type II fiber if $D = 0$;
(2) and two type I_1 fibers otherwise,

where $D = \left(b_{-6}^2 + 4b_{-5}\right)^2 - 24b_{-7}\left(b_{-6}b_{-4} + 2b_{-3}\right)$.

Notice again that according to [20, Section 4.1] this is a complete list of the possible singular fibers of elliptic fibrations without multiple fibers and having a singular fiber of type $\widetilde{E}_8$. We prove Theorem 1.3 in Section 6.

Now let us give an outline of the paper. In Section 2 we fix our notations and provide some well-known background material used later. In Section 3 we give a detailed analysis of elliptic fibrations on the rational elliptic surface with one singular fiber of type $\widetilde{E}_7$ or of type $\widetilde{E}_8$. In Section 4 we first construct the rational surface Y governing the moduli space $\mathcal{M}$ in the untwisted case. Quoting the general categorical equivalence of [21], we then achieve the proof of Theorem 1.1, up to the stability analysis of irregular Higgs bundles with reducible spectral curve. This latter, in turn, is carried out in Section 5. The analysis of the case of a type I_2 curve proceeds along the lines of Section 4 of Schaub's paper [19].

We start Section 6 by some straightforward computations expressing the coefficients of the Puiseux-expansion of the eigenvalues of the Higgs field in terms of the parameters (T). We then go on to construct the rational surface Y governing the moduli space $\mathcal{M}$ in the twisted case. Next, in Proposition 6.4 we give an analogue of the general categorical equivalence of [21] between twisted irregular Higgs bundles and some pure 1-dimensional rank one sheaves on Y. This then allows us to prove Theorem 1.3.

Let us make a few remarks on related literature. In the paper [18], spaces of initial conditions for Painlevé equations are studied using rational surfaces and root systems. In particular, in Appendix B *loc. cit.* configurations of curves similar to ours appear. In [9] the singular fiber of the Hitchin map corresponding to a singular spectral curve of type A_k is determined. Our Section 5 is reminiscent to (special cases of) their results. The work [8] (in particular, Section 9 thereof) undertakes the analysis of wall-crossing phenomena related to Hitchin systems with irregular singularities. Finally, let us mention that we hope to treat the 2-dimensional moduli spaces of rank 2 irregular Higgs bundles over $\mathbb{C}P^1$ with several marked points in the future, cf. [12].

Acknowledgments: The third author was supported by NKFIH K120697. The authors were supported by NKFIH KKP126683, and by the *Lendület* program of the Hungarian Academy of Sciences. They also want to thank the referee for many useful comments and suggestions.

2. Preparatory material

We denote by $\mathcal{O}$ and K the sheaf of regular functions and the canonical sheaf respectively. We identify holomorphic line bundles over $\mathbb{C}P^1$ with their sheaves of sections. We equally let $\mathcal{O}(1)$ stand for the ample line bundle and for $n \in \mathbb{Z}$ set $K(n) = K \otimes \mathcal{O}(n)$.

2.1. **The second Hirzebruch surface and the basic birational map.**

Throughout the paper we will consider the surface

$$X = \mathbf{P}(K(4) \oplus \mathcal{O}),$$

the fiberwise projectivization of the rank 2 holomorphic line bundle $K(4) \oplus \mathcal{O}$ over $\mathbb{C}P^1$. Given that the line bundle $K(4)$ is isomorphic to $\mathcal{O}(2)$, we get that X is biholomorphic to the Hirzebruch surface of index 2. The surface X naturally fibers over $\mathbb{C}P^1$ with fibers isomorphic to $\mathbb{C}P^1$:

$$p : X \to \mathbb{C}P^1. \tag{1}$$

This morphism is sometimes called the ruling. We denote its generic fiber by F and the homology class of F by $[F] \in H_2(X;\mathbb{Z})$.

It is known that X admits two further remarkable closed curves denoted by C_0, C_∞ and called the 0-section and section at infinity, respectively. Both C_0 and C_∞ are sections of p, in particular they are biholomorphic to $\mathbb{C}P^1$. Specifically, if we let $\mathbf{0}$ stand for the 0-section of $K(4)$ and $\mathbf{1}$ stand for the constant section equal to 1 of $\mathcal{O}$ then

$$C_0 = \{[\mathbf{0}_q : \mathbf{1}_q] \mid q \in \mathbb{C}P^1\},$$

where the subscripts q mean evaluation of the given sections at q, and as usual $[\cdot : \cdot]$ denote projective coordinates. Locally, the section at infinity can be defined similarly, however it is not possible to pick a single section of $K(4)$ because any such section vanishes at two points of $\mathbb{C}P^1$. So, letting κ stand for a local non-vanishing section of $K(4)$ on some open set $U \subset \mathbb{C}P^1$, we define

$$C_\infty \cap p^{-1}(U) = \{[\kappa(q) : \mathbf{0}] \mid q \in U\}$$

where $\mathbf{0}$ stands for the 0-section of $\mathcal{O}$. It can be checked that if V is another open subset of $\mathbb{C}P^1$ with a non-vanishing section μ then these definitions of C_∞ agree on $U \cap V$, hence these formulas give a well-defined curve. We denote the homology classes defined by these sections by $[C_0], [C_\infty]$.

The second homology $H_2(X;\mathbb{Z})$ is generated by the classes of any two of the above three curves, the relation between them being

$$[C_\infty] = [C_0] - 2[F].$$

The intersection pairing is given by the formulas

$$[C_\infty]^2 = -2, \quad [C_0]^2 = 2, \quad [F]^2 = 0, \quad [C_\infty]\cdot[C_0] = 0, \quad [C_\infty]\cdot[F] = [C_0]\cdot[F] = 1.$$

As it is well-known, X is birational to $\mathbb{C}P^2$ by the morphisms

$$\begin{array}{ccc} & \widetilde{X} & \\ \swarrow & & \searrow \\ X & \overset{\omega}{\dashrightarrow} & \mathbb{C}P^2 \end{array} \tag{2}$$

where $\widetilde{X} \to X$ is the blow-up of a point $(\kappa(q) : \mathbf{1}) \in X \setminus C_\infty$ for any $q \in U \subset \mathbb{C}P^1$ and local section $\kappa \in H^0(U; K(4))$, and $\widetilde{X} \to \mathbb{C}P^2$ is the blow-up of two infinitely close points on $\mathbb{C}P^2$. For sake of concreteness, we may take the locus of this reduced point to be $(0:0:1)$. The proper transform of the fiber F_q of the map p of Equation (1) over $q \in \mathbb{C}P^1$ is the exceptional divisor of the second blow-up of $\mathbb{C}P^2$. On the other hand, the proper (which in this case is the same as the total) transform of C_∞ in $\widetilde{X}$ is equal to the proper transform of the exceptional divisor of the first blow-up of $\mathbb{C}P^2$ under the second blow-up. Throughout the paper we will use the above ω to go back and forth between X and $\mathbb{C}P^2$.

2.2. **Elliptic fibrations and their relative compactified Picard schemes.** In this section we summarize some facts concerning families of curves that we will need in the paper.

Let B be a scheme over $\mathbb{C}$ and $X \to B$ be a flat projective map of relative dimension 1. For a geometric point b of B we call the fiber at b the base change of X under the inclusion map $b \to B$, and we denote the fiber at b by X_b. Throughout this section we assume that for each geometric point b of B the fiber X_b is reduced. We furthermore assume that each singular fiber is of the following types:

1. a simple nodal rational curve I_1;
2. two smooth rational curves meeting transversely in two distinct points I_2;
3. a cuspidal rational curve II.

(Again, for the definition of the various singularities appearing in elliptic fibrations see [14] or Section 3. The case of type III singular fibers, also needed in the proof of Theorem 1.1, will be discussed in Subsection 5.2.) In this situation there exists a relative compactified Picard scheme

$$\overline{\mathrm{Pic}}_{X|B}$$

parametrizing torsion-free sheaves $\mathcal{S}$ of $\mathcal{O}_{X_b}$-modules of rank 1. It naturally decomposes according to the (total) degree δ of $\mathcal{S}$ as

$$\overline{\mathrm{Pic}}^{\delta}_{X|B} \tag{3}$$

where the degree is defined by

$$\deg(\mathcal{S}) = \chi(\mathcal{S}) - \chi(\mathcal{O}_{X_b}) \tag{4}$$

with χ standing for Poincaré characteristic. For types I_1 and II the scheme $\overline{\mathrm{Pic}}$ was constructed by [7]. The I_2 case is a particular case of [16]; we will come back to this case in Subsection 2.2.1. In order to introduce the ideas to be used later in various other situations, let us give here the description of (3) in the cases I_1 and II according to [16, Section 13] and [6, Chapter 4]. Our argument can be made more precise using generalized parabolic line bundles on the normalization introduced by [4].

Proposition 2.1. *(Oda–Seshadri* [16]*, Altman–Kleiman* [2]*)*

(1) Let X_b be a curve of type I_1. Then for any $\delta \in \mathbb{Z}$ the scheme $\overline{\mathrm{Pic}}^{\delta}_{X_b}$ is isomorphic to a curve of type I_1.

(2) Let X_b be a curve of type II. Then for any $\delta \in \mathbb{Z}$ the scheme $\overline{\mathrm{Pic}}^{\delta}_{X_b}$ is isomorphic to a curve of type II.

Proof. We only treat part (1). Let

$$\pi : \tilde{X}_b \to X_b$$

stand for the normalization of X_b. Then $\tilde{X}_b$ is a smooth rational curve. Let us denote by $x_0 \in X_b$ the only singular point and by $0, \infty \in \tilde{X}_b$ its preimages under the map π. Then a degree 0 line bundle on X_b is the same thing as a line bundle L of degree 0 on $\tilde{X}_b$ endowed with an isomorphism

$$L_0 \cong L_\infty,$$

where L_p denotes the fiber of L over $p \in \tilde{X}_b$. Now there is just one degree 0 holomorphic line bundle on $\tilde{X}_b$, namely $L = \mathcal{O}_{\tilde{X}_b}$, so the data above reduces to just the identification of the fibers. This in turn can be described by the image $\lambda \in \mathbb{C}^\times \subset L_\infty$ of $1 \in L_0$. Intrinsically λ can be understood as an element of the projective line

$$\mathbf{P}(L_0 \oplus L_\infty).$$

Let us denote by $L(\lambda)$ the degree 0 line bundle on X_b obtained by the above identification of the fibers; clearly, for $\lambda' \neq \lambda$ the line bundle $L(\lambda')$ is not isomorphic to $L(\lambda)$. To sum up, the universal line bundle on X_b is given by

$$L(\cdot) \to \mathbb{C}^\times \times X_b \subset \mathbf{P}(L_0 \oplus L_\infty) \times X_b.$$

Our aim is to find the limit of $L(\lambda)$ as $\lambda \to 0$ or ∞ in $\mathbf{P}(L_0 \oplus L_\infty)$. In the case $\lambda = 0$ the limit consists of a line bundle on $\tilde{X}_b$ with an identification of the fiber L_0 to $0 \in L_\infty$; said differently, there is a short exact sequence

$$0 \to L(0) \to \pi_* L \to L_0 \to 0,$$

hence $L(0) = \pi_* \mathcal{O}_{\tilde{X}_b}(-\{0\})$. Similarly, the limit $\lambda \to \infty$ fits into the short exact sequence

$$0 \to L(\infty) \to \pi_* L \to L_\infty \to 0,$$

hence $L(\infty) = \pi_* \mathcal{O}_{\tilde{X}_b}(-\{\infty\})$. As $\tilde{X}_b$ is of genus 0, the bundles $\mathcal{O}_{\tilde{X}_b}(-\{0\})$ and $\mathcal{O}_{\tilde{X}_b}(-\{\infty\})$ are isomorphic to each other, therefore so are their direct images by π. The statement in the case of I_1 now follows.

As for part (2), see [2, Theorem 18]. □

2.2.1. *Oda–Seshadri stability for I_2 curves.* In this subsection we continue the summary of known results concerning compactified Picard schemes. For families with singular fibers I_n for $n \geq 2$ (and more generally, for reduced curves with only simple nodes as singular points) the compactifications of the Picard scheme were studied in [16]. In this case, the degree of the restriction of $\mathcal{S}$ to each component of X_b needs to be centered about some values. Let us restrict our attention to the case $n = 2$ and denote by X_+, X_- the irreducible components of X_b. These are smooth curves of genus 0, attached at two points. We may assume for ease of notations that the common points are $0, \infty \in X_\pm$ so that $0 \in X_+$ is identified with $0 \in X_-$ and $\infty \in X_+$ is identified with $\infty \in X_-$. We will also denote by 0 and ∞ the point of X_b obtained by the above identification. The curve

$$\tilde{X}_b = X_+ \coprod X_-$$

is called the normalization of X_b. There is an obvious map

$$\sigma : \tilde{X}_b \to X_b.$$

It turns out that in order to get a moduli scheme we need to impose a further condition of stability on the sheaves $\mathcal{S}$ that we wish to parametrize. This stability condition depends on some parameters $(\phi_+, \phi_-) \in \mathbb{R}^2$ satisfying

$$\phi_+ + \phi_- = 0.$$

For a torsion-free coherent sheaf $\mathcal{S}$ of $\mathcal{O}_{X_b}$-modules of rank 1 let us set

$$\mathcal{L}(\mathcal{S}) = \sigma^* \mathcal{S} / \mathcal{T}or^{\mathcal{O}_{\tilde{X}_b}}(\sigma^* \mathcal{S}) \tag{5}$$

with $\mathcal{T}or^{\mathcal{O}_{\tilde{X}_b}}(\sigma^*\mathcal{S})$ denoting the torsion part of the $\mathcal{O}_{\tilde{X}_b}$-module $\sigma^*\mathcal{S}$, and for $i \in \{\pm\}$ define

$$\delta_i = \deg(\mathcal{L}(\mathcal{S})|_{X_i}), \tag{6}$$

where deg stands for the degree with respect to the standard polarization on X_i. Notice that for any i there exists a canonical morphism $\mathcal{S} \to \mathcal{L}(\mathcal{S})|_{X_i}$ from the composition

$$\mathcal{S} \to \sigma^*\mathcal{S} \to \mathcal{L}(\mathcal{S}) \to \mathcal{L}(\mathcal{S})|_{X_i}. \tag{7}$$

Setting

$$J(\mathcal{S}) = \{j \in \{0,\infty\} : \quad \mathcal{S} \text{ is locally free near } j\},$$

we have a short exact sequence of coherent sheaves

$$0 \to \mathcal{S} \to \mathcal{L}(\mathcal{S})|_{X_+} \oplus \mathcal{L}(\mathcal{S})|_{X_-} \to \oplus_{j\in J(\mathcal{S})}\mathbb{C} \to 0, \tag{8}$$

hence

$$\chi(\mathcal{S}) + |J(\mathcal{S})| = \chi(\mathcal{L}(\mathcal{S})|_{X_+}) + \chi(\mathcal{L}(\mathcal{S})|_{X_-}). \tag{9}$$

Applying this formula to $\mathcal{S} = \mathcal{O}_{X_b}$ we get

$$\chi(\mathcal{O}_{X_b}) + 2 = \chi(\mathcal{O}_{X_+}) + \chi(\mathcal{O}_{X_-}). \tag{10}$$

