



CONVERGENCE ANALYSIS OF FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED NONLOCAL DIFFERENTIAL-DIFFERENCE PROBLEM

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Received 12 April, 2017

Abstract. This study is concerned with a singularly perturbed three-point boundary-value problem with delay. Firstly, bounds on the solution and its derivative of the solution to be used later in the paper are obtained. To solve it numerically, we use an exponentially fitted difference scheme on an equidistant mesh which is established by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. Then, the stability and convergence analysis of difference scheme is given and it is uniformly convergent in perturbation parameter. Furthermore, numerical results which show the effectiveness of the method are presented.

2010 *Mathematics Subject Classification:* 34B10; 34K10; 65L11; 65L12; 65L20

Keywords: singular perturbation, nonlocal condition, finite difference method, initial layer, uniform convergence

1. INTRODUCTION

If an ordinary differential equation contains both a perturbation parameter and a delay term, it is called a singularly perturbed differential-difference equation. These equations are a very important role in science and engineering field. For instance, they occur in the study of human pupil light reflex [18], first-exit problems in neurobiology [22], models of physiological processes and diseases [20], optimal control theory [13], optically bistable devices [9] and signal transmission [10], and other models [11].

On the other hand, in recent years many methods have been developed for solving singularly perturbed delay differential problems. In [8, 16] some asymptotic analysis of boundary value problems for second order singularly perturbed differential-difference equations have been considered and some numerical techniques for solving of this type of problems with large and small shifts were considered in [12, 17]. Particularly, reproducing kernel method [12], initial value technique [23], some special finite element method [25] have been used for solving these problems.

Motivated by the above works, we consider the following model problem:

$$Lu := \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-r) = f(x), \quad x \in \Omega, \quad (1.1)$$

subject to the interval and boundary conditions,

$$u(x) = \varphi(x), \quad x \in \Omega_0; \quad \gamma u(l_1) + u(l) = A, \quad l_1 \in \Omega \quad (1.2)$$

where $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (0, r]$, $\Omega_2 = (r, l)$, $\bar{\Omega} = [0, l]$, $\Omega_0 = [-r, 0]$ and $0 < \varepsilon \ll 1$ is the perturbation parameter, $\gamma > 0$ and A are given constants. $a(x) \geq \alpha > 0$, $b(x) \leq 0$, $c(x)$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and r is a constant delay, which is independent of ε .

The special case $r = 0$ and $\varphi(x) = B$ (B is a real constant) of the above problem so called nonlocal boundary value problem of ordinary differential equation, was initiated by [14]. This kind of problems occur in mathematical models of a large number of phenomena in catalytic processes in chemistry and biology, in problems of semiconductors, in problems of hydromechanics, in studying heat transfer problems and some other physical phenomena [1]. In the survey paper [19], nonlocal problems have been extensively studied many researchers. Nonetheless, the existence of positive solutions for the multipoint boundary value problem with delay is studied in [5, 7, 15, 24]. Recently, numerical study of the boundary value problem for linear singularly perturbed differential difference equation with large delay and first type condition is extensively investigated [12, 23, 25]. But, singularly perturbed nonlocal problems with delay are not studied in literature so far.

It is well known that, for small values of ε , standard numerical methods for solving such problems are unstable and do not give accurate results. For that reason, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε , i.e., methods that are convergent ε -uniformly. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special non-uniform grids which condense in the boundary layers in a special manner. The various approaches to the design and analysis of appropriate numerical methods for singularly perturbed differential equations can be found in [2, 6, 21] and the references therein. The numerical method presented here comprises a fitted difference scheme on a uniform mesh. We have derived this approach on the basis of the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first order derivatives of exact solution and hence facilitates examination of the convergence. The solution of a singularly perturbed problem of the form (1.1)-(1.2) normally has a boundary layer at $x = 0$.

This paper is organized as follows. In Section 2, we state some important properties of the exact solution. The description of the finite difference discretization

have been given in Section 3. In Section 4, we present the error analysis for the approximate solution. Uniform convergence is proved in the discrete maximum norm. In Section 5, we formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally. The paper ends with a summary of the main conclusions.

Notation. Throughout the paper, C denotes a generic positive constant independent of ε and the mesh parameter. Some specific, fixed constants of this kind are indicated by subscripting C . For any continuous function $g(x)$ denote norms which

$$\|g\|_\infty \equiv \|g\|_{\infty, \bar{\Omega}} = \max_{0 \leq x \leq l} |g(x)|, \|g\|_1 \equiv \|g\|_{1, \Omega} = \int_0^l |g(x)| dx,$$

$$\|g\|_{\infty, k} \equiv \|g\|_{\infty, \bar{\Omega}_k}, \quad \|g\|_{1, k} \equiv \|g\|_{1, \Omega_k}, \quad k = 0, 1, 2.$$

2. PROPERTIES OF EXACT SOLUTION

Here we give some properties of the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution.

