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ON A CLASS OF UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN INTEGRO-DIFFERENTIAL OPERATOR

Á. O. PÁLL-SZABÓ

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Abstract. In this paper we consider the $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$, $\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z)$ linear operator, where $\mathscr{D}^n$ is the Sălăgean differential operator and I^n is the Sălăgean integral operator. We study several differential subordinations generated by $\mathscr{L}^n$. We introduce a class of holomorphic functions $L_n^m(\beta)$, and obtain some subordination results.

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1. Preliminaries

Let U be the unit disk in the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$\mathcal{A}_m = \left\{ f \in \mathcal{H}(U) : \ f(z) = z + a_{m+1} z^{m+1} + \cdots, z \in U \right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and $m \in \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \ldots\}$ let

$$\mathcal{H}[a, m] = \left\{ f \in \mathcal{H}(U) : \ f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, z \in U \right\}.$$

Denote by

$$K = \left\{ f \in \mathcal{A} : \Re \frac{z f''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

the class of normalized convex functions in U.

Definition 1 ([5], def. 3.5.1)**.** Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z))$; $\forall z \in U$. We denote by $\prec$ the subordination relation. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a function and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

(i) $\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z)$, $(z \in U)$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i). A dominant $\widetilde{q}$, which satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). The best dominant is unique up to a rotation of U. In order to prove the original results we use the following lemmas.

Lemma 1 (Hallenbeck and Ruscheweyh, [2]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \;\; z \in U$$

then

$$p(z) \prec q(z) \prec h(z), z \in U$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \;\; z \in U.$$

Lemma 2 (Miller and Mocanu, [3]). *Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z), \;\; z \in U$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \;\; z \in U$$

is holomorphic in U and

$$p(z) + n\alpha z p'(z) \prec h(z), \;\; z \in U$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

Definition 2 ([8]). For $f \in \mathcal{A}, n \in \mathbb{N}_0$, the Sălăgean differential operator $\mathcal{D}^n$ is defined by $\mathcal{D}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{D}^0 f(z) = f(z),$$

$$\ldots$$

$$\mathcal{D}^{n+1} f(z) = z\left(\mathcal{D}^n f(z)\right)', z \in U$$

Remark 1. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

Definition 3 ([8])**.** For $f \in \mathcal{A}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}$, the operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$\ldots$$

$$I^n f(z) = I\left(I^{n-1} f(z)\right), z \in U$$

Remark 2. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k,$$

$z \in U$, $(n \in \mathbb{N}_0)$ and $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

Definition 4. Let $\lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda) \mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 3. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U. \tag{1.1}$$

2. Main results

Theorem 1. *Let q be a convex function, $q(0) = 1$ and let h be the function*

$$h(z) = q(z) + z q'(z), z \in U.$$

If $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left[\mathscr{L}^n f(z)\right]' \prec h(z), \quad z \in U \tag{2.1}$$

then

$$\frac{\mathscr{L}^n f(z)}{z} \prec q(z), \quad z \in U$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{L}^n f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n} \right] a_k z^k}{z} = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots \tag{2.2}$$

$z \in U$. From (2.2) we have $p \in \mathcal{H}[1,1]$. Let

$$\mathcal{L}^n f(z) = z p(z), z \in U. \tag{2.3}$$

Differentiating (2.3), we obtain

$$\left[\mathcal{L}^n f(z) \right]' = p(z) + z p'(z), z \in U. \tag{2.4}$$

Then (2.1) becomes

$$p(z) + z p'(z) \prec h(z), z \in U. \tag{2.5}$$

By using Lemma 2, we have

$$p(z) \prec q(z), z \in U,$$

i.e.

$$\frac{\mathcal{L}^n f(z)}{z} \prec q(z), \quad z \in U.$$

□

Remark 4. If $\lambda = 0$ we get Theorem 4 from Oros [6] and for $\lambda = 1$ we get Theorem 4 from Bălăeţi [1].