Now subtracting (10) from (9) and taking into account definitions (4) and (6), we infer

$$\deg(\mathcal{S}) = \delta_+ + \delta_- + 2 - |J(\mathcal{S})|. \tag{11}$$

The construction of Oda and Seshadri uses the dual graph $\Gamma = (V,E)$ associated to X_b: by definition, $V = \{X_+, X_-\} = \{+,-\}$ is the set of all connected components of the normalization $\tilde{X}_b$, $E = \{0,\infty\}$ is the set of all double points of X_b, and an edge j is adjacent to a vertex i if and only if the double point corresponding to j lies on the connected component corresponding to i. For $i \in \{\pm\}$ Oda and Seshadri define the value

$$d(J - J(\mathcal{S}))_i$$

as the number of edges $j \in \{0,\infty\}$ such that i is one of the end-points of j and $\mathcal{S}$ is not locally free at j. As both $i = \pm$ are end-points of both edges $j \in \{0,\infty\}$, it is obvious from this definition that the quantity $d(J - J(\mathcal{S}))_i$ does not depend on $i \in \{\pm\}$, and we have the equality

$$d(J - J(\mathcal{S}))_i = |J - J(\mathcal{S})| = 2 - |J(\mathcal{S})|.$$

Furthermore, for any non-trivial subset $I' \subset \{\pm\}$, Oda and Seshadri set $I'' = \{\pm\} - I'$ and denote by

$$(\delta_{J(\mathcal{S})}v(I''), \delta_{J(\mathcal{S})}v(I'')) \tag{12}$$

the number of edges $j \in \{0,\infty\}$ such that $\mathcal{S}$ is locally free near j and has one end-point in I' and the other one in I''. As any non-trivial $I' \subset \{\pm\}$ is necessarily of the form $I' = \{i\}$ for some $i \in \{\pm\}$ and every edge has both vertices i as end-point, clearly the last condition on the edges is vacuous. Hence (12) simply gives the number of edges such that $\mathcal{S}$ is locally free near j, said differently we find

$$(\delta_{J(\mathcal{S})}v(I''), \delta_{J(\mathcal{S})}v(I'')) = |J(\mathcal{S})|.$$

With these preliminaries Oda and Seshadri call $\mathcal{S}$ ϕ-semistable if for both $i \in \{\pm\}$ the inequalities

$$\delta_i + \frac{1}{2}d(J - J(\mathcal{S}))_i - \phi_i \le \frac{(\delta_{J(\mathcal{S})}v(I - \{i\}), \delta_{J(\mathcal{S})}v(I - \{i\}))}{2}$$

are fulfilled, and ϕ-stable if the corresponding strict inequalities hold. Plugging the formulas found above into this inequality we find that in the case of an I_2 curve X_b the semi-stability condition reads as

$$\delta_i - \phi_i \le |J(\mathcal{S})| - 1, \tag{13}$$

and stability is defined by the corresponding strict inequality. Taking into account the equality of (11), this may be equivalently rewritten as

$$\delta - 1 < \delta_i - \phi_i \leq |J(\mathcal{S})| - 1.$$

The compactified Picard scheme

$$\overline{\mathrm{Pic}}^{\delta,\phi}_{X_b}$$

of degree $\delta \in \mathbb{Z}$ is then defined as the scheme parametrizing ϕ-stable torsion-free sheaves of degree δ over X_b. More precisely, Oda and Seshadri define the Picard functor of ϕ-stable torsion-free sheaves and they show that it is representable by a scheme.

2.3. **Irregular Higgs bundles.** We study rank 2 irregular Higgs bundles $(\mathcal{E},\theta)$ defined over $\mathbb{C}P^1$, where $\mathcal{E}$ is a rank 2 vector bundle and θ is a meromorphic section of $\mathcal{E}nd(\mathcal{E})\otimes K$ called the Higgs field. We set

$$\deg(\mathcal{E}) = d.$$

We will limit ourselves to the case where θ has a single pole q of order 4:

$$\theta : \mathcal{E} \to \mathcal{E} \otimes K(4\cdot\{q\}).$$

Introduce two local charts on $\mathbb{C}P^1$: U_1 with $z_1 \in \mathbb{C}$ where $\{z_1 = 0\} = q$ and U_2 with $z_2 \in \mathbb{C}$ where $\{z_2 = \infty\} = q$. Then over $\mathbb{C}$ the line bundle $K(4\cdot\{q\})$ admits the trivializing sections κ_i over U_i given as

$$\begin{aligned}\kappa_1 &= \frac{\mathrm{d}z_1}{z_1^4},\\ \kappa_2 &= \mathrm{d}z_2.\end{aligned} \tag{14}$$

The conversion from κ_1 to κ_2 is the following:

$$\kappa_1 = \frac{\mathrm{d}z_1}{z_1^4} = -z_2^2\mathrm{d}z_2 = -z_2^2\kappa_2. \tag{15}$$

The trivialization κ_i induces a trivialization κ_i^2 on $K(4\cdot\{q\})^{\otimes 2}$, $i = 1,2$.

The Hirzebruch surface X can be covered by four charts. We will need only two of those, since we only consider curves disjoint from the section C_∞ at infinity. Let us denote $V_i \subset p^{-1}(U_i)$ the complement of the section at infinity in $p^{-1}(U_i)$ ($i = 1,2$). Let $\zeta \in \Gamma(X, p^*K(4\cdot\{q\}))$ be the canonical section, and introduce $w_i \in \Gamma(V_i,\mathcal{O})$ by

$$\zeta = w_i \otimes \kappa_i.$$

Use (15) for the conversion between w_1 to w_2:

$$w_2 \otimes \kappa_2 = \zeta = w_1 \otimes \kappa_1 = -z_2^2 w_1 \otimes \kappa_2.$$

In the κ_1 trivialization of $\mathcal{E}$ near q we have

$$\theta = \sum_{n\geq -4} A_n z_1^n \otimes \mathrm{d}z_1, \tag{16}$$

where $A_n \in \mathfrak{gl}(2,\mathbb{C})$.

For the identity automorphism $\mathrm{I}_{\mathcal{E}}$ of $\mathcal{E}$ we may consider the characteristic polynomial

$$\chi_\theta(\zeta) = \det(\zeta\mathrm{I}_{\mathcal{E}} - \theta) = \zeta^2 + \zeta F + G, \tag{17}$$

for some

$$F \in H^0(\mathbb{C}P^1, K(4\cdot\{q\})), \quad G \in H^0(\mathbb{C}P^1, K(4\cdot\{q\})^{\otimes 2}).$$

Said differently, F is a meromorphic differential and G is a meromorphic quadratic differential.

Let us set $\vartheta_1 = \sum_{n\geq 0} A_{n-4} z_1^n$ and $\vartheta_2 = \sum_{n\geq 0} B_n z_2^n$, so that we have

$$\theta = \vartheta_i \otimes \kappa_i,$$

where $i = 1, 2$. If we now factor κ_i in (17), then the characteristic polynomial can be rewritten as

$$\chi_{\vartheta_i}(w_i) = \det(w_i \mathrm{I}_{\mathcal{E}} - \vartheta_i) = w_i^2 + w_i f_i + g_i, \tag{18}$$

with

$$F = f_i \kappa_i, \quad G = g_i \kappa_i^2.$$

Now, as $K(4 \cdot \{q\}) \cong \mathcal{O}(2)$, the coefficients f_i and g_i are polynomials in z_i of degree 2 and 4, respectively:

$$f_1(z_1) = -(p_2 z_1^2 + p_1 z_1 + p_0), \tag{19}$$

$$g_1(z_1) = -(q_4 z_1^4 + q_3 z_1^3 + q_2 z_1^2 + q_1 z_1 + q_0), \tag{20}$$

where all coefficients are elements of $\mathbb{C}$. According to conversion (15):

$$f_2(z_2) = p_0 z_2^2 + p_1 z_2 + p_2, \tag{21}$$

$$g_2(z_2) = -\left(q_0 z_2^4 + q_1 z_2^3 + q_2 z_2^2 + q_3 z_2 + q_4\right). \tag{22}$$

In the next two subsections we explain how to fix the polar parts of θ depending on whether its leading-order term is regular semi-simple (the so-called untwisted case) or has a non-trivial nilpotent part (twisted case).

2.3.1. *The untwisted case.* In this case we will fix scalars $a_\pm \in \mathbb{C}$ with $a_+ \neq a_-$ and assume that the leading-order term of θ (i.e., the coefficient A_{-4} of z_1^{-4} in its Laurent series) is semi-simple with eigenvalues $a_\pm$. Then there exists a polynomial gauge transformation in the indeterminate z_1 that transforms θ into the form

$$\theta = \left[z_1^{-4} \begin{pmatrix} a_+ + b_+ z_1 + c_+ z_1^2 + \lambda_+ z_1^3 & 0 \\ 0 & a_- + b_- z_1 + c_- z_1^2 + \lambda_- z_1^3 \end{pmatrix} + \cdots \right] \otimes \mathrm{d}z_1 \tag{23}$$

in some local trivialization of $\mathcal{E}$ near q where the dots stand for higher-order matrices in z_1. Indeed, up to applying a constant base change we may assume that A_{-4} is diagonal. Furthermore the action of

$$\gamma(z_1) = 1 + \gamma_n z_1^n$$

on (16) is

$$\gamma(z_1)\theta(z_1)\gamma(z_1)^{-1} = \big(A_{-4} z_1^{-4} + \cdots + A_{n-5} z_1^{n-5} + \\ + (A_{n-4} - \mathrm{ad}_{A_{-4}}(\gamma_n)) z_1^{n-4} + O(z_1^{n-3}) \big) \otimes \mathrm{d}z_1,$$

and since the image of $\mathrm{ad}_{A_{-4}}$ is the subspace of off-diagonal matrices we can successively apply such gauge transformations with $n = 1, 2$ and 3 to cancel the off-diagonal terms of A_{-3}, then those of A_{-2} and finally those of A_{-1}.

The matrices appearing in (23) are called the *polar part* of θ at the singularity. From now on we assume that the constants $a_\pm, b_\pm, c_\pm, \lambda_\pm \in \mathbb{C}$ appearing in (23) are fixed. A necessary condition for the existence of Higgs bundles with this polar part is given by the residue theorem which states that

$$\lambda_+ + \lambda_- = 0. \tag{24}$$

We therefore assume that the parameters are fixed so that this equality holds.

We introduce

$$\mathtt{P} = 4 \cdot \{q\}, \quad \mathtt{P}_{\mathrm{red}} = \{q\};$$

$\mathtt{P}$ is called the *polar divisor* and $\mathtt{P}_{\mathrm{red}}$ the *parabolic divisor*. A parabolic structure compatible with $(\mathcal{E}, \theta)$ is a choice

$$(\alpha_q^+, \alpha_q^-) \in [0, 1)^2$$

of two distinct numbers for the singular point $q \in \mathtt{P}_{\mathrm{red}}$; the scalars $\alpha_q^\pm$ are called parabolic weights. Essentially, $\alpha_q^\pm$ are associated to the $\lambda_\pm$ in the above polar parts at q, and they correspond to the flag

$$\mathcal{E}_q \supset L_q^+ \supset \{0\}$$

invariant under the polar part of θ. The pair (α_q^+, α_q^-) is *generic* if $\alpha_q^+ \alpha_q^- \neq 0$. The parabolic weights constitute parameters appearing in the behavior of a compatible Hermitian–Einstein metric near the puncture, that one may freely prescribe independently of the eigenvalues of the residue of the Higgs field. Notice that the associated graded $\mathfrak{t}$ of this flag is a Cartan subalgebra uniquely determined by the polar part, so the only choice for the parabolic structure is that of the weights $\alpha_q^\pm$, which then singles out a Borel subalgebra containing $\mathfrak{t}$. A Higgs subbundle of $(\mathcal{E}, \theta)$ is a pair $(\mathcal{F}, \theta|_{\mathcal{F}})$ with $\mathcal{F}$ a holomorphic subbundle of $\mathcal{E}$ such that

$$\theta|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes K(\mathtt{P}).$$

One immediately sees that if this is the case then the fiber $\mathcal{F}_q$ of $\mathcal{F}$ at q must be one of the eigenlines $L_q^\pm$. In particular, if $(\mathcal{E}, \theta)$ is endowed with a compatible parabolic structure then any Higgs subbundle $(\mathcal{F}, \theta|_{\mathcal{F}})$ inherits a parabolic structure from $(\mathcal{E}, \theta)$ in a natural way: according to whether $\mathcal{F}_q = L_q^\pm$ we set

$$\alpha_q(\mathcal{F}) = \alpha_q^\pm$$

to be the parabolic weight of $(\mathcal{F}, \theta|_{\mathcal{F}})$ at $q \in \mathtt{P}_{\mathrm{red}}$. We then define

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + (\alpha_q^+ + \alpha_q^-)$$

and

$$\text{par-deg}(\mathcal{F}) = \deg(\mathcal{F}) + \alpha_q(\mathcal{F}).$$

We say that $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-semistable if and only if for all Higgs subbundles $(\mathcal{F}, \theta|_{\mathcal{F}})$ we have

$$\text{par-deg}(\mathcal{F}) \leq \frac{\text{par-deg}(\mathcal{E})}{2}$$

and $\vec{\alpha}$-stable if strict inequality holds. Observe that if $\text{par-deg}(\mathcal{E}) = 0$ then these conditions simplify to

$$\text{par-deg}(\mathcal{F}) \leq 0$$

(respectively $<$). If $(\mathcal{F}, \theta|_{\mathcal{F}})$ is a Higgs subbundle of $(\mathcal{E}, \theta)$ then θ also induces a morphism on the quotient vector bundle

$$\mathcal{Q} = \mathcal{E}/\mathcal{F},$$

and we denote the resulting Higgs field by

$$\overline{\theta} : \mathcal{Q} \to \mathcal{Q} \otimes K(\mathtt{P}).$$

In this situation we say that $(\mathcal{Q}, \overline{\theta})$ is a quotient Higgs bundle of $(\mathcal{E}, \theta)$. Furthermore, if $(\mathcal{E}, \theta)$ is endowed with a compatible parabolic structure then it induces a parabolic structure on $\mathcal{Q}$: if $\alpha_q(\mathcal{F}) = \alpha_q^\pm$ then we simply set

$$\alpha_q(\mathcal{Q}) = \alpha_q^\mp.$$

Just as above, we set

$$\text{par-deg}(\mathcal{Q}) = \deg(\mathcal{Q}) + \alpha_q(\mathcal{Q}).$$

By additivity of the degree, we have an equivalent definition of $\vec{\alpha}$-stability in terms of quotients: namely, $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-semistable if and only if for any quotient Higgs bundle $(\mathcal{Q}, \overline{\theta})$ we have

$$\text{par-deg}(\mathcal{Q}) \geq \frac{\text{par-deg}(\mathcal{E})}{2}$$

and $\vec{\alpha}$-stable if strict inequality holds. Again, if $\text{par-deg}(\mathcal{E}) = 0$ then these conditions simplify to

$$\text{par-deg}(\mathcal{Q}) \geq 0$$

(respectively $>$).

We will be interested in the moduli spaces

$$\mathcal{M}^{(s)s} = \mathcal{M}^{(s)s}(\mathbb{C}P^1, q, a_\pm, b_\pm, c_\pm, \lambda_\pm, \alpha_q^\pm)$$

of $\vec{\alpha}$-stable (resp. $\vec{\alpha}$-semi-stable) irregular Higgs bundles on $\mathbb{C}P^1$ of 0 parabolic degree with the polar parts at q as prescribed in (23), up to gauge equivalence. The spaces $\mathcal{M}^{(s)s}$ are called *irregular Dolbeault moduli spaces*. The general construction of moduli spaces $\mathcal{M}^s$ parametrizing isomorphism classes of stable objects was given in [5] using gauge theoretic methods. In particular, it is proved that if semi-stability is equivalent to stability and the adjoint orbits of the residues are closed, then the moduli space $\mathcal{M}^s$ is a complete hyper-Kähler manifold. On the other hand, in order to consider moduli spaces $\mathcal{M}^{ss}$ parametrizing equivalence classes of semi-stable objects one needs to slightly relax the notion of equivalence. Namely, to any strictly semi-stable object $(\mathcal{E},\theta)$ it is possible to find a Jordan–Hölder filtration

$$0 \subset (\mathcal{E}_1,\theta_1) \subset (\mathcal{E},\theta)$$

(in our case necessarily of length 2) such that both $(\mathcal{E}_1,\theta_1)$ and $(\mathcal{E}_2,\theta_2)$ are stable (where $\mathcal{E}_2 = \mathcal{E}/\mathcal{E}_1$ and θ_2 is the Higgs field on $\mathcal{E}_2$ induced by θ). We then call

$$(\mathcal{E}_1,\theta_1) \oplus (\mathcal{E}_2,\theta_2)$$

the associated graded irregular Higgs bundle of $(\mathcal{E},\theta)$ and we call $(\mathcal{E},\theta)$ and $(\mathcal{E}',\theta')$ S-equivalent if their associated graded irregular Higgs bundles agree. This definition reduces to isomorphism in the case of stable irregular Higgs bundles. We expect that there exists a quasi-projective smooth coarse moduli scheme $\mathcal{M}^{ss}$ parametrizing S-equivalence classes of semi-stable irregular Higgs bundles using a geometric invariant theory construction. Such a construction for the ramified irregular de Rham moduli space is given in [11]. It is highly plausible that the construction of Inaba carries over to provide a ramified irregular Dolbeault moduli space too. In this paper we indicate an alternative approach to study the irregular Dolbeault moduli space. Namely, the relative Picard scheme was constructed by Grothendieck as an algebraic variety (for an exposition of the construction by S. Kleiman, see [13, Theorem 9.4.8]). The refined BNR-correspondence [21, Theorem 5.4] is a biholomorphism between moduli spaces of irregular Higgs bundles of prescribed polar part and the Picard scheme of sheaves on ruled surfaces. The definition of this map is purely algebraic, hence the algebraic structure of the relative Picard scheme endows the complex analytic manifold $\mathcal{M}^{(s)s}$ with the structure of a complex algebraic variety. In particular, Theorems 1.1 and 1.3 provide a $\mathbb{C}$-analytic description of the corresponding moduli spaces.