Lemma 1. *Let $a(x), b(x), c(x), f(x) \in C(\bar{\Omega}), \varphi(x) \in C(\Omega_0)$ and*

$$\lambda := \alpha^{-1} \beta (\|b\|_1 + \|c\|_{1,2}) < 1. \tag{2.1}$$

Then for the solution $u(x)$ of the problem (1.1)-(1.2) the following estimates hold:

$$\|u\|_\infty \leq C_0, \tag{2.2}$$

$$|u'(x)| \leq C \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right), \quad x \in \bar{\Omega}, \tag{2.3}$$

where

$$C_0 = [|\varphi(0)| + |A| + \beta(|\varphi(0)| + \alpha^{-1} \|f\|_1 + \alpha^{-1} \|c\|_{\infty,1} \|\varphi\|_{1,0})] (1 - \lambda)^{-1},$$

$$\beta = 1 + \gamma l (\gamma l_1 + l)^{-1}.$$

Proof. Firstly, we prove (2.2). From (1.1) it follows that

$$u'(x) = u'(0) e^{-\frac{1}{\varepsilon} \int_0^x a(\eta) d\eta} - \frac{1}{\varepsilon} \int_0^x F(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^x a(\eta) d\eta} d\xi \tag{2.4}$$

with

$$F(x) = -f(x) + b(x)u(x) + c(x)u(x-r).$$

Integrating the relation (2.4) over $(0, x)$ we get

$$u(x) = \varphi(0) + u'(0) \int_0^x e^{-\frac{1}{\varepsilon} \int_0^\tau a(\eta) d\eta} d\tau - \frac{1}{\varepsilon} \int_0^x d\tau \int_0^\tau F(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^\tau a(\eta) d\eta} d\xi$$

$$= \varphi(0) + u'(0) \int_0^x e^{-\frac{1}{\varepsilon} \int_0^\tau a(\eta) d\eta} d\tau - \frac{1}{\varepsilon} \int_0^x d\xi F(\xi) \int_\xi^x e^{-\frac{1}{\varepsilon} \int_\xi^\tau a(\eta) d\eta} d\tau.$$

Using now the condition $\gamma u(l_1) + u(l) = A$, we have

$$u'(0) = \frac{A - (1 + \gamma)\varphi(0) + \gamma q(l_1) + q(l)}{\gamma p(l_1) + p(l)} \quad (2.5)$$

and so

$$u(x) = \varphi(0) + \frac{A - (1 + \gamma)\varphi(0) + \gamma q(l_1) + q(l)}{\gamma p(l_1) + p(l)} p(x) - q(x), \quad (2.6)$$

where

$$p(x) = \int_0^x e^{-\frac{1}{\varepsilon} \int_0^\tau a(\eta) d\eta} d\tau, \quad q(x) = \frac{1}{\varepsilon} \int_0^x d\xi F(\xi) \int_\xi^x e^{-\frac{1}{\varepsilon} \int_\xi^\tau a(\eta) d\eta} d\tau.$$

In order to prove (2.2), we will use appropriate Green function as an alternative to (2.6).

$$\begin{aligned} L_* v &:= \varepsilon v''(x) + a(x)v'(x) = H(x), \quad 0 < x < l, \\ v(x) &= \varphi(x), \quad -r \leq x \leq 0; \quad v(l) = A, \end{aligned} \quad (2.7)$$

can be also expressed the solution (2.7) as [3]

$$v(x) = \left(1 - \frac{p(x)}{p(l)}\right)\varphi(0) + \frac{p(x)}{p(l)}A + \int_0^l G(x, s)H(s)ds \quad (2.8)$$

where

$$G(x, s) = \frac{1}{\varepsilon} \int_s^l e^{-\frac{1}{\varepsilon} \int_s^\tau a(\eta) d\eta} d\tau \times \frac{p(x)}{p(l)} - \frac{1}{\varepsilon} T_0(x-s) \int_s^x e^{-\frac{1}{\varepsilon} \int_s^\tau a(\eta) d\eta} d\tau, \quad (2.9)$$

$$T_0(t) = 1, t \geq 0; \quad T_0(t) = 0, t < 0,$$

and

$$H(x) = -f(x) + b(x)v(x) + c(x)v(x-r).$$

On the other hand, using transformation

$$u(x) = v(x) - \frac{\gamma x}{\gamma l_1 + l} v(l_1)$$

we can rewrite for the solution of the problem (1.1)-(1.2), such that

$$u(x) = \left(1 - \frac{p(x)}{p(l)}\right)\varphi(0) + \frac{p(x)}{p(l)}A + \int_0^l G(x, s)F(s)ds - \frac{\gamma x}{\gamma l_1 + l} \int_0^l G(l_1, s)F(s)ds.$$

Also in [3] the authors show that $0 \leq G(x, s) \leq \alpha^{-1}$ for Green's function (2.9), from this point of view

$$\begin{aligned}
 |u(x)| &\leq |\varphi(0)| + |A| + \int_0^l |G(x, s)| |F(s)| ds + \frac{\gamma l}{\gamma l_1 + l} \int_0^l |G(l_1, s)| |F(s)| ds \\
 &\leq |\varphi(0)| + |A| + \max_{x, s \in \Omega} |G(x, s)| \int_0^l |F(s)| ds \\
 &\quad + \max_{l_1, s \in \Omega} |G(l_1, s)| \frac{\gamma l}{\gamma l_1 + l} \int_0^l |F(s)| ds \\
 &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \int_0^l |F(s)| ds \\
 &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \|f\|_1 + \alpha^{-1} \beta \|b\|_1 \|u\|_\infty \\
 &\quad + \alpha^{-1} \beta \int_0^l |c(s)u(s-r)| ds.
 \end{aligned}$$