Example 1. For $\lambda = 0, n = 1, f \in \mathcal{A}$ we deduce that

$$f'(z) + z f''(z) \prec \frac{1}{(1-z)^2}, \quad z \in U$$

implies

$$f'(z) \prec \frac{1}{1-z}, \quad z \in U.$$

Example 2. For $\lambda = 1, n = 1, f \in \mathcal{A}$ we deduce that

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^2}, \quad z \in U$$

implies

$$\frac{\int_0^z f(t) t^{-1} dt}{z} \prec \frac{1}{1-z}, \quad z \in U.$$

Theorem 2. *Let q be a convex function, $q(0) = 1$ and let h be the function*

$$h(z) = q(z) + zq'(z), z \in U.$$

If $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left(\frac{z\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)}\right)' \prec h(z), \quad z \in U \tag{2.6}$$

then

$$\frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \prec q(z), \quad z \in U$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} = \frac{z + \sum_{k=2}^{\infty}\left[k^{n+1}(1-\lambda) + \lambda \frac{1}{k^{n+1}}\right] a_k z^k}{z + \sum_{k=2}^{\infty}\left[k^{n}(1-\lambda) + \lambda \frac{1}{k^{n}}\right] a_k z^k}.$$

We have $p'(z) = \dfrac{\left(\mathcal{L}^{n+1} f(z)\right)'}{\mathcal{L}^n f(z)} - p(z)\dfrac{\left(\mathcal{L}^{n} f(z)\right)'}{\mathcal{L}^n f(z)}$ and
$p(z) + zp'(z) = \left(\dfrac{z\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)}\right)'$.
Relation (2.6) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z), \quad z \in U.$$

By using Lemma 2 we have

$$p(z) \prec q(z) \ \ i.e. \ \frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \prec q(z), \quad z \in U.$$

□

Theorem 3. *Let q be a convex function, $q(0) = 1$ and let h be the function*

$$h(z) = q(z) + zq'(z), z \in U.$$

If $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left(\mathcal{L}^{n+1} f(z)\right)' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] \prec h(z), \quad z \in U \tag{2.7}$$

then

$$\left[\mathcal{L}^{n} f(z)\right]' \prec q(z), \quad z \in U$$

and this result is sharp.

Proof. By using the properties of operator $\mathscr{L}^n$, we obtain

$$\mathscr{L}^{n+1} f(z) = (1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n+1} f(z), z \in U. \tag{2.8}$$

Then (2.7) becomes

$$\left[(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n+1} f(z)\right]' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] \prec h(z), \quad z \in U. \tag{2.9}$$

After computation we get

$$(1-\lambda)\left[\mathscr{D}^{n+1} f(z)\right]' + \lambda\left[I^{n-1} f(z)\right]' \prec h(z)$$

or equivalently

$$(1-\lambda)\left[z\left(\mathscr{D}^{n} f(z)\right)'\right]' + \lambda\left[z\left(I^{n} f(z)\right)'\right]' \prec h(z).$$

The above relation is equivalent to

$$(1-\lambda)\left[\left(\mathscr{D}^{n} f(z)\right)' + z\left(\mathscr{D}^{n} f(z)\right)''\right] + \lambda\left[\left(I^{n} f(z)\right)' + z\left(I^{n} f(z)\right)''\right] \prec h(z)$$

or

$$\left[\mathscr{L}^{n} f(z)\right]' + z\left[\mathscr{L}^{n} f(z)\right]'' \prec h(z), \quad z \in U. \tag{2.10}$$