2.3.2. *The twisted case.* We now consider the case where A_{-4} has non-trivial nilpotent part. In a convenient trivialization we then have

$$A_{-4} = \begin{pmatrix} b_{-8} & 1 \\ 0 & b_{-8} \end{pmatrix}$$

for some $b_{-8} \in \mathbb{C}$ (the labeling will shortly become clear). Observe that $\mathrm{im}(\mathrm{ad}_{A_{-4}})$ is spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the same argument as in the twisted case it follows that there exists a polynomial gauge transformation $\gamma(z)$ that transforms θ into the form

$$\theta = \left(\begin{pmatrix} b_{-8} & 1 \\ 0 & b_{-8} \end{pmatrix} z^{-4} + \begin{pmatrix} 0 & 0 \\ b_{-7} & b_{-6} \end{pmatrix} z^{-3} + \begin{pmatrix} 0 & 0 \\ b_{-5} & b_{-4} \end{pmatrix} z^{-2} + \begin{pmatrix} 0 & 0 \\ b_{-3} & b_{-2} \end{pmatrix} z^{-1} + O(1) \right) \otimes \mathrm{d}z. \tag{25}$$

Observe that by virtue of the residue theorem this time we have

$$b_{-2} = 0.$$

On the other hand, notice that if $b_{-7} = 0$ in the above matrix then A_{-4} can be diagonalized using the meromorphic gauge transformation

$$\gamma(z) = 1 + \begin{pmatrix} 0 & -b_6^{-1} \\ 0 & 0 \end{pmatrix} z^{-1}$$

unless b_{-6} also vanishes. Since in this section we are interested in the case where A_{-4} is not diagonalizable (even by meromorphic gauge transformations), from now on we therefore assume that

$$b_{-7} \neq 0.$$

and that the constants $b_{-8}, \dots, b_{-3} \in \mathbb{C}$ appearing in (25) are fixed.

This time the data of the parabolic structure compatible with θ is trivial, i.e. is the trivial flag

$$\mathcal{E}_q \supset \{0\}$$

with an arbitrary weight α_q. Indeed, as the rank of $\mathcal{E}$ is 2, the only other possibility would be a full flag as in the untwisted case; however, then the graded pieces of the polar parts would be of dimension 1, and we could not get nilpotent graded polar parts.

Again, we will be interested in the moduli spaces

$$\mathcal{M}^{(s)s} = \mathcal{M}^{(s)s}(\mathbb{C}P^1, q, b_{-8}, \dots, b_{-3}, \alpha_q)$$

of S-equivalence classes of (semi-)stable irregular Higgs bundles on $\mathbb{C}P^1$ with polar part at q with respect to some trivialization as prescribed in (25). We will see that in this case the weight α_q actually plays no role. The existence of a moduli space parametrizing isomorphism classes of stable objects should follow from [5], and we again expect that there should exist a quasi-projective smooth coarse moduli scheme $\mathcal{M}^{ss}$ parametrizing S-equivalence classes of semi-stable objects.

2.4. **Spectral data of irregular Higgs bundles and the irregular Hitchin map.** A categorical equivalence between the groupoid of irregular Higgs bundles with semi-simple polar part and the relative Picard functor of a Hilbert scheme of curves on a certain multiple blow-up Y of the Hirzebruch surface X from Subsection 2.1 was described in [21]. We will refer to this equivalence as the refined Beauville–Narasimhan–Ramanan (BNR-) correspondence. The sheaf associated to an irregular Higgs bundle by this correspondence is called its *spectral sheaf*, usually denoted by $\mathcal{S}$. The general formula relating the degrees appearing in the two setups is

$$\delta = d + \frac{1}{2} r(r-1)\deg(K(4)) = d + 2, \tag{26}$$

where $d = \deg(\mathcal{E})$ and δ denotes the degree of $\mathcal{S}$ defined in (4). (Recall that in the latter formula X_b denotes the support of $\mathcal{S}$.) We refer the reader to [21] for the general correspondence; in Subsection 4.1 we will spell it out explicitly in the untwisted case. In the twisted case we prove an analogous result in Section 6. We expect that such a result should hold in general, and not only in the particular case we are treating here.

A closely related concept is that of the *irregular Hitchin map*. Namely, to an irregular Higgs bundle one may associate the support $\widetilde{\Sigma}$ of $\mathcal{S}$, called the *spectral curve*. With the notations of Subsection 2.2, when $\widetilde{\Sigma}$ is singular it is an instance of one of the curves X_b. Roughly speaking, in the untwisted case it turns out that the prescription (23) on the eigenvalues of the polar parts amounts to requiring the two branches of the spectral curve X_b to pass through the points $a_\pm$ in the fiber of X over q (with respect to a natural fiber coordinate), with first-, second- and third-order holomorphic derivatives with respect to z equal to $b_\pm, c_\pm, \lambda_\pm$ respectively. Said differently, if one defines Y as the 8-times blow-up of X along the corresponding non-reduced subscheme, then the proper transform of $\widetilde{\Sigma}$ naturally lies within Y. Moreover, it turns out that the proper transform of $\widetilde{\Sigma}$ must intersect the cycles in second homology with prescribed intersection numbers. To sum up, these conditions mean that the curve $\widetilde{\Sigma}$ belongs to a complete linear system $|D|$ of curves on Y determined by the map $Y \to X$. Finally, this curve must not intersect set-theoretically a given divisor (called *divisor at infinity*); this then shows that the natural map

$$(\mathcal{E}, \theta) \mapsto \widetilde{\Sigma}$$

obtained by composing the refined BNR-correspondence above and the forgetful functor mapping a sheaf to its support, actually takes values in an affine subspace $|D|_0 \subset |D|$. For more details, see Proposition 4.2 or [21, Theorem 4.3]. For an extension to the unramified case, see Proposition 6.4. Therefore, the above association gives rise to the *irregular Hitchin map*

$$H : \mathcal{M}^{ss} \to |D|_0.$$

We call H the irregular Hitchin map because it is a straightforward analogue of the map defined in [10]. It follows from [5] that for generic choices of the singularity parameters (namely, assuming that the adjoint orbits of the residues are closed), the irregular Dolbeault moduli spaces are complete holomorphic-symplectic smooth manifolds. Based on this fact and the above analogy, it is therefore natural to expect that H is a proper map which endows $\mathcal{M}^{ss}$ with the structure of an algebraically completely integrable system.

3. Elliptic fibrations on rational elliptic surfaces

In this section we will study singular fibers of elliptic fibrations on rational elliptic surfaces. As 4-manifolds, these surfaces are diffeomorphic to the 9-fold blow-up $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ of the complex projective plane $\mathbb{C}P^2$. The potential singular fibers are classified by Kodaira [14]. Here we will concentrate only on those fibrations which contain singular fibers of types $\tilde{E}_8$ and $\tilde{E}_7$. (For the plumbing description of these singular fibers see Figure 1.)

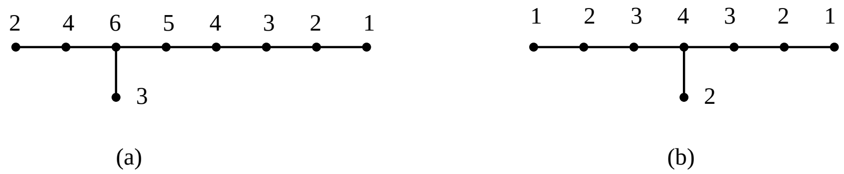

FIGURE 1. Plumbings of singular fibers of types (a) $\tilde{E}_8$ and (b) $\tilde{E}_7$ (integers next to vertices indicate the multiplicities of the corresponding homology classes in the fiber). All curves are rational, all intersections are transverse, and all self-intersections are equal to -2.

One way to construct an elliptic fibration on the rational elliptic surface is by giving a pencil of cubic curves in $\mathbb{C}P^2$ (with the additional property that the pencil contains at least one smooth cubic) and then blowing up the basepoints of the pencil. In turn, the pencil can be given by specifying two degree-3 homogeneous polynomials p_0 and p_1 in three variables and considering the curves $C(p_t)$ corresponding to the polynomials $p_t = t_0 p_0 + t_1 p_1$ for $t = [t_0 : t_1] \in \mathbb{C}P^1$. The pencil will not contain smooth curves if p_0 and p_1 admit common singular points, hence this case will be avoided.

Recall that the singular fiber in an elliptic fibration with a single node is called I_1 (or a fishtail fiber), the fiber with a cusp singularity (which can be modeled by the cone on the trefoil knot $T_{2,3}$, or can be given by the local equation $y^2 = x^3$) is a cusp fiber (also denoted by II). A singular fiber with two rational curves intersecting each other in two distinct points (and having self-intersection -2) is an I_2 fiber. If the two rational curves are tangent to each other (still with self-intersection -2) then we have a type III fiber. (There are further singular fibers in the Kodaira list, but we will not meet them in our subsequent arguments.)

The determination of the type of all singular fibers in an elliptic fibration specified by two cubic polynomials p_0, p_1 can be a rather tedious problem. By choosing specific polynomials, the existence of two singular fibers is quite transparent, but the identification of the further ones usually requires further computations.

3.1. **The case of singular fibers of type $\widetilde{E}_8$.** Suppose first that we have an elliptic fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with a singular fiber of type $\widetilde{E}_8$. We will also assume that the fibration comes from blowing up a pencil, hence it admits a section. This section then necessarily intersects the $\widetilde{E}_8$-fiber in the unique curve with multiplicity 1. Consider a generic fiber C of the fibration, and blow down the section and then consecutively the next six curves of the $\widetilde{E}_8$-fiber. The image of C (now of self-intersection 7) will intersect two curves E_1, E_2 (both of self-intersection (-1)) from the fiber, one of which (say E_2) is further intersected by the leaf E_3 of the $\widetilde{E}_8$ fiber, and is of multiplicity 2. (We point out that, as it is obvious from the construction, the two curves E_1, E_2 intersect C at the same point, cf. the left diagram of Figure 2.)

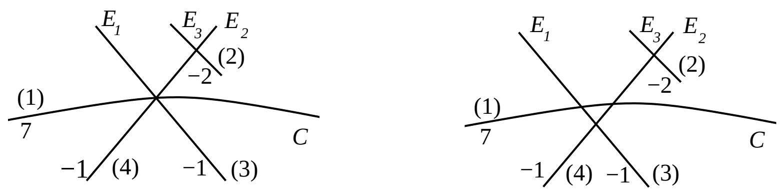

FIGURE 2. Curve configurations when blowing down a section and a singular fiber of type (a) $\widetilde{E}_8$ and (b) $\widetilde{E}_7$. Integers next to the curves indicate self-intersections, while integers in brackets are multiplicities.

There is a choice in continuing the blow-down process. If we blow down E_1, then we get a configuration of curves in the second Hirzebruch surface, where the image of E_2 is a fiber, E_3 is the section at infinity, and C blows down to a multisection, intersecting the generic fiber twice and being tangent to E_2. On the other hand, blowing down E_2 first, and then E_3, the curve C blows down to a cubic curve C_0 in $\mathbb{C}P^2$, and the image of E_1 will be a projective line, triply tangent to C_0 (at one of its inflection points). The two results are related by the birational morphism ω of Equation (2).

In conclusion,

Theorem 3.1. *Any elliptic fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with a section and with a singular fiber of type $\widetilde{E}_8$ can be blown up from a pencil defined by either*

(1) the union of the infinity section (with multiplicity 2) with a fiber (with multiplicity 4) in the second Hirzebruch surface, and with a double section which is tangent to the chosen fiber, or

(2) a cubic curve in $\mathbb{C}P^2$, with a triple tangent line (at one of the inflection points of the cubic), the latter with multiplicity three.

The converse statements also hold: pencils given by (1) or (2) above give rise to fibrations (after the infinitely close blow-ups of the base point) to elliptic fibrations containing an $\widetilde{E}_8$ fiber. □

3.2. **The case of singular fibers of type $\widetilde{E}_7$.** Next we would like to analyze pencils resulting in fibrations with singular fibers of type $\widetilde{E}_7$. Assume therefore that the fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ contains such a singular fiber, and that the fibration results from a pencil, hence it also admits a section. Indeed, since the pencil should have at least two basepoints (otherwise the fibration has a singular fiber which contains a chain of 8 curves with self-intersection (-2), which is impossible next to a fiber of type $\widetilde{E}_7$), we can assume that there are two sections, intersecting the type $\widetilde{E}_7$ singular fibers in the two (-2)-curves with multiplicity 1. As before, let C be a regular fiber of the fibration.

After 7 blow-downs (by blowing down the two sections and two, respectively three curves from the two long arms of the $\widetilde{E}_7$-fiber) we get a configuration of 4 curves: the image of the fiber C, two (-1)-curves (called E_1 and E_2) intersecting it in two distinct points (and also intersecting each other) and a (-2)-curve E_3 intersecting E_2 only, cf. the right diagram of Figure 2. As in the case of an $\widetilde{E}_8$-fiber, we have a choice in performing the next blow-down. If we blow down E_1, we get a configuration again in the second Hirzebruch surface, while if we blow down E_2 (and then E_3), we get a configuration in $\mathbb{C}P^2$. Consequently we get

Theorem 3.2. *Any elliptic fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with two sections and with a singular fiber of type $\widetilde{E}_7$ can be blown up from a pencil defined by either*

(1) the union of the infinity section (with multiplicity 2) with a fiber (with multiplicity 4) in the second Hirzebruch surface, and with a double section which intersects the distinguished fiber in two distinct points, or

(2) a cubic in $\mathbb{C}P^2$, with a tangent line which intersects the cubic in one further point; the tangent line with multiplicity three.

The converse of this statement also holds: the pencils specified in (1) or (2) above — after infinitely close blow-ups of the base points — give rise to elliptic fibrations containing an $\widetilde{E}_7$ fiber. □

Assume now that the elliptic fibration contains (besides the type $\widetilde{E}_7$-fiber) a further singular fiber which is either of type I_2 or of type III. By further inspecting the blow-down process, now choosing the curve C to be a singular fiber of type I_2 or III we get:

Proposition 3.3. *If an elliptic fibration with a fiber of type $\widetilde{E}_7$ and two sections contains a further singular fiber either of type I_2 or of type III, then the pencil of curves resulting from the repeated blow-down in the second Hirzebruch surface contains a double section which is the union of two sections of the ruling of the surface.* □

The same argument (now by blowing down the configuration to $\mathbb{C}P^2$) shows that the pencil in $\mathbb{C}P^2$ can be chosen to be generated by a projective line ℓ (with multiplicity three, just as before) and another curve, which has two components, a line ℓ_1 and a quadric q, where ℓ intersects ℓ_1 in one point P, while ℓ is tangent to the quadric q (in a point distinct from P). The pencil gives rise to a fibration which has (besides a type $\widetilde{E}_7$ fiber) an I_2 fiber if ℓ_1 intersects q in two distinct points, and a type III fiber if ℓ_1 is tangent to q.