After, by means of changing integral variable by $t = s - r$, we find that

$$\begin{aligned}
 |u(x)| &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \|f\|_1 + \alpha^{-1} \beta \|b\|_1 \|u\|_\infty \\
 &\quad + \alpha^{-1} \beta \int_{-r}^{l-r} |c(r+t)u(t)| dt \\
 &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \|f\|_1 + \alpha^{-1} \beta \|b\|_1 \|u\|_\infty \\
 &\quad + \alpha^{-1} \beta \int_{-r}^0 |c(r+t)\varphi(t)| dt + \alpha^{-1} \beta \int_0^{l-r} |c(r+t)u(t)| dt \\
 &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \|f\|_1 + \alpha^{-1} \beta \|b\|_1 \|u\|_\infty \\
 &\quad + \alpha^{-1} \beta \|c\|_{\infty,1} \|\varphi\|_{1,0} + \alpha^{-1} \beta \|u\|_\infty \int_r^l |c(t)| dt \\
 &\leq |\varphi(0)| + |A| + \alpha^{-1} \beta \|f\|_1 + \alpha^{-1} \beta \|b\|_1 \|u\|_\infty \\
 &\quad + \alpha^{-1} \beta \|c\|_{\infty,1} \|\varphi\|_{1,0} + \alpha^{-1} \beta \|c\|_{1,2} \|u\|_\infty
 \end{aligned}$$

from which (2.2) follows immediately. We have now to prove the estimate (2.3). If we reconsider the terms in (2.5), we obtain the following relations:

$$\begin{aligned} \gamma p(l_1) + p(l) &= \gamma \int_0^{l_1} e^{-\frac{1}{\varepsilon} \int_0^\tau a(\eta) d\eta} d\tau + \int_0^l e^{-\frac{1}{\varepsilon} \int_0^\tau a(\eta) d\eta} d\tau \\ &\geq \gamma \int_0^{l_1} e^{-\frac{a^* \tau}{\varepsilon}} d\tau + \int_0^l e^{-\frac{a^* \tau}{\varepsilon}} d\tau = \frac{\varepsilon}{a^*} [\gamma(1 - e^{-\frac{a^* l_1}{\varepsilon}}) + 1 - e^{-\frac{a^* l}{\varepsilon}}] \\ &\geq \frac{\varepsilon}{a^*} [\gamma(1 - e^{-a^* l_1}) + 1 - e^{-a^* l}] \equiv c_1 \varepsilon, (a^* = \|a\|_\infty), \end{aligned}$$

and

$$\begin{aligned} \gamma |q(l_1)| + |q(l)| &\leq \frac{\gamma}{\varepsilon} \int_0^{l_1} d\xi |F(\xi)| \int_\xi^{l_1} e^{-\frac{1}{\varepsilon} \int_\xi^\tau a(\eta) d\eta} d\tau \\ &\quad + \frac{1}{\varepsilon} \int_0^l d\xi |F(\xi)| \int_\xi^l e^{-\frac{1}{\varepsilon} \int_\xi^\tau a(\eta) d\eta} d\tau \\ &\leq \frac{\gamma}{\varepsilon} \int_0^{l_1} d\xi |F(\xi)| \int_\xi^{l_1} e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}} d\tau + \frac{1}{\varepsilon} \int_0^l d\xi |F(\xi)| \int_\xi^l e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}} d\tau \\ &= \frac{\gamma}{\varepsilon} \int_0^{l_1} d\xi |F(\xi)| [\alpha^{-1} \varepsilon (1 - e^{-\frac{\alpha(l_1-\xi)}{\varepsilon}})] \\ &\quad + \frac{1}{\varepsilon} \int_0^l d\xi |F(\xi)| [\alpha^{-1} \varepsilon (1 - e^{-\frac{\alpha(l-\xi)}{\varepsilon}})] \\ &\leq \alpha^{-1} (\gamma \int_0^{l_1} |F(\xi)| d\xi + \int_0^l |F(\xi)| d\xi). \end{aligned}$$

For $l_1 \leq r$ we get

$$\gamma |q(l_1)| + |q(l)| \leq \alpha^{-1} \gamma \|f\|_1 + \alpha^{-1} \gamma \|b\|_\infty \int_0^{l_1} |u(s)| ds$$

$$\begin{aligned}
 & + \alpha^{-1}(1 + \gamma) \|c\|_\infty \int_{-r}^0 |\varphi(s)| ds + \alpha^{-1} \|c\|_\infty \int_0^{l-r} |u(s)| ds \\
 \leq & \alpha^{-1} \gamma (\|f\|_1 + \|b\|_\infty C_0 l_1) + \alpha^{-1} (1 + \gamma) \|c\|_\infty \|\varphi\|_{1,0} \\
 & + \alpha^{-1} \|c\|_\infty C_0 (l - r) \equiv C_1
 \end{aligned}$$

and when $l_1 > r$ we have

$$\begin{aligned}
 \gamma |q(l_1)| + |q(l)| & \leq \alpha^{-1} \gamma \|f\|_1 + \alpha^{-1} \gamma \|b\|_\infty \int_0^{l_1} |u(s)| ds \\
 & + \alpha^{-1} (1 + \gamma) \|c\|_\infty \int_{-r}^0 |\varphi(s)| ds \\
 & + \alpha^{-1} \gamma \|c\|_\infty \int_0^{l_1-r} |u(s)| ds + \alpha^{-1} \|c\|_\infty \int_0^{l-r} |u(s)| ds \\
 \leq & \alpha^{-1} \gamma (\|f\|_1 + \|b\|_\infty C_0 l_1) + \alpha^{-1} (1 + \gamma) \|c\|_\infty \|\varphi\|_{1,0} \\
 & + \alpha^{-1} \|c\|_\infty C_0 (\gamma (l_1 - r) + (l - r)) \equiv C_2.
 \end{aligned}$$

Thus, from (2.5) we are led to

$$\begin{aligned}
 |u'(0)| & \leq \frac{|A| + (1 + \gamma) |\varphi(0)| + \gamma |q(l_1)| + |q(l)|}{\gamma p(l_1) + p(l)} \\
 & \leq \frac{c_1^{-1} (|A| + (1 + \gamma) |\varphi(0)| + \max\{C_1, C_2\})}{\varepsilon} \equiv \frac{C_3}{\varepsilon}.
 \end{aligned}$$