Let

$$p(z) = (1-\lambda)\left[\mathscr{D}^{n} f(z)\right]' + \lambda\left[I^{n} f(z)\right]' = \left[\mathscr{L}^{n} f(z)\right]', \quad z \in U \tag{2.11}$$

$$= (1-\lambda)\left[z + \sum_{k=2}^{\infty} k^n a_k z^k\right]' + \lambda\left[z + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^k\right]' =$$

$$= (1-\lambda)\left[1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1}\right] + \lambda\left[1 + \sum_{k=2}^{\infty} \frac{1}{k^{n-1}} a_k z^{k-1}\right] =$$

$$= 1 + \sum_{k=2}^{\infty}\left[k^{n+1}(1-\lambda) + \lambda\frac{1}{k^{n-1}}\right] a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots$$

In view of (2.11), we deduce that $p \in \mathscr{H}[1,1]$. Using the notation in (2.11), the (2.10) differential subordination becomes

$$p(z) + z p'(z) \prec h(z) = q(z) + z q'(z), \quad z \in U.$$

By using Lemma 2 we have

$$p(z) \prec q(z) \;\; i.e. \;\; \left[\mathscr{L}^{n} f(z)\right]' \prec q(z), \quad z \in U.$$

□

Remark 5. If $\lambda = 0$ we get Theorem 2 from Oros [6] and for $\lambda = 1$ we get Theorem 2 from Bălăeţi [1].

Example 3. For $\lambda = 0$, $n = 1$, $f \in \mathcal{A}$ we deduce that

$$f'(z) + 3zf''(z) + z^2 f'''(z) \prec 1 + 2z, \ \ z \in U$$

implies

$$f'(z) + zf''(z) \prec 1 + z, \ \ z \in U.$$

Theorem 4. *Let $h \in \mathcal{H}(U)$ such that $h(0) = 1$ and*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \ \ z \in U.$$

If $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{L}^{n+1} f(z)\right)' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] \prec h(z), \ \ z \in U \tag{2.12}$$

then

$$\left[\mathcal{L}^{n} f(z)\right]' \prec q(z), \ \ z \in U$$

where q is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and is the best dominant.

Proof. If we use the differential subordination technique we can see that the function g is convex.[3], p. 66 By using (2.11) we obtain

$$\left(\mathcal{L}^{n+1} f(z)\right)' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] = p(z) + zp'(z), \ \ z \in U$$

Then (2.12) becomes

$$p(z) + zp'(z) \prec h(z), \ \ z \in U.$$

Since $p \in \mathcal{H}[1,1]$, we deduce that $p(z) \prec q(z)$, i.e.

$$\left[\mathcal{L}^{n} f(z)\right]' \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \ \ z \in U$$

and q is the best dominant. □

Remark 6. If $\lambda = 0$ we get Theorem 3 from Oros [6].

Example 4. For $\lambda = 0$, $n = 0$, $h(z) = \frac{1+z}{1-z}$ we deduce that

$$f'(z) + zf''(z) \prec \frac{1+z}{1-z}, \ \ z \in U,$$

implies

$$f'(z) \prec 1 - \frac{2}{z} ln(1-z), \ \ z \in U.$$

Theorem 5. *Let $h \in \mathcal{H}(U)$ such that $h(0) = 1$ and*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in \mathcal{A}$ satisfies the differential subordination

$$\left[\mathcal{L}^n f(z)\right]' \prec h(z), \quad z \in U \tag{2.13}$$

then

$$\frac{\mathcal{L}^n f(z)}{z} \prec q(z), \quad z \in U$$

where q is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and is the best dominant.

Proof. If we use the differential subordination technique we can see that the function g is convex. [3], p. 66. Differentiating both sides in (2.2) we obtain

$$\left[\mathcal{L}^n f(z)\right]' = p(z) + zp'(z), \quad z \in U$$

Then (2.13) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Since $p \in \mathcal{H}[1,1]$, we deduce that $p(z) \prec q(z)$, i.e.

$$\frac{\mathcal{L}^n f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U$$

and q is the best dominant. □

Remark 7. If $\lambda = 0$ we get Theorem 5 from Oros [6] and for $\lambda = 1$ we get Theorem 5 from Bălăeţi [1].