4. THE UNTWISTED CASE

4.1. **The refined BNR-correspondence.** We start by applying the refined BNR-correspondence of [21] to describe a certain blow-up Y of the surface $\widetilde{X}$ whose geometry governs $\mathcal{M}$. We have already referred to Y in Subsection 2.4; here we will make its construction rigorous. Namely, a local trivialization of $K(4) \cong K \otimes \mathcal{O}(4\cdot\{q\})$ near $z_1 = 0$ is given by $z_1^{-4}\mathrm{d}z_1$, so the expressions $z_1^{-4}(a_\pm + b_\pm z_1 + c_\pm z_1^2 + \lambda_\pm z_1^3)\mathrm{d}z_1$ specify non-reduced subschemes of dimension 0 and length 4 in X. We define Y as the blow-up of X along these subschemes, with $\widetilde{X}$ being an intermediate step in the blowing up.

In concrete terms, as in Section 2 q denotes the point with $z_1 = 0$, $U_1 = \mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$, $\kappa_1 = z_1^{-4}\mathrm{d}z_1$, and parametrize $p^{-1}(U_1) \setminus C_\infty$ by coordinates $(z_1, w_1) \in \mathbb{C}^2$ as follows: we let the point of X corresponding to these parameters be $[w_1\kappa_1 : \mathbf{1}]$. We may assume that $\widetilde{X}$ is the blow-up of X in the point $[a_+\kappa_1(0) : \mathbf{1}]$, i.e. over $p^{-1}(U_1)$ the surface $\widetilde{X}$ is defined by

$$(z_1w_1' - (w_1 - a_+)z_1') \subset \mathbb{C}^2 \times \mathbb{C}P^1$$

where $[z_1' : w_1'] \in \mathbb{C}P^1$ are homogeneous coordinates corresponding to the direction of tangent vectors at $z_1 = 0, w_1 = a_+$. We denote this blow-up by

$$\sigma_{1+} : X_{1+} = \widetilde{X} \to X$$

and its exceptional divisor by

$$E_{1+} = \{z_1 = 0, w_1 = a_+, [z_1' : w_1']\}.$$

According to [21, (4.25)], we now need to blow up $\widetilde{X}$ in the point

$$[z_1' : w_1'] = [1 : b_+] \in E_{1+}.$$

For this purpose, we introduce the local chart U_{1+}' of $\widetilde{X}$ given by $z_1' \neq 0$. Here we may normalize $z_1' = 1$, and so a local coordinate chart of U_{1+}' is given by z_1, w_1'. The blow-up

$$\sigma_{2+} : X_{2+} \to X_{1+}$$

we consider is then the blow-up of the point with coordinates $z_1 = 0, w_1' = b_+$. Similarly to the above, we denote the exceptional divisor of σ_{2+} by E_{2+}, and we get canonical coordinates $[z_1'' : w_1'']$ parametrizing E_{2+} starting from the coordinates z_1, z_1'. Again by [21, (4.25)], we now blow up the point

$$[z_1'' : w_1''] = [1 : c_+] \in E_{2+}$$

and call the corresponding birational map

$$\sigma_{3+} : X_{3+} \to X_{2+}.$$

Finally, just as above we get canonical coordinates $[z_1''' : w_1''']$ on the exceptional divisor E_{3+} of σ_{3+}, and we define the blow-up

$$\sigma_{4+} : X_{4+} \to X_{3+}$$

of the point with coordinates

$$[z_1''' : w_1'''] = [1 : \lambda_+] \in E_{3+}.$$

We then let $X_{0-} = X_{4+}$ and carry out a similar procedure for the length 4 non-reduced subschemes corresponding to the expression $z_1^{-4}(a_- + b_- z_1 + c_- z_1^2 + \lambda_- z_1^3)\mathrm{d}z_1$. We denote the birational maps and their exceptional divisors by

$$\sigma_{i-} : X_{i-} \to X_{(i-1)-}$$

and E_{i-} for $1 \leq i \leq 4$. By an abuse of notation, we will continue to denote the proper transforms of E_{i+} and E_{i-} along the subsequent maps σ_{j+} and σ_{j-} by the same symbols. The surface of interest to us is

$$Y = X_{4-} \xrightarrow{\sigma} X. \tag{27}$$

Clearly then there is a diagram

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ X & \overset{\omega}{\dashrightarrow} & \mathbb{C}P^2 \end{array}$$

where the left-hand map is a blow-up of X in 8 points and the right-hand map is a blow-up of $\mathbb{C}P^2$ in 9 points. In particular, as a smooth 4-manifold Y is diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$. By an abuse of notation, we will denote the composition of $\widetilde{X} \to X$ with $p : X \to \mathbb{C}P^1$ by $p : \widetilde{X} \to \mathbb{C}P^1$ and also the composition of $Y \to X$ with $p : X \to \mathbb{C}P^1$ by $p : Y \to \mathbb{C}P^1$.

It follows from [21, Theorem 4.3] that irregular rank 2 Higgs bundles on $\mathbb{C}P^1$ with a pole of order 4 of the local form (23) are in one-to-one correspondence with data of the form $(\widetilde{\Sigma}, \mathcal{S})$ where $\widetilde{\Sigma}$ is a closed holomorphic curve in Y satisfying certain properties and $\mathcal{S}$ is a torsion-free sheaf of $\mathcal{O}_{\widetilde{\Sigma}}$-modules of some given degree δ.

Definition 4.1. *Let $|D|_0$ denote the set of closed holomorphic curves in Y satisfying the following three conditions:*

- (a) $\widetilde{\Sigma}$ *is disjoint from the proper transform of C_∞ in Y;*
- (b) $p : \widetilde{\Sigma} \to \mathbb{C}P^1$ *is a double ramified cover;*
- (c) $\widetilde{\Sigma}$ *intersects the exceptional divisors $E_{4\pm}$ in one point each, away from their "points at infinity" $[z_1^{(iv)} : w_1^{(iv)}] = [0 : 1] \in E_{4\pm}$.*

In particular, conditions (b)–(c) imply that any $\widetilde{\Sigma} \in |D|_0$ intersects neither the proper transform $\tilde{F}_0$ of the fiber F_0 in Y nor the exceptional divisors $E_{i\pm}$ with $1 \le i \le 3$.

Proposition 4.2. *There exists an elliptic fibration $Y \to \mathbb{C}P^1$ with an $\widetilde{E}_7$ singular fiber Y_∞ over $\infty \in \mathbb{C}P^1$, such that $\mathcal{M}^{ss}$ is a relative compactified Picard scheme of torsion-free sheaves of relative degree 1 over $Y \setminus Y_\infty$.*

Proof. Let F denote the fiber class of the Hirzebruch surface, $\tilde{F}_0$ the proper transform under the map (27) of the fiber F_0 of p over q, and recall again our convention that $E_{i\pm}$ stands for the proper transform in Y of the exceptional divisor of the blow-up $\sigma_{i\pm}$. The Picard group of Y is generated by the classes $F, C_\infty, E_{i\pm}(1 \le i \le 4)$, with only non-zero intersection numbers among these classes

$$\begin{aligned} C_\infty^2 &= -2 \\ F \cdot C_\infty &= 1 \\ E_{i\pm}^2 &= -2 \quad (1 \le i \le 3) \\ E_{4\pm}^2 &= -1 \\ E_{i+} \cdot E_{(i+1)+} &= 1 \quad (1 \le i \le 3) \\ E_{i-} \cdot E_{(i+1)-} &= 1 \quad (1 \le i \le 3). \end{aligned}$$

We note the relation

$$F = \tilde{F}_0 + \sum_{i=1}^{4} (E_{i+} + E_{i-}). \tag{28}$$

Consider the divisor

$$Y_\infty = 2C_\infty + 4\tilde{F}_0 + 3(E_{1+} + E_{1-}) + 2(E_{2+} + E_{2-}) + (E_{3+} + E_{3-})$$

of Y and the linear system $|D|$ generated by Y_∞ in Y. A straightforward check using the above intersection numbers shows that Y_∞ is of type $\widetilde{E}_7$, in particular its self-intersection number is 0.

For completing the proof of the proposition, we need a few lemmas.

Lemma 4.3. *A projective curve $\widetilde{\Sigma} \subset Y$ belongs to $|D|$ if and only if*

- $\widetilde{\Sigma} \cdot C_\infty = 0$;
- $\widetilde{\Sigma} \cdot F = 2$;
- $\widetilde{\Sigma} \cdot E_{4+} = 1 = \widetilde{\Sigma} \cdot E_{4-}$.

Proof. An easy check shows that for $\widetilde{\Sigma} = Y_\infty$, the algebraic intersection numbers satisfy all the asserted requirements. For any curve $\widetilde{\Sigma} \in |D|$ the line bundles $\mathcal{O}_Y(\widetilde{\Sigma})$ and $\mathcal{O}_Y(D)$ are linearly equivalent. On the other hand, for any other projective curve $C \subset Y$ we have

$$\widetilde{\Sigma} \cdot C = \langle c_1(\mathcal{O}_Y(\widetilde{\Sigma})), [C] \rangle.$$

Since the first Chern class only depends on the linear equivalence class, the above observation implies the "only if" direction.

For the other direction, note that any curve $\widetilde{\Sigma}$ with given intersection numbers is homologous to Y_∞ because the intersection lattice of Y is non-degenerate and generated by

$F, C_\infty, E_{i\pm}(1 \leq i \leq 4)$. Said differently, the line bundles $\mathcal{O}_Y(D)$ and $\mathcal{O}_Y(\widetilde{\Sigma})$ have the same first Chern class

$$c_1(\mathcal{O}_Y(D)) = c_1(\mathcal{O}_Y(\widetilde{\Sigma})). \tag{29}$$

Now, the Picard group $\mathrm{Pic}(Y)$ can be written as an extension

$$0 \to \mathrm{Pic}^0(Y) \to \mathrm{Pic}(Y) \xrightarrow{c_1} H^2(Y, \mathbb{Z}) \to 0$$

with

$$\mathrm{Pic}^0(Y) = H^{1,0}(Y)/H^1(Y, \mathbb{Z}).$$

Taking into account that $H^1(Y, \mathbb{C}) = 0$, this implies that $\mathrm{Pic}^0(Y) = 0$. Then (29) implies that $\mathcal{O}_Y(D) = \mathcal{O}_Y(\widetilde{\Sigma})$. □

The conditions of Lemma 4.3 are counterparts in terms of algebraic intersection numbers of the geometric conditions (a)–(c) of Definition 4.1. (Just as there, it follows from these requirements and the relation (28) that $\widetilde{\Sigma} \cdot E_{i\pm} = 0$ for all $1 \leq i \leq 3$.) From this, we see that $|D|_0 \subseteq |D|$. The base of $|D|$ is $P(H^0(Y, \mathcal{O}_Y(D)))$.

Lemma 4.4. *We have* $\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(D)) = 2$*, i.e.* $|D|$ *is a pencil.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(D) \to \mathcal{O}_D(D) \to 0$$

of sheaves on Y, and its associated long exact sequence in cohomology

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(D)) \to H^0(Y, \mathcal{O}_D(D)) \to H^1(Y, \mathcal{O}_Y) = 0.$$

Since $D \cdot D = 0$, we have

$$H^0(Y, \mathcal{O}_D(D)) = H^0(Y, \mathcal{O}_D) = H^0(D, \mathcal{O}_D) \cong \mathbb{C}.$$

This implies the assertion. □

Lemma 4.5. *Let* $\widetilde{\Sigma} \in |D|_0$*. Then,*

(1) the restriction of the birational map $Y \to \mathbb{C}P^2$ *establishes a biholomorphism between* $\widetilde{\Sigma}$ *and a cubic curve in* $\mathbb{C}P^2$*;*
(2) the restriction of the birational map (27) *establishes a biholomorphism between* $\widetilde{\Sigma}$ *and a closed holomorphic curve in* X*.*

In particular, by (1) $\widetilde{\Sigma}$ *is of arithmetic genus* 1.

Proof. Under the map $Y \to \mathbb{C}P^2$ the generic fibers of $p : Y \to \mathbb{C}P^1$ get mapped to curves of self-intersection number 1, i.e. to lines ℓ in $\mathbb{C}P^2$ passing through $[0 : 0 : 1]$. Thus the image of a curve $\widetilde{\Sigma}$ is a curve in $\mathbb{C}P^2$ intersecting the generic such line ℓ in two points distinct from $[0 : 0 : 1]$ (corresponding to the intersection points of $\widetilde{\Sigma}$ with the generic fiber of Y). Furthermore it is easy to see that the point $[0 : 0 : 1]$ is a base point of such curves $\widetilde{\Sigma}$, but blowing it up once is sufficient to separate them. In different terms, $\widetilde{\Sigma}$ intersects the generic line ℓ passing through $[0 : 0 : 1]$ in 3 points (counted with multiplicity). By the conditions, no component of $\widetilde{\Sigma}$ gets contracted to a point and moreover no two points of $\widetilde{\Sigma}$ get identified. We infer that the restriction is one to one. This proves part (1).

For part (2), it is sufficient to prove that the centers of the quadratic transformations $\sigma_{i\pm}$ are smooth points of $\sigma(\widetilde{\Sigma})$ and its proper transforms. This immediately follows as $\sigma(\widetilde{\Sigma})$ transversely intersects the fiber of X over q in two distinct points. □

Lemma 4.6. *The map* $Y \to |D|$ *is a fibration.*

Proof. The union of the curves $\widetilde{\Sigma} \in |D|$ is of dimension 2, so it is equal to Y because this latter is irreducible. Since the curves in $|D|$ have zero self-intersection, the pencil is indeed a fibration. □

Lemma 4.7. *The curve* Y_∞ *is the only element of* $|D| \setminus |D|_0$.

Proof. The curve E_{4+} results from the last blow-up, it is a section of the elliptic fibration $Y \to |D|$. Through any point of E_{4+} there passes a unique curve $\widetilde{\Sigma} \in |D|$. Now, Y_∞ is the curve passing through the point $[0:1] \in E_{4+}$. Therefore, any fiber $\widetilde{\Sigma} \in |D| \setminus \{Y_\infty\}$ intersects E_{4+} transversely in a point different from $[0:1] \in E_{4+}$, and is distinct from the fiber Y_∞. This shows that the geometric conditions (a)–(c) listed in Definition 4.1 are fulfilled. □

With the above lemmas at hand, now we are ready to return to the proof of Proposition 4.2. [21, Theorem 4.3] now implies that $\mathcal{M}^{ss}$ is a relative compactified Picard scheme of torsion-free sheaves of relative degree 1 over $Y \setminus Y_\infty \to |D|_0$. This concludes the proof of Proposition 4.2. □

4.2. **Local description of irregular Higgs bundles.** Next we will start identifying the singular fibers of the resulting elliptic fibration. In the untwisted case, the matrices in (23) gave a local form for θ. The matrix A_{-4} will encode the base locus of a pencil and the matrices A_{-3}, A_{-2} and A_{-1} will represent the tangents and the higher order derivatives of the curves of a pencil.