Using the preceding in (2.4) and arguing as above we arrive at (2.3). □

3. DISCRETE PROBLEM

Henceforth, we denote by ω a uniform mesh on Ω :

$$\omega = \{x_i = ih, i = 1, 2, \dots, N - 1; h = l/N\}$$

and $\bar{\omega} = \omega \cup \{x = 0, l\}$. For simplicity, we will suppose that $\frac{r}{l} N = N_0$ and $\frac{l_1}{l} N = N_1$ are integers, i.e., $x_{N_0} = r$ and $x_{N_1} = l_1$. Before describing our numerical method, we introduce some notation for the mesh functions. For any mesh function $g(x)$, we use

$$\begin{aligned}
 g_i & = g(x_i), \quad g_{\bar{x},i} = (g_i - g_{i-1})/h, \quad g_{x,i} = (g_{i+1} - g_i)/h, \\
 g_{0,x,i} & = (g_{x,i} + g_{\bar{x},i})/2, \quad g_{\bar{x}x,i} = (g_{x,i} - g_{\bar{x},i})/h,
 \end{aligned}$$

$$\|g\|_{\infty} \equiv \|g\|_{\infty, \bar{\omega}} = \max_{0 \leq i \leq N} |g_i|, \quad \|g\|_1 = \|g\|_{1, \omega} = \sum_{i=1}^{N-1} h |g_i|.$$

The approach of generating difference method is through the integral identity

$$h^{-1} \int_0^l Lu(x)\psi_i(x)dx = h^{-1} \int_0^l f(x)\psi_i(x)dx, \quad i = 1, 2, \dots, N-1 \quad (3.1)$$

with basis functions

$$\psi_i(x) = \begin{cases} \psi_i^{(1)}(x), & x_{i-1} < x < x_i, \\ \psi_i^{(2)}(x), & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ are the solutions of the following problems, respectively

$$\varepsilon \psi''(x) - a_i \psi'(x) = 0, \quad x_{i-1} < x < x_i, \quad \psi(x_{i-1}) = 0, \quad \psi(x_i) = 1,$$

$$\varepsilon \psi''(x) - a_i \psi'(x) = 0, \quad x_i < x < x_{i+1}, \quad \psi(x_i) = 1, \quad \psi(x_{i+1}) = 0.$$

The functions $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ can be explicitly expressed as

$$\psi_i^{(1)}(x) = \frac{e^{a_i(x-x_{i-1})/\varepsilon} - 1}{e^{a_i h/\varepsilon} - 1}, \quad \psi_i^{(2)}(x) = \frac{1 - e^{-a_i(x_{i+1}-x)/\varepsilon}}{1 - e^{-a_i h/\varepsilon}},$$

which, obviously, satisfy

$$h^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x) dx = 1.$$

Rearranging (3.1), it gives

$$-\varepsilon h^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x) u'(x) dx + a_i h^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x) u'(x) dx + b_i u_i + c_i u_{i-N_0} = f_i - R_i,$$

$$i = 1, 2, \dots, N-1$$

with

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)} + R_i^{(4)}, \quad (3.2)$$

$$R_i^{(1)} = h^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)] u'(x) \psi_i(x) dx,$$

$$R_i^{(2)} = h^{-1} \int_{x_{i-1}}^{x_{i+1}} [b(x)u(x) - b(x_i)u(x_i)] \psi_i(x) dx$$

$$\begin{aligned}
 &= h^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \psi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} (b(x)u(x)) K_{0,i}^*(x, \xi) d\xi, \\
 R_i^{(3)} &= h^{-1} \int_{x_{i-1}}^{x_{i+1}} [c(x)u(x-r) - c(x_i)u(x_i-r)] \psi_i(x) dx \\
 &= h^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \psi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} (c(x)u(x-r)) K_{0,i}^*(x, \xi) d\xi, \\
 K_{0,i}^*(x, \xi) &= T_0(x-\xi) - T_0(x_i-\xi), \quad 1 \leq i \leq N-1, \\
 R_i^{(4)} &= h^{-1} \int_{x_{i-1}}^{x_{i+1}} [f(x_i) - f(x)] \psi_i(x) dx.
 \end{aligned}$$

To be consistent with [3], we present the following approach for exact solution

$$\ell u_i \equiv \varepsilon \theta_i u_{\bar{x}x,i} + a_i u_{0,x,i} + b_i u_i + c_i u_{i-N_0} = f_i - R_i, \quad i = 1, 2, \dots, N-1.$$

where

$$\theta_i = \rho_i \coth(\rho_i), \quad \rho_i = \frac{a_i h}{2\varepsilon}. \tag{3.3}$$

Herefrom, we propose the following difference scheme for approximating (1.1)-(1.2)

$$\ell y_i \equiv \varepsilon \theta_i y_{\bar{x}x,i} + a_i y_{0,x,i} + b_i y_i + c_i y_{i-N_0} = f_i, \quad 0 < i < N, \tag{3.4}$$

$$y_i = \varphi_i, \quad -N_0 \leq i \leq 0; \quad \gamma y_{N_1} + y_N = A, \tag{3.5}$$

where θ is given by (3.3).

4. STABILITY BOUND AND CONVERGENCE

In this section, we show the convergence of our method. Note that the error function $z_i = y_i - u_i$ is the solution of the discrete problem

$$\ell z_i = R_i, \quad 0 < i < N, \tag{4.1}$$

$$z_i = 0, \quad -N_0 \leq i \leq 0; \quad \gamma z_{N_1} + z_N = 0. \tag{4.2}$$

where the truncation error R_i is given by (3.2).