Example 5. For $\lambda = 0$, $n = 1$, $h(z) = \frac{1}{(1+z)^2}$ we deduce that

$$f'(z) \prec \frac{1}{(1+z)^2}, \quad z \in U,$$

implies

$$\frac{f(z)}{z} \prec \frac{1}{1+z}, \quad z \in U.$$

We get the same result as [4].

Definition 5 ([7], [9], [1], [6]). If $0 \leq \beta < 1$ and $n \in \mathbb{N}$, we let $L_n^m(\beta)$ stand for the class of functions $f \in \mathcal{A}_m$, which satisfy the inequality

$$\Re\left[\mathcal{L}^n f(z)\right]' > \beta, \quad (z \in U).$$

Remark 8. For $n = 0$ we obtain $\Re f'(z) > \beta$.

Theorem 6. *The set $L_n^m(\beta)$ is convex.*

Proof. Let the function

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k_i} z^k, \quad i = 1,2 \quad z \in U$$

be in the class $L_n^m(\beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$$

with $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$ is in $L_n(\beta)$. Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^k, \quad z \in U$$

then

$$\mathscr{L}^n h(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n} \right] (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^k, \quad z \in U. \tag{2.14}$$

Differentiating (2.14), we get

$$\left[\mathscr{L}^n h(z)\right]' = 1 + \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}} \right] (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^{k-1}.$$

Hence

$$\Re \left[\mathscr{L}^n h(z)\right]' = 1 + \Re \left\{ \mu_1 \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_1} z^{k-1} \right\} +$$

$$+ \Re \left\{ \mu_2 \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_2} z^{k-1} \right\}. \tag{2.15}$$

Since $f_1, f_2 \in L_n^m(\beta)$, we obtain

$$\Re \left\{ \mu_i \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_i} z^{k-1} \right\} > \mu_i (\beta - 1), \quad i = 1,2. \tag{2.16}$$

Using (2.16) we get from (2.15)

$$\Re \left[\mathscr{L}^n h(z)\right]' > 1 + \mu_1 (\beta - 1) + \mu_2 (\beta - 1),$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\Re \left[\mathscr{L}^n h(z)\right]' > \beta, \quad (z \in U)$$

i.e. $L_n^m(\beta)$ is convex. □

Theorem 7. *If* $0 \le \beta < 1$ *and* $m, n \in \mathbb{N}$ *then we have*

$$L_n^m(\beta) \subset L_{n+1}^m(\delta),$$

where $\delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right)$ *and* $\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt$. *The result is sharp.*

Proof. Assume that $f \in L_n^m(\beta)$. Let $\mathscr{L}^n f(z) = zp(z), z \in U$. Differentiating, we obtain

$$\left[\mathscr{L}^n f(z)\right]' = p(z) + zp'(z), z \in U.$$

Since $f \in L_n^m(\beta)$, from Definition 5 we have

$$\Re\left(p(z) + zp'(z)\right) > \beta, z \in U$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\beta - 1)z}{1+z} \equiv h(z), z \in U$$

By using Lemma 1, we have:

$$p(z) \prec q(z) \prec h(z), z \in U,$$

where

$$q(z) = \frac{1}{mz^{\frac{1}{m}}}\int_0^z \frac{1 + (2\beta-1)t}{1+t} t^{\frac{1}{m}-1} dt =$$

$$= \frac{1}{mz^{\frac{1}{m}}}\int_0^z \left[2\beta - 1 + 2(1-\beta)\frac{1}{1+t}\right] t^{\frac{1}{m}-1} dt =$$

$$= \frac{1}{mz^{\frac{1}{m}}}\int_0^z (2\beta-1) t^{\frac{1}{m}-1} dt + \frac{2(1-\beta)}{mz^{\frac{1}{m}}}\int_0^z \frac{t^{\frac{1}{m}-1}}{1+t} dt =$$

$$= 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right)\frac{1}{z^{\frac{1}{m}}}, z \in U.$$

The function q is convex and is the best dominant. From $p(z) \prec q(z)$ follows that

$$\Re p(z) > \Re q(1) = \delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right),$$

from which we deduce that $L_n^m(\beta) \subset L_{n+1}^m(\delta)$. □

Remark 9. If $\lambda = 0$ we get Theorem 1 from Oros [6] and for $\lambda = 1$ we get Theorem 1 from Bălăeţi [1].