We wrote $\chi_{\vartheta_i}(w_i)$ as the characteristic polynomial of θ in the trivialization given by κ_i and κ_i^2 ($i = 1, 2$), see (18), cf. also Subsection 2.3. The polynomials $\chi_{\vartheta_i}(w_i)$ are the local forms of the spectral curves in X. In concrete terms, using Equations (19), (20), (21) and (22):

$$\chi_{\vartheta_1}(z_1, w_1) = w_1^2 - \left(p_2 z_1^2 + p_1 z_1 + p_0\right) w_1 - \left(q_4 z_1^4 + q_3 z_1^3 + q_2 z_1^2 + q_1 z_1 + q_0\right),$$
$$\chi_{\vartheta_2}(z_2, w_2) = w_2^2 + \left(p_0 z_2^2 + p_1 z_2 + p_2\right) w_2 - \left(q_0 z_2^4 + q_1 z_2^3 + q_2 z_2^2 + q_3 z_2 + q_4\right). \tag{30}$$

The roots of the characteristic polynomial in w_1 have expansions with respect to z_1 near q. The first several terms of the expansion are the same as the diagonal elements of the matrix in (23). More precisely, the series of the "negative" root of $\chi_\vartheta(w_1)$ up to third order is equal to $a_- + b_- z_1 + c_- z_1^2 + \lambda_- z_1^3$ and the "positive" root up to third order is equal to $a_+ + b_+ z_1 + c_+ z_1^2 + \lambda_+ z_1^3$. From these equations we get the following expressions:

$$\begin{aligned} f_1(z_1) &= -\left((c_- + c_+) z_1^2 + (b_- + b_+) z_1 + (a_- + a_+)\right), \\ g_1(z_1) &= -q_4 z_1^4 + (a_-\lambda_+ + a_+\lambda_- + b_+c_- + b_-c_+) z_1^3 + \\ &\quad + (a_+c_- + a_-c_+ + b_-b_+) z_1^2 + (a_+b_- + a_-b_+) z_1 + a_-a_+. \end{aligned}$$

According to the residue theorem (24) we know that $\lambda_+ + \lambda_- = 0$, hence we can eliminate $\lambda_- = -\lambda_+$. It turns out that these equations do not depend on q_4 (the coefficient of g_1 and g_2), thus we set

$$t = q_4.$$

Hence we get a pencil parametrized by t with base locus $(0, a_+)$ and $(0, a_-)$ in $\mathbb{C}^2$:

$$\begin{aligned} \chi_{\vartheta_1}(z_1, w_1, t) =& w_1^2 - \left((c_- + c_+) z_1^2 + (b_- + b_+) z_1 + a_- + a_+\right) w_1 - \\ & - t z_1^4 + ((a_- - a_+) \lambda_+ + b_+c_- + b_-c_+) z_1^3 + \\ & + (a_+c_- + a_-c_+ + b_-b_+) z_1^2 + (a_+b_- + a_-b_+) z_1 + a_-a_+ = 0. \end{aligned}$$

We note that $\chi_{\vartheta_1}(z_1, w_1, t)$ intersects the fiber component of the fiber with multiplicity 4 in two distinct points and every spectral curve is a double section.

If we rewrite the Equation (30), then we get a pencil on the chart U_2:

$$\begin{aligned} \chi_{\vartheta_2}(z_2, w_2, t) =& w_2^2 + f_2(z_2) w_2 + g_2(z_2, t) = \\ =& w_2^2 + \left((a_- + a_+) z_2^2 + (b_- + b_+) z_2 + c_- + c_+\right) w_2 + \\ & + a_-a_+ z_2^4 + (a_+b_- + a_-b_+) z_2^3 + (a_+c_- + a_-c_+ + b_-b_+) z_2^2 + \\ & + ((a_- - a_+) \lambda_+ + b_+c_- + b_-c_+) z_2 - t = 0. \end{aligned}$$

More precisely, the pencil in the Hirzebruch surface X is defined by $\chi_{\vartheta_1}(z_1, w_1, t)$ and the union of the section at infinity with fiber F_0. According to the converse direction of Theorem 3.2, the pencil gives rise to an elliptic fibration in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with a singular fiber of type $\widetilde{E}_7$.

Our goal is to find the other singular fibers in the pencil. For this reason, we will identify the singular points on the spectral curves. The spectral curves intersect the fiber component of the curve C_∞ at infinity (whose fiber component is of multiplicity 4) in two distinct points and according to Condition (c) of Definition 4.1, the pencil has no singular point on the distinguished fiber F_0. Thus it is sufficient to consider the κ_2 trivialization, i. e. the chart (z_2, w_2). For identifying the singular fibers in the pencil, we look for triples (z_2, w_2, t) such that (z_2, w_2) fits the curve with parameter t and the partial derivatives below vanish:

$$\chi_{\vartheta_2}(z_2, w_2, t) = 0, \tag{31a}$$

$$\frac{\partial \chi_{\vartheta_2}(z_2, w_2, t)}{\partial w_2} = 0, \tag{31b}$$

$$\frac{\partial \chi_{\vartheta_2}(z_2, w_2, t)}{\partial z_2} = 0. \tag{31c}$$

These triples are in one-to-one correspondence with singular points of singular fibers. Every spectral curve X_b is a double section of the ruling on the Hirzebruch surface X, thus every triple (z_2, w_2, t) satisfying Equations (31) maps to distinct points under the ruling p. Indeed, if one fiber (with fixed t value) contains two singular points with the same z_2 coordinate then the corresponding fiber of p would intersect X_b with multiplicity higher than two. Furthermore, it cannot happen that two singular points with the same z_2 coordinate lie on distinct fibers (two distinct t values): we will see in Equation (32) that the t values are determined by the z_2 values. Consequently the z_2-values from the triples (z_2, w_2, t) are in one-to-one correspondence with singular points.

Computing the partial derivatives and expressing w_2 from Equation (31a) and t from Equation (31b) by z_2 we get:

$$\begin{aligned} w_2(z_2) &= -\frac{1}{2}\left((a_- + a_+) z_2^2 + (b_- + b_+) z_2 + c_- + c_+\right), \\ t(z_2) &= -\frac{1}{4}\Big((a_- - a_+)^2 z_2^4 + 2(a_- - a_+)(b_- - b_+) z_2^3 + \\ &\quad + \left(2(a_- - a_+)(c_- - c_+) + (b_- - b_+)^2\right) z_2^2 + \\ &\quad + \left(2(b_- - b_+)(c_- - c_+) - 4(a_- - a_+)\lambda_+\right) z_2 + (c_- + c_+)^2\Big). \end{aligned} \tag{32}$$

Substitute the resulting expression into the Equation (31c) and get

$$\begin{aligned} 0 =& 2(a_- - a_+)^2 z_2^3 + 3(a_- - a_+)(b_- - b_+) z_2^2 + \\ &+ \left(2(a_- - a_+)(c_- - c_+) + (b_- - b_+)^2\right) z_2 + 2(a_+ - a_-)\lambda_+ + (b_- - b_+)(c_- - c_+). \end{aligned} \tag{33}$$

The roots of this polynomial correspond to the z_2 values of the singular points in the singular curves on the Hirzebruch surface X, which become fibers on the 8-fold blow up. Since we have a cubic polynomial in Equation (33), generally we get three distinct roots, and this corresponds to the fact that there are at most three singular fibers in the fibration (next to $\widetilde{E}_7$).

The cubic polynomial of (33) with variable z_2 has multiple roots if and only if its discriminant

$$(a_- - a_+)^2\left(\left((b_- - b_+)^2 - 4(a_- - a_+)(c_- - c_+)\right)^3 - 432(a_- - a_+)^4\lambda_+^2\right)$$

vanishes.

With the choice $a_- = a_+$ the configuration reduces to the case of a type $\widetilde{E}_8$ singular fiber (to be treated in Section 6), therefore we can assume that $a_- \neq a_+$. We define

$$\Delta := \left((b_- - b_+)^2 - 4(a_- - a_+)(c_- - c_+)\right)^3 - 432(a_- - a_+)^4 \lambda_+^2.$$

We analyze the cases depending on how many singular points are in the fibration.

4.2.1. *One root.* The cubic in (33) has one root if and only if the discriminant Δ vanishes, and the derivative of (33) with respect to z_2 has one root, hence the discriminant of the latter quadratic equation also vanishes. This means that $\Delta_0 = 0$ with

$$\Delta_0 := 3(a_- - a_+)^2 \left((b_- - b_+)^2 - 4(a_- - a_+)(c_- - c_+)\right).$$

It is easy to see $\Delta = \Delta_0 = 0$ is equivalent to $\Delta = \lambda_\pm = 0$.

When (33) has one root then the pencil has one singular curve in the corresponding chart. By the classification of singular fibers in elliptic fibrations, the unique singular fiber with a single singular point besides an $\widetilde{E}_7$-fiber must be of type III.

Conversely, if the fibration has a type III fiber, then the pencil has no other singular point. This requires that the cubic in (33) has only one root, which is equivalent to $\Delta = \lambda_\pm = 0$. Hence we get part (1) of Theorem 1.1.

4.2.2. *Two roots.* At the same time we verified part (2) as well, since $\Delta = 0$ and $\Delta_0 \neq 0$ is equivalent to $\Delta = 0$ and $\lambda_\pm \neq 0$, furthermore Equation (33) has two distinct roots. By the classification the only possibility is the fibration has fibers of types II and I_1.

4.2.3. *Three roots.* Now, we consider the $\Delta \neq 0$ case, where the cubic in (33) has three distinct roots and the fibration has three singularities.

Lemma 4.8. $\Delta \neq 0$ *and* $\lambda_\pm = 0$ *holds if and only if the pencil has* I_2 *and* I_1 *singularities.*

Proof. The first direction is simple, because the equation of (33) can be easily solved and the three z_2 values can be substituted into Equation (32). We get two distinct t values, which correspond to one curve with two singularities and another curve with one singularity. The three z_2 values are:

$$(z_2)_1 = \frac{b_+ - b_-}{2(a_- - a_+)},$$
$$(z_2)_{2,3} = \frac{b_+ - b_- \pm \sqrt{(b_- - b_+)^2 - 4(a_- - a_+)(c_- - c_+)}}{2(a_- - a_+)}.$$

The two t values are:

$$t_1 = -\frac{1}{64(a_- - a_+)^2}\Big(16(a_- - a_+)^2 c_-^2 + 8(a_- - a_+)c_- \cdot$$
$$\cdot \left(4(a_- - a_+)c_+ - (b_- - b_+)^2\right) + \left((b_- - b_+)^2 + 4(a_- - a_+)c_+\right)^2\Big),$$
$$t_{2,3} = -c_- c_+.$$

Let us see now the converse direction. If the pencil contains an I_2 and an I_1 curve then the equation of (33) has three distinct roots. Let us denote these roots by y_1, y_2, y_3. Denote the value of t by t_i after the substitution of z_2 with y_i in Equation (32). Two roots (say y_1 and y_2) provide singularities on the same curve, that is, $t_1 = t_2$. Equivalently

$$0 = t_1 - t_2 = 4(a_- - a_+)\lambda_+ - \left((a_- - a_+)(y_1 + y_2) + b_- - b_+\right) \cdot$$
$$\cdot \left((a_- - a_+)\left(y_1^2 + y_2^2\right) + (b_- - b_+)(y_1 + y_2) + 2(c_- - c_+)\right),$$

where we simplify with $\frac{1}{4}(y_1 - y_2)$.

Obviously, the three distinct roots provide two values for t if and only if

$$0 = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1).$$

This expression is a symmetric polynomial in y_1, y_2, y_3, hence can be written as a polynomial of the elementary symmetric polynomials $\sigma_1 = y_1 + y_2 + y_3$, $\sigma_2 = y_1 y_2 + y_2 y_3 + y_3 y_1$ and $\sigma_3 = y_1 y_2 y_3$. The above vanishing condition would yield a long expression, but $\sigma_1, \sigma_2, \sigma_3$ can be determined from the coefficient of equation (33) by Vieta's formulas. The relations between the symmetric polynomials and the coefficients then provide

$$\begin{aligned} \sigma_1 &= -\frac{3\,(b_- - b_+)}{2\,(a_- - a_+)}, \\ \sigma_2 &= \frac{2\,(a_- - a_+)\,(c_- - c_+) + (b_- - b_+)^2}{2\,(a_- - a_+)^2}, \\ \sigma_3 &= \frac{2\,(a_- - a_+)\,\lambda_+ - (b_- - b_+)\,(c_- - c_+)}{2\,(a_- - a_+)^2}. \end{aligned}$$

After simplifications, we get a condition for fibration to contain an I_2 curve:

$$\frac{\lambda_+ \left(\left((b_- - b_+)^2 - 4\,(a_- - a_+)\,(c_- - c_+) \right)^3 - 432\,(a_- - a_+)^4 \lambda_+^2 \right)}{16\,(a_- - a_+)} = 0. \tag{34}$$

Now Δ is in the nominator, and λ_+ is a multiplication factor, hence Equation (34) becomes the following:

$$-\frac{\Delta \lambda_+}{16\,(a_- - a_+)} = 0.$$

Since the pencil has three singularities, we have that $\Delta \neq 0$, consequently $\lambda_+ = 0$ concluding the proof of Lemma 4.8. □

Remark 4.9. *Note that according to Proposition 3.3 the fibration has the fiber of type III or $I_2 + I_1$ if and only if the pencil contains a double section which is the union of two sections, and this last condition is easily seen to be equivalent to $\lambda_\pm = 0$.*

We need to examine the last case when $\Delta \neq 0$ and $\lambda_\pm \neq 0$. By process of elimination there is a single possibility for the singular fibers: there are three I_1 fibers in the fibration.

In summary, so far we have identified all possible singular curves in the pencil on X. We summarize the cases in Table 1.

	$\lambda_+ = 0$	$\lambda_+ \neq 0$
$\Delta = 0$	III	$II + I_1$
$\Delta \neq 0$	$I_2 + I_1$	$3I_1$

TABLE 1. The type of singular curves in untwisted case

By Lemma 4.5 the same classification applies for the fibers of $p \circ \sigma: Y \to \mathbb{C}P^1$. The fibration obtained from the pencil has a section (actually, even two sections). Let $|D|_0^{sm}$ be the subset of $|D|_0$ parametrizing smooth curves. The relative Abel–Jacobi map gives an algebraic isomorphism between the restriction of the fibration to $|D|_0^{sm}$ and its relative Picard scheme. Therefore, in order to conclude the proof of Theorem 1.1, one merely needs to study torsion-free sheaves on the singular fibers of H. In the cases of curves of types I_1, II this was carried out in Proposition 2.1. For curves of types I_2 and III, the analysis is carried out in Section 5.

5. STABILITY ANALYSIS IN THE UNTWISTED CASE

In cases (2) and (4) of Theorem 1.1 the singular fibers of the elliptic pencil (except for the type $\widetilde{E}_7$ fiber at infinity) are integral (i.e. irreducible and reduced), so the Hitchin fiber of the moduli space corresponding to the singular fibers is just the usual compactified Picard scheme of degree δ. In the other cases however we need to determine the Hitchin fibers of $\mathcal{M}$ corresponding to the reducible singular fibers of the pencil.

5.1. **Stability analysis in the case $\widetilde{E}_7 + I_2 + I_1$.** We use the results and notations of Subsection 2.2.1. We let $b \in B$ denote the point whose preimage in the pencil is the singular fiber of type I_2. We assume that

$$\mathcal{E} = p_*(\mathcal{S})$$

for some torsion-free sheaf $\mathcal{S}$ of $\mathcal{O}_{X_b}$-modules of rank 1 and use the definitions (6). By assumption we have

$$0 = \text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \alpha_0^+ + \alpha_0^-. \tag{35}$$

Then in view of (11) and (26) the above formula may be rewritten as

$$0 = (\delta_+ + \alpha_0^+) + (\delta_- + \alpha_0^-) - |J(\mathcal{S})|. \tag{36}$$

For any non-trivial Higgs subbundle $(\mathcal{F}, \theta|_{\mathcal{F}})$ of $(\mathcal{E}, \theta)$ the scheme

$$(\theta|_{\mathcal{F}} - \lambda) \subset X$$

is a sub-scheme of X_b that is a one-to-one cover of $\mathbb{C}P^1$. Clearly the same also holds for non-trivial quotient Higgs bundles $(\mathcal{Q}, \overline{\theta})$. On the other hand, for any $i \in \{\pm\}$ the functor p_* applied to the morphism (7) gives rise to a quotient Higgs bundle $(\mathcal{Q}_i, \overline{\theta})$. Again by (26) the degree of this quotient is given by δ_i so its parabolic degree is

$$\delta_i + \alpha_0^i.$$

It is easy to see that these are the only quotient Higgs bundles of $(\mathcal{E}, \theta)$, because the support of the spectral sheaf of any such quotient is a component of X_b, and there are exactly two such components. We infer that $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-stable if and only if the two inequalities

$$\delta_i + \alpha_0^i > 0$$

for $i \in \{\pm\}$ hold. Taking into account the formula (36) these inequalities are also equivalent to

$$0 < \delta_i + \alpha_0^i < |J(\mathcal{S})| \tag{37}$$

for $i \in \{\pm\}$. Let us point out that this can only have a solution if $|J(\mathcal{S})| \in \{1, 2\}$. Now, setting

$$\phi_i = 1 - \alpha_0^i,$$

we see that the stability condition (37) transforms into (13), which is the Oda–Seshadri stability condition for the values (ϕ_+, ϕ_-). (Notice however that the equality

$$\phi_- + \phi_+ = 0$$

holds if and only if

$$\alpha_0^+ + \alpha_0^- = 2,$$

which is incompatible with our assumption that $0 \leq \alpha_0^{\pm} < 1$.)