Lemma 2. *If $a(x), b(x), c(x), f(x) \in C^1(\bar{\Omega})$ and $\varphi(x) \in C^1(\Omega_0)$, then for the truncation error R_i we have*

$$\|R\|_1 \leq Ch. \tag{4.3}$$

Proof. Using the mean value theorem, we get

$$|a(x) - a(x_i)| = |a'(\xi)| |x - x_i| \leq Ch, \quad x, \xi \in [x_{i-1}, x_{i+1}].$$

Hence

$$\left| R_i^{(1)} \right| \leq Ch(h^{-1} \int_{x_{i-1}}^{x_{i+1}} |u'(x)| \psi_i(x) dx)$$

and taking also into account that $0 \leq \psi_i(x) \leq 1$, we have

$$\left\| R^{(1)} \right\|_1 \leq Ch \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} |u'(x)| dx \leq 2Ch \int_0^l |u'(x)| dx.$$

We now use the estimate (2.3) to establish that

$$\left\| R^{(1)} \right\|_1 \leq Ch \left(1 + \frac{1}{\varepsilon} \int_0^l e^{-\frac{\alpha x}{\varepsilon}} dx \right) \leq Ch(1 + \alpha^{-1}(1 - e^{-\frac{\alpha l}{\varepsilon}})) = O(h).$$

For $R_i^{(2)}$, in view of $b \in C^1(\overline{\Omega})$ and using Lemma 1, we obtain

$$\left| R_i^{(2)} \right| \leq \int_{x_{i-1}}^{x_{i+1}} \left| \frac{d}{dx} [b(x)u(x)] \right| (\xi) d\xi \leq C \left(1 + \int_{x_{i-1}}^{x_{i+1}} |u'(\xi)| d\xi \right).$$

Hence

$$\left\| R^{(2)} \right\|_1 \leq Ch \left(1 + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} |u'(\xi)| d\xi \right) \leq Ch \left(1 + \int_0^l |u'(\xi)| d\xi \right).$$

Using the (2.3), we get

$$\left\| R^{(2)} \right\|_1 \leq Ch \left(1 + \frac{1}{\varepsilon} \int_0^l e^{-\frac{\alpha \xi}{\varepsilon}} d\xi \right) \leq Ch(1 + \alpha^{-1}(1 - e^{-\frac{\alpha l}{\varepsilon}})) = O(h).$$

For $R_i^{(3)}$, in virtue of $c \in C^1(\overline{\Omega})$ and using Lemma 1, we get

$$\left| R_i^{(3)} \right| \leq \int_{x_{i-1}}^{x_{i+1}} \left| \frac{d}{dx} [c(x)u(x-r)] \right| (\xi) d\xi \leq C \left(1 + \int_{x_{i-1}}^{x_{i+1}} |u'(\xi-r)| d\xi \right).$$

Hence

$$\left\| R^{(3)} \right\|_1 \leq Ch \left(1 + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} |u'(\xi-r)| d\xi \right) \leq Ch \left(1 + \int_0^l |u'(\xi-r)| d\xi \right)$$

and, after replacing $s = \xi - r$ this reduces to

$$\|R^{(3)}\|_1 \leq Ch(1 + \int_{-r}^{l-r} |u'(s)| ds) = Ch(1 + \int_{-r}^0 |\varphi'(s)| ds + \int_0^{l-r} |u'(s)| ds)$$

which yields

$$\|R^{(3)}\|_1 \leq Ch(1 + \|\varphi'\|_{1,0} + \alpha^{-1}(1 - e^{-\frac{\alpha(l-r)}{\varepsilon}})) = O(h).$$

The same estimate is obtained for $R_i^{(4)}$ in the similar manner as above. □

Lemma 3. *Let z_i be the solution (4.1)-(4.2) and (2.1) holds true. Then*

$$\|z\|_{\infty,\omega} \leq \|R\|_{1,\omega}. \tag{4.4}$$

Proof. We here will use the discrete Green's function $G^h(x_i, s_j)$ for the operator

$$L_*^h \bar{z}_i := -\varepsilon \theta_i \bar{z}_{\bar{x}x,i} - a_i \bar{z}_{x,i}, \quad 1 \leq i \leq N-1, \\ \bar{z}_0 = 0, \quad \bar{z}_N = 0.$$

As a function of x_i for fixed s_j this function is being defined as

$$L_*^h G^h(x_i, s_j) = \delta^h(x_i, s_j), \quad x_i \in \omega, \quad s_j \in \omega, \\ G^h(0, s_j) = G^h(l, s_j), \quad s_j \in \omega,$$

where $\delta^h(x_i, s_j) = h^{-1} \delta_{ij}$ and δ_{ij} is the Kronecker delta. Using Green's function and below transformation

$$z_i = \bar{z}_i - \frac{\gamma x_i}{\gamma x_{N_1} + x_N} \bar{z}_{N_1},$$

we can write down the following relation for solution of problem (4.1)-(4.2)

$$z_i = \sum_{j=1}^{N-1} h G^h(x_i, s_j) (b_j z_j + c_j z_{j-N_0} - R_j) \\ - \frac{\gamma x_i}{\gamma x_{N_1} + x_N} \sum_{j=1}^{N-1} h G^h(x_{N_1}, s_j) (b_j z_j + c_j z_{j-N_0} - R_j), \quad x_i \in \omega. \tag{4.5}$$

In the analogous manner as in [4] one can show that $0 \leq G^h(x_i, s_j) \leq \alpha^{-1}$. Then from (4.5) it follows that

$$\|z\|_{\infty,\omega} \leq \alpha^{-1} \{ \|z\|_{\infty,\omega} \sum_{j=1}^{N-1} h(|b_j| + |c_j|) + \|R\|_{1,\omega} \} \\ + \frac{\gamma x_N}{\gamma x_{N_1} + x_N} \alpha^{-1} \{ \|z\|_{\infty,\omega} \sum_{j=1}^{N-1} h(|b_j| + |c_j|) + \|R\|_{1,\omega} \}$$

$$\begin{aligned} &\leq \alpha^{-1} \beta \{ \|z\|_{\infty, \omega} (\|b\|_{1, \omega} + \|c\|_{1, \omega}) + \|R\|_{1, \omega} \} \\ &\leq \lambda \|z\|_{\infty, \omega} + \alpha^{-1} \beta \|R\|_{1, \omega}, \end{aligned}$$

which implies the validity of (4.4).