Theorem 8. *Let q be a convex function in U with $q(0)=1$ and let*

$$h(z)=q(z)+\frac{1}{c+2}zq'(z), z\in U,$$

where c is a complex number, with $\Re c>-2$.
If $f\in L_n^m(\beta)$ and $F=I_c(f)$, where

$$F(z)=I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)dt,\ \ \Re c>-2, \tag{2.17}$$

then

$$\left[\mathscr{L}^n f(z)\right]'\prec h(z),\ \ z\in U, \tag{2.18}$$

implies

$$\left[\mathscr{L}^n F(z)\right]'\prec q(z),\ \ z\in U,$$

and this result is sharp.

Proof. From (2.17), we have

$$z^{c+1}F(z)=(c+2)\int_0^z t^c f(t)dt,\ \ \Re c>-2,\ \ z\in U. \tag{2.19}$$

Differentiating, with respect to z, we obtain

$$(c+1)F(z)+zF'(z)=(c+2)f(z),\ \ z\in U$$

and

$$(c+1)\mathscr{L}^n F(z)+z\left[\mathscr{L}^n F(z)\right]'=(c+2)\mathscr{L}^n f(z),\ \ z\in U. \tag{2.20}$$

Differentiating (2.20), we obtain

$$\left[\mathscr{L}^n F(z)\right]'+\frac{z}{c+2}\left[\mathscr{L}^n F(z)\right]''=\left[\mathscr{L}^n f(z)\right]',\ \ z\in U. \tag{2.21}$$

Using (2.21), the differential subordination (2.18) becomes

$$\left[\mathscr{L}^n F(z)\right]'+\frac{1}{c+2}z\left[\mathscr{L}^n F(z)\right]''\prec h(z)=q(z)+\frac{1}{c+2}zq'(z),\ \ z\in U. \tag{2.22}$$

Let

$$p(z)=\left[\mathscr{L}^n F(z)\right]'=\left\{z+\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]a_k z^k\right\}'= \tag{2.23}$$

$$=1+p_1z+p_2z^2+\cdots,\ \ z\in U,\ \ p\in\mathscr{H}[1,1].$$

Replacing (2.23) in (2.22) we obtain

$$p(z)+\frac{1}{c+2}zp'(z)\prec h(z)=q(z)+\frac{1}{c+2}zq'(z),\ \ z\in U,$$

Using Lemma 1, we obtain $p(z)\prec q(z)$ i.e.

$$\left[\mathscr{L}^n F(z)\right]'\prec q(z),\ \ z\in U,$$

and q is the best dominant. □

Remark 10. If $\lambda = 0$ we get Theorem 2.2 from Tăut et alii [9].

Example 6. If we take $c = 1+2i$ and $q(z) = \dfrac{1+z}{1-z}$ then
$h(z) = \dfrac{\left(1-z^2\right)(3+2i)+2z}{(3+2i)(1-z)^2}$.
From Theorem 8 we deduce

$$\left[\mathscr{L}^n f(z)\right]' \prec \frac{\left(1-z^2\right)(3+2i)+2z}{(3+2i)(1-z)^2}, \quad z \in U,$$

implies

$$\left[\mathscr{L}^n F(z)\right]' \prec \frac{1+z}{1-z}, \quad z \in U,$$

where F is given by (2.17).

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Author's address

Á. O. Páll-Szabó
Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail address: pallszaboagnes@math.ubbcluj.ro