Let us explicitly write down the corresponding Hitchin fibers. For simplicity let us set

$$\alpha^i = \alpha_0^i.$$

Since

$$\alpha^+ + \alpha^- = -\deg(\mathcal{E})$$

is an integer and $\alpha^+, \alpha^- \in [0, 1)$, it follows that

- either we have $\deg(\mathcal{E}) = -1$ and

$$\alpha^+ + \alpha^- = 1 \tag{38}$$

- or we have $\deg(\mathcal{E}) = 0$ and

$$\alpha^+ = 0 = \alpha^-. \tag{39}$$

5.1.1. *Case of degree* -1. Assume that $d = \deg(\mathcal{E}) = -1$. By virtue of (26) this amounts to $\delta = \deg(\mathcal{S}) = 1$. Let us first study the sheaves with $|J(\mathcal{S})| = 2$, i.e. invertible sheaves on X_b. Assumptions (38) and $\alpha_i \in [0,1)$ imply that $\alpha_i \in (0,1)$, therefore, by condition (37) we have either

$$\delta_+ = 0, \delta_- = 1$$

or

$$\delta_+ = 1, \delta_- = 0.$$

Let us introduce the notation

$$\mathcal{L}_i = \mathcal{L}(\mathcal{S})|_{X_i}$$

and fix one of the two conditions on the degrees spelled out above. Then, as $X_\pm$ are rational curves, the isomorphism class of $\mathcal{L}_\pm$ is completely determined. Moreover, according to (8), $\mathcal{S}$ is obtained by first identifying the fibers $(\mathcal{L}_+)_0$ and $(\mathcal{L}_-)_0$ by an isomorphism, then identifying the fibers $(\mathcal{L}_+)_\infty$ and $(\mathcal{L}_-)_\infty$ by an isomorphism. The possible identifications between these pairs of lines are parametrized by $\mathbb{C}^\times \times \mathbb{C}^\times$. Indeed, for trivializations σ_i over open affine subsets of X_i, we have

$$\sigma_+(0) = \lambda_0 \sigma_-(0), \quad \sigma_+(\infty) = \lambda_\infty \sigma_-(\infty)$$

for some

$$(\lambda_0, \lambda_\infty) \in \mathbb{C}^\times \times \mathbb{C}^\times \subset \mathbb{C}^2.$$

However, we may act on this space of identifications by constant automorphisms of one of the bundles $\mathcal{L}_i$ (say $\mathcal{L}_+$) without changing the isomorphism class of the sheaf $\mathcal{S}$ obtained by the identifications. Constant automorphisms are isomorphic to $\mathbb{C}^\times$ and $t \in \mathbb{C}^\times$ obviously acts by

$$t(\lambda_0, \lambda_\infty) = (t\lambda_0, t\lambda_\infty).$$

Therefore, we are left with a parameter space

$$\mathrm{Pic}^{\delta_+,\delta_-} = \mathbb{C}^\times \times \mathbb{C}^\times / \mathbb{C}^\times = \mathbb{C}^\times \subset \mathbb{C}P^1$$

for such invertible sheaves. It is easy to see that these sheaves are all non-isomorphic. This implies that the universal line bundle on X_b of bidegree (δ_+, δ_-) is given by

$$L^{\delta_+,\delta_-}(\cdot) \to \mathrm{Pic}^{\delta_+,\delta_-} \times X_b = \mathbb{C}^\times \times X_b.$$

Now let us consider the case of sheaves $\mathcal{S}$ with $|J(\mathcal{S})| = 1$. These sheaves are locally free in a neighborhood of exactly one of the two points $\{0, \infty\}$. Clearly, if $\mathcal{S}$ is locally free near 0 and not locally free near ∞ then $\mathcal{S}$ cannot be isomorphic to a sheaf $\mathcal{S}'$ that is locally free near ∞ and not locally free near 0. Thus there exist at least 2 points in

$$\overline{\mathrm{Pic}}^{\delta,\phi}_{X_b} \setminus (\mathrm{Pic}^{0,1} \cup \mathrm{Pic}^{1,0}).$$

Our aim is to show that there exist exactly 2 points in this complement. Indeed, we first observe that if $|J(\mathcal{S})| = 1$ then (37) only allows for

$$\delta_+ = 0 = \delta_-.$$

As $X_\pm$ are rational curves, the isomorphism class of line bundles of degree 0 on $X_\pm$ is unique, they are given by $\mathcal{L}_\pm = \mathcal{O}_{X_\pm}$. Now, assume that $\mathcal{S}$ is locally free near 0. Then $\mathcal{S}$ is obtained by identifying the fibers $(\mathcal{L}_+)_0$ and $(\mathcal{L}_-)_0$ by a linear isomorphism. The choices for such an isomorphism are parametrized by $\mathbb{C}^\times$. However, we again get isomorphic sheaves if we apply a constant automorphism to one of $\mathcal{L}_\pm$. It follows that there exists a single stable sheaf $\mathcal{S}_0$ that is locally free near 0 but not locally free near ∞. Similarly, there exists a unique stable sheaf $\mathcal{S}_\infty$ that is locally free near ∞ but not locally free near 0.

Finally, we show that both $\mathcal{S}_0$ and $\mathcal{S}_\infty$ are in the closure of both $\mathrm{Pic}^{0,1}$ and $\mathrm{Pic}^{1,0}$ in $\overline{\mathrm{Pic}}^{\delta,\phi}_{X_b}$. The argument closely follows the one in the proof of Proposition 2.1. Let us for instance work in the chart $\lambda_\infty = 1$ of $\mathbb{C}P^1$, and fix one of the two conditions on the degrees spelled out above, say $(1,0)$. We will consider the limit $L^{1,0}(0)$ of the line bundles

$L^{1,0}(\lambda)$ as $\lambda = \lambda_0 \to 0$. Let us denote the two preimages of $0 \in X_b$ by $0_+ \in X_+, 0_- \in X_-$ respectively. For $\lambda_0 = 0$ we get

$$\sigma_+(0_+) = 0 \cdot \sigma_-(0_-),$$

hence

$$\mathcal{T}or^{\mathcal{O}_{X_+,0_+}}(\pi^* L^{1,0}(0)) \cong \mathbb{C}_{0_+}$$

is generated by $\sigma_-(0_-)$, and

$$\mathcal{T}or^{\mathcal{O}_{X_+,0_+}}(\pi^* L^{1,0}(0)) \cong 0.$$

At the points $\infty_\pm$, $L^{1,0}(0)$ is locally free. We infer that the line bundle $\mathcal{L}_+(0)$ of (5) over X_+ associated to $L^{1,0}(0)$ fits into the short exact sequence

$$0 \to \mathcal{L}_+(0) \to \mathcal{O}_{X_+}(1) \to \mathbb{C}_{0_+} \to 0,$$

and that $\mathcal{L}_-(0) = \mathcal{O}_{X_-}$; in other words, these line bundles are both of degree 0. As we have already shown, $\mathcal{S}_\infty$ is up to isomorphism the unique sheaf of bidegree $(0,0)$ which is locally free near ∞ but not locally free near 0. We infer that

$$L^{1,0}(0) = \mathcal{S}_\infty.$$

A similar argument for $L^{0,1}$ over the affine chart $\lambda_\infty = 1$ now shows that the limit of $L^{0,1}(\lambda)$ as $\lambda \to 0$ is a sheaf of bidegree $(-1,1)$, locally free near ∞ but not locally free near 0. Let us denote by X_0 the partial normalization of X_b at the point $0 \in X_b$. By the uniqueness of $\mathcal{S}_\infty$ we see that

$$L^{0,1}(0) \cong \mathcal{S}_\infty \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-\{0_+\} + \{0_-\}).$$

However, as the arithmetic genus of X_0 is 0, the latter sheaf is trivial. Hence, $L^{0,1}(0)$ is also isomorphic to $\mathcal{S}_\infty$.

The case of $\mathcal{S}_0$ can then be obtained by exchanging the roles of 0 and ∞.

We infer from the discussion above that the moduli space has the structure of an elliptic fibration near the point $b \in B$ corresponding to the singular fiber. Furthermore, it is easy to check (using the fact that the parabolic weights are non-zero) that in this case semi-stability is equivalent to stability. Therefore, by [5] the moduli space is complete. It then follows that the fiber of the Hitchin map H over b is either a smooth elliptic curve or one of the singular fibers on Kodaira's list. As we have shown above, this fiber is homeomorphic to two copies of $\mathbb{C}P^1$ attached at two different points. In particular, the fiber is singular, and as the only fiber on Kodaira's list homeomorphic to two copies of $\mathbb{C}P^1$ attached at two points is I_2, we conclude that $H^{-1}(b)$ is a type I_2 curve.

5.1.2. *Case of degree* 0. The analysis is similar to the case of degree -1, hence we only give the outline. In the case $|J(\mathcal{S})| = 2$ of invertible sheaves, we obtain

$$\delta_+ = 1 = \delta_-,$$

and if $|J(\mathcal{S})| = 1$ then no (δ_+, δ_-) solves (37). We infer that stable sheaves are parametrized by $\mathbb{C}^\times$. Let us now consider strictly semi-stable sheaves. Then, the solutions in the case $|J(\mathcal{S})| = 2$ are

$$\delta_+ \in \{0, 1, 2\},$$

with $\delta_- = 2 - \delta_+$. The parameter space consists of 3 copies of $\mathbb{C}^\times$. The solutions (δ_+, δ_-) with $|J(\mathcal{S})| = 1$ are

$$(0,1), \quad (1,0),$$

each being parametrized by a point. The point corresponding to bidegree $(0,1)$ is both a limit point of the $\mathbb{C}^\times$ parametrizing invertible sheaves of bidegree $(0,2)$ and the one parametrizing invertible sheaves of bidegree $(1,1)$. Similarly, the point corresponding to bidegree $(1,0)$ is both a limit point of the $\mathbb{C}^\times$ parametrizing invertible sheaves of bidegree $(2,0)$ and the one parametrizing invertible sheaves of bidegree $(1,1)$. All the semi-stable solutions are parametrized by a copy of $\mathbb{C}P^1$ with two copies of $\mathbb{C}$ attached to it at two different points of $\mathbb{C}P^1$. In contrast with the case of degree -1, this time there do exist strictly semi-stable Higgs bundles, and in addition the parabolic weights are not all distinct. Hence, we cannot use a completeness argument to determine the algebraic type of the singular fiber.

5.2. **Stability analysis in the case $\widetilde{E}_7 + III$.** We now let $b \in B$ be the point whose preimage in the pencil is the singular fiber of type III. We again have (35).

We assume that

$$\mathcal{E} = p_*(\mathcal{S})$$

and use the definitions of (6). The curve X_b has a single singular point x which is a tacnode (an A_3-singularity). It is known that there exists a fractional ideal

$$\mathcal{O}_{X_b,x} \subseteq I \subseteq \mathcal{O}_{\tilde{X}_b,x}$$

of $\mathcal{O}_{X_b,x}$ such that

$$\mathcal{S}_x \cong I.$$

The length of $\mathcal{S}$ at x is by definition

$$l(\mathcal{S}) = \dim_{\mathbb{C}}(I/\mathcal{O}_{X_b,x}),$$

and we have the inequalities

$$0 \le l(\mathcal{S}) \le \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{X}_b,x}/\mathcal{O}_{X_b,x}) = 2.$$

Now there exists a short exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{L}(\mathcal{S})|_{X_+} \oplus \mathcal{L}(\mathcal{S})|_{X_-} \to \mathbb{C}_x^{2-l(\mathcal{S})} \to 0,$$

hence

$$\chi(\mathcal{S}) + 2 - l(\mathcal{S}) = \chi(\mathcal{L}(\mathcal{S})|_{X_+}) + \chi(\mathcal{L}(\mathcal{S})|_{X_-}).$$

Applying this to $\mathcal{O}_{X_b}$ in the place of $\mathcal{S}$ we get

$$\chi(\mathcal{O}_{X_b}) + 2 = \chi(\mathcal{O}_{X_+}) + \chi(\mathcal{O}_{X_-}).$$

Subtracting the second formula from the first we infer

$$\delta - l(\mathcal{S}) = \delta_+ + \delta_-,$$

with $\delta, \delta_+, \delta_-$ the degrees of $\mathcal{S}, \mathcal{L}(\mathcal{S})|_{X_+}$ and $\mathcal{L}(\mathcal{S})|_{X_-}$, respectively. Using this formula and (26) we can rewrite (35) as

$$0 = \delta_+ + \delta_- + l(\mathcal{S}) - 2 + \alpha^+ + \alpha^-. \tag{40}$$

The canonical morphisms (7) give quotient irregular parabolic Higgs bundles $\mathcal{E}_i$ of $\mathcal{E}$ of rank 1 and degree

$$d_i = \delta_i$$

for $i \in \{\pm\}$. Furthermore, these are again the only non-trivial Higgs quotient bundles of $\mathcal{E}$. The parabolic weight associated to $\mathcal{E}_i$ is α^i, so the parabolic degree of $\mathcal{E}_i$ is

$$\text{par-deg}(\mathcal{E}_i) = \delta_i + \alpha^i.$$

It follows that the parabolic stability of $(\mathcal{E}, \theta)$ is equivalent to the inequalities

$$0 < \delta_i + \alpha^i$$

for $i \in \{\pm\}$. Taking (40) into account, this is equivalent to

$$\delta_+ + \alpha^+ + 2l(\mathcal{S}) - 2 < \delta_- + \alpha^- + l(\mathcal{S}) < \delta_+ + \alpha^+ + 2. \tag{41}$$

This time this inequality immediately implies that there exist no stable Higgs bundles with spectral sheaf $\mathcal{S}$ of length 2.

We again set

$$\alpha^i = \alpha_0^i$$

and we need to distinguish two cases:

- either we have $\deg(\mathcal{E}) = -1$ and

$$\alpha^+ + \alpha^- = 1 \tag{42}$$

- or we have $\deg(\mathcal{E}) = 0$ and

$$\alpha^+ = 0 = \alpha^-. \tag{43}$$

5.2.1. *Case of degree* -1. Let us first treat the case of (42). Assume first $l(\mathcal{S}) = 0$, i.e. $\mathcal{S}$ is an invertible sheaf on X_b. Then, independently of the values of $\alpha^\pm$ satisfying (42), condition (41) implies either

$$\delta_+ = 0, \delta_- = 1$$

or

$$\delta_+ = 1, \delta_- = 0.$$

Therefore, such sheaves are parametrized by $\mathbb{C} \coprod \mathbb{C}$, as it readily follows from the long exact sequence associated to

$$0 \to \mathcal{O}_{X_b} \to \mathcal{O}_{\tilde{X}_b} \to \mathcal{O}_{\tilde{X}_b,x}/\mathcal{O}_{X_b,x} \to 0$$

using the fact that $\tilde{X}_b$ has two connected components.

If, on the other hand, we have $l(\mathcal{S}) = 1$ then the only solution is

$$\delta_+ = 0 = \delta_-,$$

again independently of the values of $\alpha^\pm$. This latter sheaf is in the closure of both components $\mathbb{C}$ parametrizing invertible sheaves. We infer that up to homeomorphism, the Hitchin fiber over the point b is parametrized by two copies of $\mathbb{C}P^1$ attached at one point. As the generic fiber of the Hitchin-fibration is an elliptic curve and the moduli space is complete by [5], the fiber over b must be again one of the fibers of Kodaira's list. However, the only singular fiber on the list that is homeomorphic to two copies of $\mathbb{C}P^1$ glued at one point is the fiber of type III. Therefore, the Hitchin fiber $H^{-1}(b)$ is a singular curve of type III.