Now, we give the main result on ε -uniform convergence of the proposed method for solving the problem (1.1)-(1.2). \square

Theorem 1. *Under the condition of Lemma 1, the solution of (1.1)-(1.2) and the solution (3.4)-(3.5) satisfy the estimate*

$$\|y - u\|_{\infty, \bar{\omega}} \leq Ch. \quad (4.6)$$

Proof. Combining the previous lemmas, we immediately have (4.6). \square

5. ALGORITHM AND NUMERICAL RESULTS

In this section, we shall propose the following iterative technique for solving problem (3.4)-(3.5). In addition, we demonstrate the effectiveness of the present method by applying it to three examples of problem (1.1)-(1.2).

$$\varepsilon \theta_i y_{\bar{x}x, i}^{(n)} + a_i y_{0, x, i}^{(n)} + b_i y_i^{(n)} = f_i^{(1)}, \quad f_i^{(1)} = f_i - c_i y_{i-N_0}, \quad 1 \leq i \leq N_0 - 1,$$

$$y_i^{(n)} = \varphi_i, \quad -N_0 \leq i \leq 0; \quad y_{N_0}^{(n)} = \kappa_{n-1},$$

$$\varepsilon \theta_i y_{\bar{x}x, i}^{(n)} + a_i y_{0, x, i}^{(n)} + b_i y_i^{(n)} = f_i^{(2)}, \quad f_i^{(2)} = f_i - c_i y_{i-N_0}, \quad N_0 + 1 \leq i \leq N - 1,$$

$$y_{N_0}^{(n)} = \kappa_{n-1}, \quad y_N^{(n)} = A - \gamma \mu_{n-1},$$

where κ_n and μ_n are values of (3.4) for y_{N_0} and y_{N_1} so that

$$\kappa_n = [\varepsilon \theta_{N_0} (y_{N_0+1}^{(n)} + y_{N_0-1}^{(n)}) + h^2 (a_{N_0} y_{0, x, N_0}^{(n)} + c_{N_0} \varphi_0 - f_{N_0})] (2\varepsilon \theta_{N_0} - b_{N_0} h^2)^{-1},$$

$$\mu_n = \begin{cases} \frac{\varepsilon \theta_{N_1} (y_{N_1+1}^{(n)} + y_{N_1-1}^{(n)}) + h^2 (a_{N_1} y_{0, x, N_1}^{(n)} + c_{N_1} \varphi_{N_1-N_0} - f_{N_1})}{2\varepsilon \theta_{N_1} - b_{N_1} h^2}, & N_0 \geq N_1, \\ \frac{\varepsilon \theta_{N_1} (y_{N_1+1}^{(n)} + y_{N_1-1}^{(n)}) + h^2 (a_{N_1} y_{0, x, N_1}^{(n)} + c_{N_1} y_{N_1-N_0}^{(n)} - f_{N_1})}{2\varepsilon \theta_{N_1} - b_{N_1} h^2}, & N_0 < N_1, \end{cases}$$

$$n = 1, 2, 3, \dots,$$

$\kappa_0 = \mu_0 = C_0$, and C_0 is given by (2.2).

We consider the following test problems:

Example 1. (Case 1: $l_1 < r$)

$$\begin{cases} \varepsilon u''(x) + 64u'(x) - u(x) + 0.25u(x-1) = 0.25x - 1.25, & 0 < x < \frac{3}{2} \\ u(x) = x, & -1 \leq x \leq 0; \quad \frac{1}{4}u(\frac{1}{2}) + u(\frac{3}{2}) = 2, \end{cases}$$

the exact solution is given by

$$u(x) = \begin{cases} 1 + d_1 e^{m_1 x} + d_2 e^{m_2 x}, & x \in [0, 1] \\ d_3 e^{m_1(x-1)} + d_4 e^{m_2(x-1)} - \frac{59}{4}, & x \in (1, \frac{3}{2}] \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d} + d_1 e^{m_1(x-1)} - d_2 e^{m_2(x-1)}), & x \in (1, \frac{3}{2}] \end{cases}$$

where

$$d_1 = \frac{\frac{133}{8} + \beta_1 e^{\frac{m_1}{2}} - (\beta_2 - e^{m_2} - \frac{1}{8\sqrt{d}} - \frac{1}{4})e^{\frac{m_2}{2}}}{\sigma_3 - \sigma_4 + \frac{1}{4}(e^{\frac{m_1}{2}} - e^{\frac{m_2}{2}})},$$

$$d_2 = -1 - d_1, \quad d_3 = \alpha_1 d_1 - \beta_1, \quad d_4 = -\alpha_2 d_1 + \beta_2 - e^{m_2}.$$

Example 2. (Case 2: $l_1 = r$)

$$\begin{cases} \varepsilon u''(x) + 64u'(x) - u(x) + 0.25u(x-1) = 0.25x - 1.25, & 0 < x < \frac{3}{2} \\ u(x) = x, & -1 \leq x \leq 0; \quad \frac{1}{2}u(1) + u(\frac{3}{2}) = 2, \end{cases}$$

the exact solution is given by

$$u(x) = \begin{cases} 1 + d_1 e^{m_1 x} + d_2 e^{m_2 x}, & x \in [0, 1] \\ d_3 e^{m_1(x-1)} + d_4 e^{m_2(x-1)} - \frac{59}{4}, & x \in (1, \frac{3}{2}] \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d} + d_1 e^{m_1(x-1)} - d_2 e^{m_2(x-1)}), & x \in (1, \frac{3}{2}] \end{cases}$$

where

$$d_1 = \frac{\frac{135}{8} + \beta_1 e^{\frac{m_1}{2}} - \frac{1}{2}(1 - e^{m_2}) - (\beta_2 - e^{m_2} - \frac{1}{8\sqrt{d}})e^{\frac{m_2}{2}}}{\frac{1}{2}(e^{m_1} - e^{m_2}) + \sigma_3 - \sigma_4},$$

$$d_2 = -1 - d_1, \quad d_3 = \alpha_1 d_1 - \beta_1, \quad d_4 = -\alpha_2 d_1 + \beta_2 - e^{m_2}.$$

Example 3. (Case 3: $l_1 > r$)

$$\begin{cases} \varepsilon u''(x) + 64u'(x) - u(x) + 0.25u(x-1) = 0.25x - 1.25, & 0 < x < \frac{3}{2} \\ u(x) = x, & -1 \leq x \leq 0; \quad \frac{1}{2}u(\frac{5}{4}) + u(\frac{3}{2}) = 2, \end{cases}$$

the exact solution is given by

$$u(x) = \begin{cases} 1 + d_1 e^{m_1 x} + d_2 e^{m_2 x}, & x \in [0, 1] \\ d_3 e^{m_1(x-1)} + d_4 e^{m_2(x-1)} - \frac{59}{4}, & x \in (1, \frac{3}{2}] \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d} + d_1 e^{m_1(x-1)} - d_2 e^{m_2(x-1)}), & x \in (1, \frac{3}{2}] \end{cases}$$

where

$$d_1 = \frac{\frac{777}{32} + \beta_1 e^{\frac{m_1}{4}} (\frac{1}{2} + e^{\frac{m_1}{4}}) - (\beta_2 - e^{m_2} - \frac{1}{8\sqrt{d}})e^{\frac{m_2}{2}}}{\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \frac{(\frac{\beta_2}{2} - \frac{e^{m_2}}{2} - \frac{1}{32\sqrt{d}})e^{\frac{m_2}{4}}}{\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4}},$$

$$d_2 = -1 - d_1, d_3 = \alpha_1 d_1 - \beta_1, d_4 = -\alpha_2 d_1 + \beta_2 - e^{m_2},$$

and the other coefficients for all examples are as follows

$$\sigma_k = \begin{cases} \left(\frac{\alpha_k}{2} + (-1)^k \frac{1}{32\sqrt{d}}\right) e^{\frac{m_k}{4}}, & k = 1, 2, \\ \left(\alpha_{k-2} + (-1)^k \frac{1}{8\sqrt{d}}\right) e^{\frac{m_{k-2}}{2}}, & k = 3, 4, \end{cases}$$

$$\alpha_{1,2} = e^{m_{1,2}} + \frac{\varepsilon}{2d}, \beta_{1,2} = \frac{\varepsilon}{4\sqrt{d}} \left(63m_{2,1} - \frac{1}{\sqrt{d}} - 1\right),$$

$$m_1 = \frac{-64 + \sqrt{d}}{2\varepsilon}, m_2 = \frac{-64 - \sqrt{d}}{2\varepsilon}, d = 2^{12} + 4\varepsilon.$$

TABLE 1. Errors and convergence rates for Example 1

| ε | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---------------|------------|-------------|-------------|-------------|--------------|
| 2^{-2} | 3.53104E-5 | 1.02837E-5 | 2.69286E-6 | 6.83444E-7 | 3.60809E-7 |
| | 1.78 | 1.93 | 1.98 | 1.92 | |
| 2^{-4} | 5.75563E-5 | 2.46914E-5 | 8.83647E-6 | 2.57391E-6 | 6.74033E-7 |
| | 1.22 | 1.48 | 1.78 | 1.93 | |
| 2^{-6} | 6.37278E-5 | 3.08407E-5 | 1.43934E-5 | 6.17426E-6 | 2.20966E-6 |
| | 1.05 | 1.10 | 1.22 | 1.48 | |
| 2^{-8} | 6.52702E-5 | 3.23831E-5 | 1.59358E-5 | 7.71119E-6 | 3.59865E-6 |
| | 1.01 | 1.02 | 1.05 | 1.10 | |
| 2^{-10} | 6.56558E-5 | 3.27687E-5 | 1.63214E-5 | 8.09677E-6 | 3.98423E-6 |
| | 1.00 | 1.01 | 1.01 | 1.02 | |
| 2^{-12} | 6.57522E-5 | 3.28651E-5 | 1.64178E-5 | 8.19317E-6 | 4.08062E-6 |
| | 1.00 | 1.00 | 1.00 | 1.01 | |
| 2^{-14} | 6.57763E-5 | 3.28892E-5 | 1.64419E-5 | 8.21727E-6 | 4.10472E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| 2^{-16} | 6.57820E-5 | 3.28949E-5 | 1.64475E-5 | 8.22291E-6 | 4.11037E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| 2^{-18} | 6.57837E-5 | 3.28967E-5 | 1.64493E-5 | 8.22470E-6 | 4.11216E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| 2^{-20} | 6.57842E-5 | 3.28971E-5 | 1.64498E-5 | 8.22515E-6 | 4.11260E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| e^N | 6.57842E-5 | 3.28971E-5 | 1.64498E-5 | 8.22515E-6 | 4.11260E-6 |
| p^N | 1.00 | 1.00 | 1.00 | 1.00 | |

Now, we define the exact error e_ε^N and the computed parameter-uniform maximum pointwise error e^N as follows:

$$e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}}, e^N = \max_\varepsilon e_\varepsilon^N,$$

where y is the numerical approximation to u for various values of N, ε . We also define the computed parameter-uniform rate of convergence to be

$$p^N = \log_2 \left(e^N / e^{2N} \right).$$

The values of ε and N_0 ($N = 3N_0/2$) for which we solve the test problems are $\varepsilon = 2^{-2i}, i = 1, 2, \dots, 10; N_0 = 64, 128, 256, 512, 1024$. The resulting errors e^N and the corresponding numbers p^N after 20 iterations are given in Tables 1-3.