5.2.2. *Case of degree* 0. Let us now study the case of (43): in this case, by virtue of (40) we have

$$2 - l(\mathcal{S}) = \delta_+ + \delta_-.$$

If $l(\mathcal{S}) = 0$ then we readily see that the only solution to equation (41) is

$$\delta_+ = 1 = \delta_-,$$

and just as above one can show that such sheaves are parametrized by $\mathbb{C}$.

On the other hand, if $l(\mathcal{S}) = 1$ then (41) has no solutions; however, if we relax the inequalities in (41) to not necessarily strict ones, then there exist two solutions:

$$\delta_+ = 0, \delta_- = 1 \tag{44}$$

and

$$\delta_+ = 1, \delta_- = 0. \tag{45}$$

The sheaves with these properties are parametrized by one point in each of the two cases.

Let us first analyze the case (44): in this case, the destabilizing quotient of $(\mathcal{E}, \theta)$ is $\mathcal{E}_+$: indeed, we have

$$0 = \delta_+ = \deg(\mathcal{E}_+) = \text{par-deg}(\mathcal{E}_+) = \deg(\mathcal{E}) = \text{par-deg}(\mathcal{E}),$$

since the parabolic weights vanish. The destabilizing Higgs subbundle of $\mathcal{E}$ is

$$\ker(\mathcal{E} \to \mathcal{E}_+),$$

which is a lower elementary transformation of $\mathcal{E}_-$:

$$\ker(\mathcal{E} \to \mathcal{E}_+) = \mathcal{E}_-(-\{t\}),$$

where $t \in \mathbb{C}P^1$ is the image under p of the singular point of X_b. Indeed, we have

$$\deg(\mathcal{E}_-(-\{t\})) = \deg(\mathcal{E}_-) - 1 = 0, \tag{46}$$

and $\mathcal{E}_-(-\{t\})$ is preserved by θ simply because the image by θ of vanishing sections of $\mathcal{E}$ at t also vanish at t, in particular, they belong to $\mathcal{E}_-$. The Jordan–Hölder filtration of $(\mathcal{E},\theta)$ is therefore given by

$$\mathcal{E}_-(-\{t\}) \subset \mathcal{E},$$

with associated graded

$$\mathcal{E}_-(-\{t\}) \oplus \mathcal{E}_+$$

endowed with the action

$$\begin{pmatrix} \theta_- & 0 \\ 0 & \theta_+ \end{pmatrix}, \tag{47}$$

where $\theta_\pm$ are the morphisms induced by θ on the two direct summands. According to (46), the vector bundle underlying this graded Higgs bundle is isomorphic to the trivial bundle of rank 2 over $\mathbb{C}P^1$. Moreover, the action of $\theta_\pm$ in the above matrix clearly has spectral curve $X_\pm$ respectively.

The case of (44) can be treated in a very similar manner, except that one needs to exchange the roles of $\mathcal{E}_+$ and $\mathcal{E}_-$. It then follows that the destabilizing Higgs subbundle of $\mathcal{E}$ is

$$\mathcal{E}_+(-\{t\}),$$

and that the graded Higgs bundle associated to the Jordan–Hölder filtration is

$$\mathcal{E}_- \oplus \mathcal{E}_+(-\{t\}),$$

the trivial vector bundle of rank 2 over $\mathbb{C}P^1$, with Higgs field given by the formula (47).

The upshot is that in both cases (44) and (45), the associated graded Higgs bundles for the Jordan–Hölder filtration have isomorphic underlying vector bundles, and the Higgs-field splits as a direct sum. Moreover, the spectral curves of θ_+ are equal in both cases, and the same holds for θ_-. We infer that the associated graded Higgs bundles of the Higgs bundles coming from spectral sheaves satisfying (44) and (45) are isomorphic. Said differently, the Higgs bundles associated to (44) and (45) are S-equivalent, therefore they are represented by the same point in $\mathcal{M}$.

To sum up, in the degree 0 case the Hitchin fiber over the point b is homeomorphic to the compactification of $\mathbb{C}$ (corresponding to invertible sheaves) by a unique point (corresponding to sheaves of length 1). However, the parabolic weights are equal and there exist strictly semi-stable Higgs bundles, so we cannot use a completeness argument to determine algebraically the special fiber of the Hitchin map.

5.3. The proof of Theorem 1.1.

Now we are in a position of proving our first main result.

Proof of Theorem 1.1. The polar part of an irregular Higgs bundle depends on the parameters listed in (U). The case-analysis in Subsection 4.2 describes all possible singular fibers in the Hirzebruch surface X. The blow-up procedure in Subsection 4.1 and Lemma 4.5 provide a biholomorphism between X and the constructed rational surface Y. Proposition 4.2 guarantees the existence of an elliptic fibration on Y with an $\widetilde{E}_7$ singular fiber and describes the moduli space $\mathcal{M}^{ss}$. Finally, Proposition 2.1 and the analysis in Section 5 identify the Hitchin fibers in $\mathcal{M}^s$ and hence verify Theorem 1.1. □

6. The twisted case

In this section we determine a certain blow-up Y of $\widetilde{X}$ depending on the parameters appearing in (T) with the property that certain sheaves on Y are in one-to-one correspondence with Higgs bundles of the local form (25). We need two preliminary lemmas.

Lemma 6.1. *Let θ be a Higgs field of the local form (25). Let us denote by ζdz the eigenvalues of θ; ζ is a ramified bi-valued meromorphic function of z_1.*

(1) *Assume that* $b_{-7} \neq 0$. *Then, for* $-8 \leq n \leq -3$ *the coefficients of the Puiseux expansion*

$$\zeta = \sum_{n=-8}^{\infty} a_n z_1^{\frac{n}{2}}. \tag{48}$$

admit expressions

$$a_n = a_n(b_{-8}, \sqrt{b_{-7}}, b_{-6}, \dots, b_n) \in \mathbb{C}[b_{-8}, b_{-7}^{\pm 1/2}, b_{-6}, \dots, b_n]$$

in the parameters b_n, *and* $a_{-7} \neq 0$.

(2) *Vice versa, if* θ *is of the local form (25) and* $a_{-7} \neq 0$ *then the parameters* $b_{-8}, \dots, b_{-3}$ *admit polynomial expressions*

$$b_n = b_n(a_{-8}, \dots, a_n) \in \mathbb{C}[a_{-8}, \dots, a_n]$$

in function of the Puiseux coefficients of ζ, *and* $b_{-7} \neq 0$.

Proof. This is a straightforward computation. Specifically, we have

$$a_{-8} = b_{-8},$$

for $n \in \{-6, -4\}$ we have

$$a_n = \frac{b_n}{2},$$

and the coefficients with odd indices are given by

$$\begin{aligned} a_{-7} &= \sqrt{b_{-7}}, \\ a_{-5} &= \frac{1}{8\sqrt{b_{-7}}}(b_{-6}^2 + 4b_{-5}), \\ a_{-3} &= \frac{1}{8\sqrt{b_{-7}}}(2b_{-4}b_{-6} + 4b_{-3}) - \frac{1}{128 b_{-7}\sqrt{b_{-7}}}(b_{-6}^2 + 4b_{-5})^2, \end{aligned}$$

(the square root of b_{-7} depending on the choice of square root of z in the Puiseux series). The inverse transformations are given by

$$\begin{aligned} b_{-7} &= a_{-7}^2 \\ b_{-5} &= 2a_{-5}a_{-7} - a_{-6}^2 \\ b_{-3} &= 2a_{-3}a_{-7} - 2a_{-4}a_{-6} + a_{-5}^2. \end{aligned}$$

□

In the lemma below we follow the conventions and notations introduced in Sections 2 and 4. In particular, in view of the definition of the affine coordinate system (z_1, w_1) near $p^{-1}(q) \setminus C^\infty$ and Lemma 6.1, the equation of the spectral curve of a Higgs field of the local form (25) reads as

$$w_1 = \sum_{n=0}^{\infty} a_{n-8} z_1^{\frac{n}{2}}.$$

Lemma 6.2. *Assume the above Puiseux expansion holds.*

(1) *If* $a_{-7} \neq 0$, *then for* $2 \leq n \leq 6$ *there exist polynomials*

$$d_n = d_n(a_{-7}, \dots, a_{n-9}) \in \mathbb{C}[a_{-7}^{\pm 1}, a_{-6}, \dots, a_{n-9}]$$

such that we have the Taylor series

$$z_1 = d_2(w_1 - a_{-8})^2 + \cdots + d_6(w_1 - a_{-8})^6 + O((w_1 - a_{-8})^7). \tag{49}$$

Moreover, $d_2 \neq 0$.

(2) *Conversely, the value* a_{n-9} *is a polynomial in* $d_2^{\pm 1/2}, d_3, \dots, d_n$, *and* $a_{-7} \neq 0$.

Proof. By assumption we have

$$\frac{w_1 - a_{-8}}{a_{-7}} = \sum_{n=1}^{\infty} \frac{a_{n-8}}{a_{-7}} z_1^{\frac{n}{2}} = z_1^{\frac{1}{2}} + O(z_1).$$

By formally inverting this series and then squaring the result we obtain the first claim. In concrete terms we find

$$\begin{aligned} d_1 &= 0 \\ d_2 &= \frac{1}{a_{-7}^2} \\ d_3 &= -2\frac{a_{-6}}{a_{-7}^4} \\ d_4 &= \frac{5a_{-6}^2 - 2a_{-5}a_{-7}}{a_{-7}^6} \\ d_5 &= \frac{-14a_{-6}^3 + 12a_{-5}a_{-6}a_{-7} - 2a_{-4}a_{-7}^2}{a_{-7}^8} \\ d_6 &= \frac{42a_{-6}^4 - 56a_{-7}a_{-5}a_{-6}^2 + 14a_{-7}^2a_{-4}a_{-6} + 7a_{-7}^2a_{-5}^2 - 2a_{-7}^3a_{-3}}{a_{-7}^{10}}. \end{aligned}$$

The converse statement follows directly. □

Lemma 6.3. *Assume that Lemmas 6.1 and 6.2 hold.*

(1) *If $b_{-7} \neq 0$, then for $2 \leq n \leq 6$ there exist polynomials*

$$d_n = d_n(b_{-7}, \dots, b_{n-9}) \in \mathbb{C}[b_{-7}^{\pm 1}, b_{-6}, \dots, b_{n-9}]$$

such that we have the Taylor series

$$z_1 = d_2(w_1 - b_{-8})^2 + \dots + d_6(w_1 - b_{-8})^6 + O((w_1 - b_{-8})^7).$$

Moreover, $d_2 \neq 0$.

(2) *Conversely, the value b_{n-9} is a polynomial in $d_2^{\pm 1}, d_3, \dots, d_n$, and $b_{-7} \neq 0$.*

Proof. The lemma directly follows from the previous two lemmas. □

We now proceed to construct the surface Y with a birational morphism to $\widetilde{X}$ whose geometry governs $\mathcal{M}$. The idea is similar to the untwisted case: we use the above expansions to recursively find the point on the exceptional divisor that we blow up in the following step. We assume that $\sigma_1 : \widetilde{X} \to X$ is the blow-up of X in the point

$$[a_{-8}\kappa_1(0) : \mathbf{1}]. \tag{50}$$

Let $E_1 \subset \widetilde{X}$ denote the corresponding exceptional divisor, see Figure 2. Observe that the coordinates $[z_1' : w_1']$ on E_1 now satisfy

$$\frac{z_1'}{w_1'} = \frac{z_1}{w_1 - a_{-8}},$$

so on a curve $\widetilde{\Sigma}$, having the expansion of (49), we have

$$\frac{z_1'}{w_1'} = \sum_{n=2}^{\infty} d_n (w_1 - a_{-8})^{n-1}.$$

We define

$$\sigma_2 : X_2 \to \widetilde{X}$$

as the blow-up of the point

$$[z_1' : w_1'] = [0 : 1] \in E_1.$$

In concrete terms, on the affine chart $V_1 = \{w_1' \neq 0\} \subset \widetilde{X}$ we normalize $w_1' = 1$ and in the affine coordinates (z_1', w_1) on V_1 we consider

$$\{(z_1', w_1, [z_1'' : w_1'']) \in V \times \mathbb{C}P^1 | \quad w_1'' z_1' - z_1''(w_1 - a_{-8}) = 0\}.$$

(Observe that we have met the exceptional divisor of σ_2 in Figure 2 under the name E_4.) With these definitions, over $V_2 = \{w_1'' \neq 0\} \subset X_2$ on a curve $\widetilde{\Sigma}$ having the expansion of (49) we have

$$\begin{aligned} \frac{z_1''}{w_1''} &= \frac{z_1'}{w_1 - a_{-8}} \\ &= \frac{z_1 w_1'}{(w_1 - a_{-8})^2} \\ &= \frac{z_1}{(w_1 - a_{-8})^2} \\ &= \sum_{n=2}^{\infty} d_n (w_1 - a_{-8})^{n-2} \end{aligned}$$

(recall we have set $w_1' = 1$).

From this point on, the pattern of the construction of Y is clear and similar to the construction in the untwisted case. Namely, for $3 \leq n \leq 8$ we successively consider the blow-up

$$\sigma_n : X_n \to X_{n-1}$$

of the point

$$[z_1^{(n-1)} : w_1^{(n-1)}] = [d_n : 1] \in E_{n+1}$$

and denote by E_{n+2} the exceptional divisor of σ_n. We set

$$Y = X_8,$$

and define

$$\sigma = \sigma_8 \circ \cdots \circ \sigma_1 : Y \to X. \tag{51}$$

Proposition 6.4. *There exists an equivalence of categories between the groupoids of*

1. *Higgs bundles on $\mathbb{C}P^1$ with one singular point $q = 0$ and local form given by (25) with $b_{-7} \neq 0$, and*
2. *pure sheaves of dimension 1 and rank 1 on Y supported on a curve $\widetilde{\Sigma}$ which is disjoint from $E_1, \dots, E_9$ and intersects E_{10} with algebraic multiplicity 1.*

Proof. Let $(\mathcal{E}, \theta)$ be a Higgs-field as in part (1). Consider its spectral sheaf

$$\mathcal{S}_0 = \operatorname{coker}\left(p^*(\mathcal{E} \otimes \Theta_C(-4 \cdot \{0\})) \xrightarrow{\xi \otimes p^*\theta + \zeta} p^*(\mathcal{E}) \otimes \mathcal{O}_Z(1) \right),$$

where $\Theta_C(-4 \cdot \{0\})$ is the dual bundle of $K_C(4 \cdot \{0\})$, and $\xi \in H^0(Z, \mathcal{O}_Z(1)), \zeta \in H^0(Z, p^*(K_C(4 \cdot \{0\})) \otimes \mathcal{O}_Z(1))$ are the canonical sections. Let us denote by Σ_0 the support of $\mathcal{S}_0$. Assume that Σ_0 is integral (i.e. irreducible and reduced). Then, by [3], we have

- Σ_0 is disjoint from C_∞,
- p is finite over Σ_0,
- $\mathcal{S}_0$ is torsion-free on Σ_0,
- $p_* \mathcal{S}_0 = \mathcal{E}$,
- the direct image of multiplication by ζ on $\mathcal{S}_0$ induces θ.

Conversely, any sheaf $\mathcal{S}_0$ satisfying the first three of these properties is the spectral sheaf of an irregular Higgs bundle $(\mathcal{E}, \theta)$. The integrality requirement on Σ_0 was later lifted in [19].

The idea of the proof is to use the properties of proper transform functor of coherent sheaves under the blow-up introduced in [1]. Namely, for any smooth surface W and a point $w \in W$, let us denote by $\tau : \widetilde{W} \to W$ the blow-up of w and by E the exceptional divisor. Now, given any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_W$-modules we set

$$\mathcal{F}^E := \mathcal{T}or_1^{\mathcal{O}_{\widetilde{W}}}(\tau^*\mathcal{F}, \mathcal{O}_{\widetilde{W}}(E)_E)$$

and

$$\mathcal{F}^\tau = \tau^*\mathcal{F}/\mathcal{F}^E.$$

With these notations, we have the following result.

Lemma 6.5 (Lemma 5.12 [1]). *Suppose that the homological dimension of $\mathcal{F}$ at x satisfies $dh(\mathcal{F}_x) = 1$.*

(1) If $\mathcal{F}_x$ is torsion, then $dh(\mathcal{F}_y^\tau) = 1$ for any $y \in E$.
(2) We have $R^0\sigma_*(\mathcal{F}^\tau) = \mathcal{F}$ and $R^i\tau_*(\mathcal{F}^\tau) = 0$ for all $i > 0$.
(3) If $\mathcal{F}$ is pure of dimension 1 then $E \notin supp(\mathcal{F}^\tau)$. □

The definition of $\mathcal{S}_0$ makes it clear that it is a torsion module, of homological dimension 1. As the surface X is regular, according to the Auslander–Buchsbaum formula we also get that $\mathcal{S}_0$ is pure of dimension 1. Let us write

$$\mathcal{S}_1 = (\mathcal{S}_0)^{\sigma_1}.$$

Then part (1) of the lemma applied to $W = X$, $w \in X$ the point given by (50) and $\mathcal{F} = \mathcal{S}_0$ implies that $\mathcal{S}_1$ is also of homological dimension 1, and as above we also get that it is pure of dimension 1. Furthermore, part (2) of the lemma implies that

$$R^0(\sigma_1)_*(\mathcal{S}_1) = \mathcal{S}_0.$$

We recursively define for all $n \in \{2, \dots, 8\}$ the coherent sheaf

$$\mathcal{S}_n = (\mathcal{S}_{n-1})^{\sigma_n}$$

on X_n. Recursive application of part (1) of the lemma then implies that $\mathcal{S}_n$ is of homological dimension 1 and pure of dimension 1, and by part (2) it satisfies

$$R^0(\sigma_n)_*(\mathcal{S}_n) = \mathcal{S}_{n-1}.$$

Let us set $\mathcal{S} = \mathcal{S}_8$. It then follows that using the map of (51) we have

$$R^0\sigma_*(\mathcal{S}) = \mathcal{S}_0.$$

Using the properties of $\mathcal{S}_0$ we then get that

$$R^0(p \circ \sigma)_*(\mathcal{S}) = \mathcal{E}. \tag{52}$$

We now show that

$$\mathcal{E} \mapsto \mathcal{S} = \mathcal{S}_8 \tag{53}$$

gives a map from the set of objects of (1) to the set of objects of (2). Indeed, purity follows from Lemma 6.5 as observed above. The rank of $\mathcal{S}$ is equal to 1 because of (52), given that the rank of $\mathcal{E}$ is 2 and that $p \circ \sigma|_{\widetilde{\Sigma}}$ is a double cover of $\mathbb{C}P^1$. Finally, by part (3) of the lemma, the exceptional divisors $E_1, \dots, E_{10}$ are not contained in $\widetilde{\Sigma}$. Moreover, according to part (1) of Lemma 6.1 and part (1) of Lemma 6.2 for each n the center of the blow-up σ_n is the only intersection point of the proper transform of Σ_0 in X_{n-1} with the exceptional divisor E_{n+1}. This implies the statement about intersections.

Conversely, suppose that a sheaf $\mathcal{S}$ fulfilling the properties of (2) is given. Then, we define a holomorphic vector bundle $\mathcal{E}$ by (52), and we define a Higgs field θ as the direct image of multiplication by $\zeta \mathrm{d}z$ on $\mathcal{S}_0 = R^0\sigma_*(\mathcal{S})$. If the curve $\widetilde{\Sigma}$ is disjoint from $E_1, \dots, E_9$ and intersects E_{10} with algebraic multiplicity 1, then the expansion of its image $\sigma(\widetilde{\Sigma})$ near q is given by (49). By virtue of part (2) of Lemma 6.2, this implies the converse expansion (48) with $a_{-7} \neq 0$. Then, according to part (2) of Lemma 6.1, the coefficients in the form (25) are as required. This then gives the inverse map of (53) on objects.

Now, let us consider the map on morphisms. Recall that an isomorphism $(\mathcal{E}_1,\theta_1) \cong (\mathcal{E}_2,\theta_2)$ amounts to an isomorphism of vector bundles

$$\Psi : \mathcal{E}_1 \to \mathcal{E}_2$$

such that

$$\theta_2 \circ \Psi = (\Psi \otimes \mathrm{I}_K) \circ \theta_1,$$

where I_K stands for the identity map of the canonical bundle $K_{\mathbb{C}P^1}$. Therefore, if $(\mathcal{E}_1,\theta_1)$ and $(\mathcal{E}_2,\theta_2)$ are isomorphic, then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E}_1 & \xrightarrow{\xi\otimes p^*\theta_1+\zeta} & p^*(E_1\otimes K(4\cdot\{0\})) & \longrightarrow & \mathcal{S}_0(\mathcal{E}_1,\theta_1)\otimes p^*(K(4\cdot\{0\})) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi\otimes\mathrm{I}_K} & & & & \\ 0 & \longrightarrow & p^*\mathcal{E}_2 & \xrightarrow{\xi\otimes p^*\theta_2+\zeta} & p^*(E_2\otimes K(4\cdot\{0\})) & \longrightarrow & \mathcal{S}_0(\mathcal{E}_2,\theta_2)\otimes p^*(K(4\cdot\{0\})) & \longrightarrow & 0 \end{array}$$

It follows from this diagram that there exists a morphism of sheaves of $\mathcal{O}_{\widetilde{X}}$-modules

$$\mathcal{S}_0(\mathcal{E}_1,\theta_1) \to \mathcal{S}_0(\mathcal{E}_2,\theta_2),$$

which is an isomorphism with inverse induced by Ψ^{-1} in the same way. This isomorphism in turn induces isomorphisms

$$\mathcal{S}_8(\mathcal{E}_1,\theta_1) \cong \mathcal{S}_8(\mathcal{E}_2,\theta_2)$$

by functoriality of the proper transform operation. On the other hand, such an isomorphism of spectral sheaves gives an isomorphism of Higgs bundles by functoriality of the direct image functor. This finishes the proof of Proposition 6.4. □

6.1. **The proof of Theorem 1.3.** Now we are ready to give the proof of our second main result.

Proof of Theorem 1.3. According to Proposition 6.4, describing the moduli space of irregular Higgs bundles with local form given by (25) is equivalent to describing the relative Picard scheme of degree 1 torsion-free sheaves on curves satisfying the properties listed in part (2) of Proposition 6.4.

As in the untwisted case, we write the characteristic polynomial of θ in the trivialization given by κ_1 and κ_1^2 of (18). The polynomials f_1 and g_1 are given in (19) and (20), and the characteristic polynomial is:

$$\chi_{\vartheta_1}(z_1,w_1) = w_1^2 - \left(p_2z_1^2+p_1z_1+p_0\right)w_1 - \left(q_4z_1^4+q_3z_1^3+q_2z_1^2+q_1z_1+q_0\right).$$

Now the roots of $\chi_{\vartheta_1}(0,w_1)$ in w_1 are equal, because the curve intersects the $z_1 = 0$ line in one point. This requirement is satisfied if the discriminant of χ_{ϑ_1} vanishes at $z_1 = 0$, that is,

$$p_0^2 + 4q_0 = 0.$$

After this simplification, we consider the expansions of the roots of $\chi_{\vartheta_1}(z_1,w_1)$ with respect to z_1. It is enough to consider the positive root, because the expansions of the two roots differ in a negative sign in certain terms. The expansion is:

$$\begin{aligned} w_1 =& \frac{p_0}{2} + \frac{1}{2}\sqrt{2p_0p_1+4q_1}\sqrt{z_1} + \frac{p_1}{2}z_1 + \frac{p_1^2+2p_0p_2+4q_2}{4\sqrt{2p_0p_1+4q_1}}z_1^{3/2} + \\ &+ \frac{p_2}{2}z_1^2 + \frac{1}{4}\sqrt{2p_0p_1+4q_1}\left(\frac{p_1p_2+2q_3}{p_0p_1+2q_1} - \frac{\left(p_1^2+2p_0p_2+4q_2\right)^2}{16\left(p_0p_1+2q_1\right)^2}\right)z_1^{5/2}. \end{aligned}$$

We write the local form of θ in the twisted case as in (25). We described the matrix eigenvalues in Lemma 6.1 by the Puiseux expansion, with coefficients a_n.

These two expansions are the same, hence by comparing the coefficients we get the following:

$$\begin{aligned}\chi_{\vartheta_1}(z_1,w_1,t) =& w_1^2 - \left(b_{-4}z_1^2 + b_{-6}z_1 + 2b_{-8}\right)w_1 - tz_1^4 - b_{-3}z_1^3 + \\ &+ \left(b_{-8}b_{-4} - b_{-5}\right)z_1^2 + \left(b_{-8}b_{-6} - b_{-7}\right)z_1 + b_{-8}^2,\end{aligned}$$

where (as in the untwisted case) we denote q_4 by t, and the degree of the polynomial is 2 in the variable w_1 and 4 in z_1. Therefore, we get a pencil parametrized by t with base locus $(0, b_{-8})$ in $\mathbb{C}^2$.

As in the untwisted case, we consider the characteristic polynomial in the chart U_2 with trivialization κ_2. The polynomials f_2 and g_2 are given in (21) and (22).

$$\begin{aligned}\chi_{\vartheta_2}(z_2,w_2,t) =& w_2^2 + f_2(z_2)w_2 + g_2(z_2,t) = \\ =& w_2^2 + \left(2b_{-8}z_2^2 + b_{-6}z_2 + b_{-4}\right)w_2 + b_{-8}^2z_2^4 + \left(b_{-8}b_{-6} - b_{-7}\right)z_2^3 + \\ &+ \left(b_{-8}b_{-4} - b_{-5}\right)z_2^2 - b_{-3}z_2 - t.\end{aligned}$$

The pencil in the Hirzebruch surface X is defined by $\chi_{\vartheta_1}(z_1,w_1,t)$ and the union of the section at infinity with fiber F_0. According to the converse direction of Theorem 3.1, the pencil gives rise to an elliptic fibration in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with a singular fiber of type $\widetilde{E}_8$.

The pencil determines the types of further singular fibers in the elliptic fibration. In the following we will identify the types of these further singular fibers in terms of the defining constants of the pencil. The spectral curves intersect the fiber component F_0 of the curve C_∞ at infinity (whose fiber is with multiplicity 4) in one point and according to Condition (c) of Definition 4.1 the pencil has no singular point on the distinguished fiber F_0. Thus it is sufficient to consider the κ_2 trivialization, i. e. the chart (z_2, w_2) (see Equation (14)). For identifying the singular fibers in the pencil, we look for triples (z_2, w_2, t) such that (z_2, w_2) fits the curve with parameter t and the partial derivatives below vanish:

$$\begin{aligned}\chi_{\vartheta_2}(z_2,w_2,t) &= 0,\\ \frac{\partial\chi_{\vartheta_2}(z_2,w_2,t)}{\partial w_2} &= 0,\\ \frac{\partial\chi_{\vartheta_2}(z_2,w_2,t)}{\partial z_2} &= 0.\end{aligned}$$

Notice that the second and third equations do not involve t, hence we can solve this system for w_2 and z_2. Indeed, by solving the second equations for the variable w_2 we get

$$w_2 = -\frac{1}{2}\left(2b_{-8}z_2^2 + b_{-6}z_2 + b_{-4}\right).$$

We substitute the resulting expression into the third equation, leading to

$$0 = 6b_{-7}z_2^2 + \left(b_{-6}^2 + 4b_{-5}\right)z_2 + b_{-6}b_{-4} + 2b_{-3}. \tag{54}$$

This polynomial is quadratic in z_2 and has one root if and only if the discriminant

$$D = \left(b_{-6}^2 + 4b_{-5}\right)^2 - 24b_{-7}\left(b_{-6}b_{-4} + 2b_{-3}\right)$$

vanishes. In this case the pencil has a single further singular fiber, which has a cusp singularity. If $D \neq 0$ then the fibration has two I_1 singular fibers.

The fibration obtained from the pencil has a section, so just as in the proof of Theorem 1.1 we may apply the relative Abel–Jacobi map to identify the fibration and its relative Picard scheme over the locus of smooth curves. Thus it is sufficient to describe the singular fibers of H. By Proposition 2.1, these latter are as stated in Theorem 1.3, concluding the proof. □

REFERENCES

[1] K. Aker and Sz. Szabó. Algebraic Nahm transform for parabolic Higgs bundles on $\mathbf{P}^1$. *Geometry and Topology*, 18(5):2487–2545, 2014.
[2] A. Altman and S. Kleiman. The presentation functor and the compactified Jacobian. In *The Grothendieck Festschrift*, volume 86 of *Progress in Mathematics*, pages 15–32. Birkhauser, 1990.
[3] A. Beauville, M. Narasimhan, and S. Ramanan. Spectral curves and the generalised theta divisor. *J. reine angew. Math.*, 398:169–179, 1989.
[4] U. Bhosle. Generalised parabolic bundles and applications to torsionfree sheaves on nodal curves. *Ark. Mat.*, 30(2):187–215, 1992.
[5] O. Biquard and Ph. Boalch. Wild non-abelian Hodge theory on curves. *Compos. Math.*, 140(1):179–204, 2004.
[6] P. Cook. *Local and Global aspects of the Module Theory of Singular Curves*. PhD thesis, University of Liverpool, 1993.
[7] C. D'Souza. Compactification of generalised Jacobians. *Proc. Ind. Acad. Sci.*, 88 A(5):419–457, 1979.
[8] D. Gaiotto, G. Moore, and A. Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013.
[9] P. Gothen and A. Oliveira. The singular fiber of the Hitchin map. *Int. Math. Res. Not. IMRN*, 5:1079–1121, 2013.
[10] N Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.
[11] M. Inaba. Moduli space of irregular singular parabolic connections of generic ramified type on a smooth projective curve. arXiv:1606.02369.
[12] P. Ivanics, A. Stipsicz, and Sz. Szabó. Hitchin fibrations on moduli of irregular higgs bundles and motivic wall-crossing. arXiv:1710.09922.
[13] S. Kleiman. The Picard scheme. In *Fundamental Algebraic Geometry, Grothendieck's FGA Explained*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2005.
[14] K. Kodaira. On compact analytic surfaces: II. *Ann. Math.*, 77:563–626, 1963.
[15] R. Miranda. Persson's list of singular fibers for a rational elliptic surface. *Math. Z.*, 205(2):191–211, 1990.
[16] T. Oda and C. Seshadri. Compactifications of the generalized Jacobian variety. *Transactions of the AMS*, 253, 1979.
[17] U. Persson. Configurations of Kodaira fibers on rational elliptic surfaces. *Math. Z.*, 205(1):1–47, 1990.
[18] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Comm. Math. Phys.*, 220(1):165–229, 2001.
[19] D. Schaub. Courbes spectrales et compactifications de Jacobiennes. *Math. Z.*, 227:295–312, 1998.
[20] A. Stipsicz, Z. Szabó, and Á. Szilárd. Singular fibers in elliptic fibrations on the rational elliptic surface. *Periodica Mathematica Hungarica*, 54:137–162, 2007.
[21] Sz. Szabó. The birational geometry of unramified irregular Higgs bundles on curves. *International Journal of Mathematics*, 28(6), 2017.

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, 1111. BUDAPEST, EGRY JÓZSEF UTCA 1. H ÉPÜLET, HUNGARY

RÉNYI INSTITUTE OF MATHEMATICS, 1053. BUDAPEST, REÁLTANODA UTCA 13-15. HUNGARY
E-mail address: ipe@math.bme.hu

RÉNYI INSTITUTE OF MATHEMATICS, 1053. BUDAPEST, REÁLTANODA UTCA 13-15. HUNGARY
E-mail address: stipsicz@renyi.hu

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, 1111. BUDAPEST, EGRY JÓZSEF UTCA 1. H ÉPÜLET, HUNGARY

RÉNYI INSTITUTE OF MATHEMATICS, 1053. BUDAPEST, REÁLTANODA UTCA 13-15. HUNGARY
E-mail address: szabosz@math.bme.hu