TABLE 2. Errors and convergence rates for Example 2

| ε | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---------------|--------------------|--------------------|--------------------|--------------------|--------------|
| 2^{-2} | 2.36474E-5 1.78 | 6.88645E-6 1.93 | 1.8032E-6 1.98 | 4.56462E-7 1.76 | 1.34503E-7 |
| 2^{-4} | 3.85540E-5 1.22 | 1.65391E-5 1.48 | 5.91880E-6 1.78 | 1.72400E-6 1.93 | 4.51464E-7 |
| 2^{-6} | 4.26883E-5 1.05 | 2.06587E-5 1.10 | 9.64138E-6 1.22 | 4.13580E-6 1.48 | 1.48013E-6 |
| 2^{-8} | 4.37216E-5 1.01 | 2.16919E-5 1.02 | 1.06746E-5 1.05 | 5.16533E-6 1.10 | 2.41053E-6 |
| 2^{-10} | 4.39799E-5 1.00 | 2.19502E-5 1.01 | 1.09329E-5 1.01 | 5.42362E-6 1.02 | 2.66881E-6 |
| 2^{-12} | 4.40444E-5 1.00 | 2.20148E-5 1.00 | 1.09974E-5 1.00 | 5.48819E-6 1.01 | 2.73338E-6 |
| 2^{-14} | 4.40606E-5 1.00 | 2.20309E-5 1.00 | 1.10136E-5 1.00 | 5.50433E-6 1.00 | 2.74952E-6 |
| 2^{-16} | 4.40643E-5 1.00 | 2.20346E-5 1.00 | 1.10173E-5 1.00 | 5.50805E-6 1.00 | 2.75325E-6 |
| 2^{-18} | 4.40655E-5 1.00 | 2.20359E-5 1.00 | 1.10185E-5 1.00 | 5.50930E-6 1.00 | 2.75449E-6 |
| 2^{-20} | 4.40659E-5 1.00 | 2.20362E-5 1.00 | 1.10189E-5 1.00 | 5.50961E-6 1.00 | 2.75480E-6 |
| e^N | 4.40659E-5 | 2.20362E-5 | 1.10189E-5 | 5.50961E-6 | 2.75480E-6 |
| p^N | 1.00 | 1.00 | 1.00 | 1.00 | |

6. CONCLUSION

In this paper, we have developed a finite difference method for solving the singularly perturbed nonlocal boundary-value problem for a second order delay differential equation. This method was based on an exponentially fitted difference scheme on a uniform mesh. The method was proved to be first order accuracy in the discrete maximum norm, with respect to the perturbation parameter ε . Test problems were

TABLE 3. Errors and convergence rates for Example 3

| ε | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---------------|--------------------|--------------------|--------------------|--------------------|--------------|
| 2^{-2} | 3.67457E-5 1.78 | 1.07011E-5 1.94 | 2.80203E-6 1.65 | 8.90948E-7 1.48 | 3.19656E-7 |
| 2^{-4} | 5.99030E-5 1.22 | 2.57004E-5 1.48 | 9.19987E-6 1.78 | 2.68102E-6 1.93 | 7.89100E-7 |
| 2^{-6} | 6.63273E-5 1.05 | 3.21016E-5 1.10 | 1.49846E-5 1.22 | 6.43084E-6 1.48 | 2.30438E-6 |
| 2^{-8} | 6.79328E-5 1.01 | 3.37071E-5 1.02 | 1.65901E-5 1.05 | 8.03061E-6 1.10 | 3.75060E-6 |
| 2^{-10} | 6.83341E-5 1.00 | 3.41084E-5 1.01 | 1.69915E-5 1.01 | 8.43195E-6 1.02 | 4.15194E-6 |
| 2^{-12} | 6.84344E-5 1.00 | 3.42087E-5 1.00 | 1.70918E-5 1.00 | 8.53229E-6 1.01 | 4.25228E-6 |
| 2^{-14} | 6.84595E-5 1.00 | 3.42338E-5 1.00 | 1.71169E-5 1.00 | 8.55737E-6 1.00 | 4.27736E-6 |
| 2^{-16} | 6.84656E-5 1.00 | 3.42397E-5 1.00 | 1.71227E-5 1.00 | 8.56318E-6 1.00 | 4.28313E-6 |
| 2^{-18} | 6.84673E-5 1.00 | 3.42416E-5 1.00 | 1.71246E-5 1.00 | 8.56511E-6 1.00 | 4.28510E-6 |
| 2^{-20} | 6.84677E-5 1.00 | 3.42420E-5 1.00 | 1.71251E-5 1.00 | 8.56558E-6 1.00 | 4.28557E-6 |
| e^N | 6.84677E-5 | 3.42420E-5 | 1.71251E-5 | 8.56558E-6 | 4.28557E-6 |
| p^N | 1.00 | 1.00 | 1.00 | 1.00 | |

addressed for different cases of l_1 and r . The case of $l_1 = r$ was obtained better results than in other cases. Furthermore, the approximate errors and the rates of convergence for test problems was computed for different values of ε and N_0 in Tables 1-3. Numerical results were carried out to show the efficiency and accuracy of the method. Theoretical results represent undergoing more complicated delay problems.

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